A Basic Course in Real Analysis

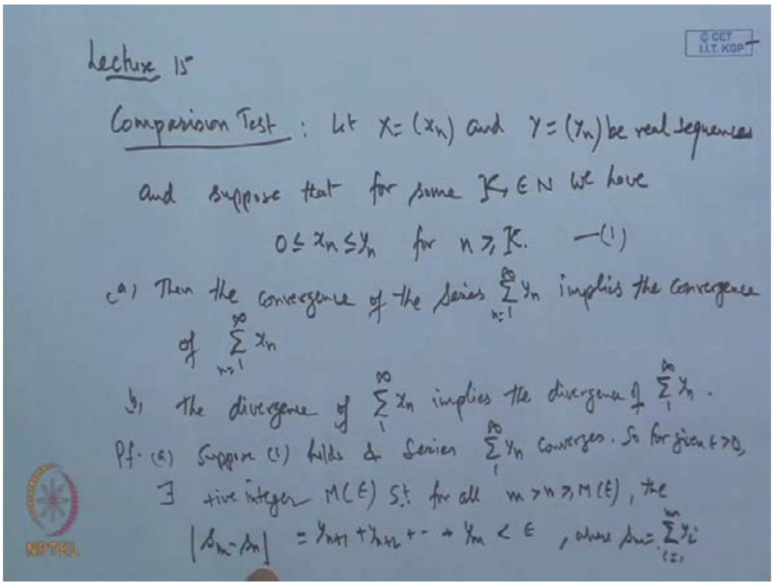
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 22

Comparison test for series, Absolutely convergent and Conditional convergent series

Last time we have discussed the various test for the convergence of the series. Now, in continuation we will do few more test, discuss a few more test, like a comparison test and limit comparison test. And after that we will see come to the conditional and absolute convergence of the series.

(Refer Slide Time: 00:49)



So, first test which in continuation we have another test which is known as the comparison test. The test says, let X which is say x n, and Y say y n be real sequences. And suppose for suppose that for some K, which is of course, a positive integer. We have this inequality 0 is less than or equal to x n less than or equal to y n. And this is valid for n greater than equal to K. It may be true for all n’s also. If not possible, then at least after certain stage, if this inequality holds, then the series the convergence of the series, then convergence of the series sigma y n; n is of course, 1 to infinity. Convergence of the series implies the convergence of the series sigma x n 1 to infinity, 1 to infinity.

And the divergence of the series sigma x n, n is 1 to infinity implies the divergence of the series sigma y n. In fact, this type result we have discussed already in case of the sequences. So, similar results we have for a sequence of non-negative series, we have sequence of the non-negative terms or terms are non-negative.

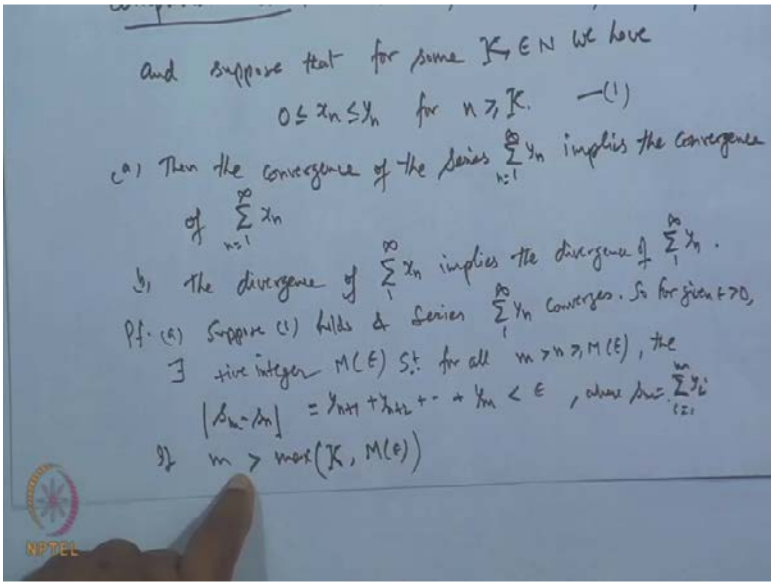
So, what the comparison test says? If we are having a series say sigma n of each terms is non-negative. Then to discuss the convergence of the series or to find out the nature of the series, sigma x n. If we are able to identify some sequence by n for which this lesson 0 is less than equal to x n less than equal to y n holds for either for all n or may be after certain stage.

Then the convergence of this y n will implies the convergence of the series sigma x n. It means with the help of x n. If we find out a suitable y n, for which the convergence of the series sigma y n nature is known, then one can easily establish the convergence of y n. And in case if this relation is true, but if the series is diverging sigma x n, then the divergence of y n will be there. It means, if we are interested to know the, expect or something that the series is diverging, then we have to find this relation where this is diverging, then this will diverge. So, this is the main motto for this comparison test. The proof of course, is very straight forward.

Suppose, we have this series converges, given the relations hold 1, holds and the series sigma y n, 1 to infinity converges, this is given. So, by definition of the convergence a series is said to be convergent if and only if it is Cauchy, So, for a given epsilon greater than 0, there exist a positive integer say capital M which will depend on epsilon such that for all m which is greater than say n and greater than equal to M epsilon. Of course, all m n greater than equal to M epsilon I am choosing m to be larger than n for all m n greater than equal to M.

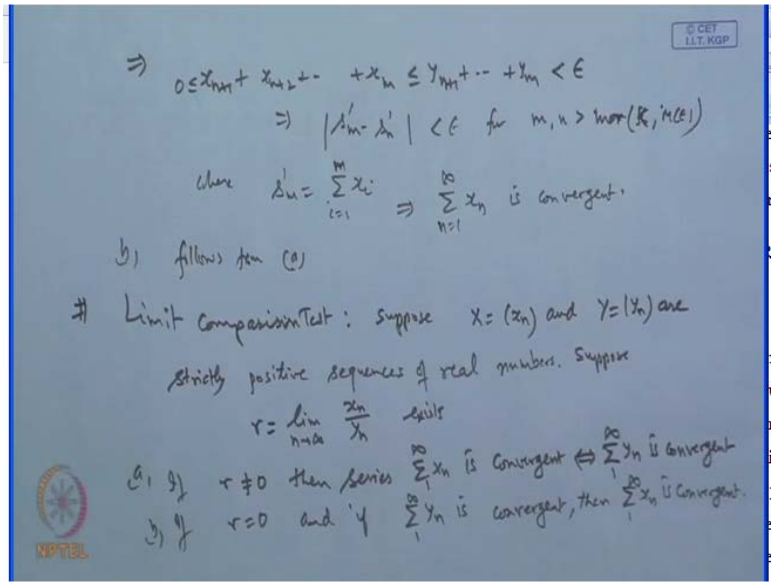
The condition that y n plus 1, y n plus 2, up to say y m is less than epsilon, because basically this condition is what? This is nothing but the s m minus s n, where s m is the sigma y i, i is 1 to m. So, basically this is a Cauchy convergence criteria, so a series because it is giving to be convergence. So, by means of Cauchy convergence criteria for any epsilon one can identify a positive integer. So, there this result holds.

(Refer Slide Time: 06:48)



Now, let us choose m to be greater than the maximum value of this K, for which this result is true. This inequality is valid as well as the M which we have already got it, because of this convergence, If I choose M to be greater than this, then in that case the result this thing. Because y x n is less than equal to y n. So, this part will give that x n plus 1, y n x n plus 2 etcetera.

(Refer Slide Time: 07:24)



This will imply that x n plus 1 x n plus 2 and so on, up to say x n. This will remain less than y n plus 1, y n plus 2, y n because x n is less, because m is greater than the number.

So, it satisfies both the conditions. And then this will be the less than epsilon, this is true for all m greater than the maximum of (( )). And this is greater than 0, because all terms are positive. So, it will be greater than or at the most equal to 0. So, what this shows? This shows that this sequence s dash m minus s dash n. This remains less than epsilon, for all m n, greater than equal to capital N capital say K maximum of K M epsilon in some integer k, m epsilon. And where s dash m is what? s dash m stands for sigma x i, i is 1 to m.

So, this shows the sequence x, s dash n is a Cauchy sequence, therefore, it is convergent. So, this implies the series, sigma x n, 1 to infinity is convergent. So, the proof is very simple. The statement b follows from the first it follows from what the, we says the series diverges, then sigma b n diverging. Suppose, the series given, sigma x n to a divergent series, and let this series sigma y n converges. But the relation one also holds. So, it means, if a series y n is convergent, then according to the part a, x n must be convergent, sigma x n must be convergent. But it is contradicts the result, given thing because the given is sigma x n is divergent. Therefore, a contradiction is because our assumption this series convergence so wrong. So, b follows from a immediately nothing to prove here is.

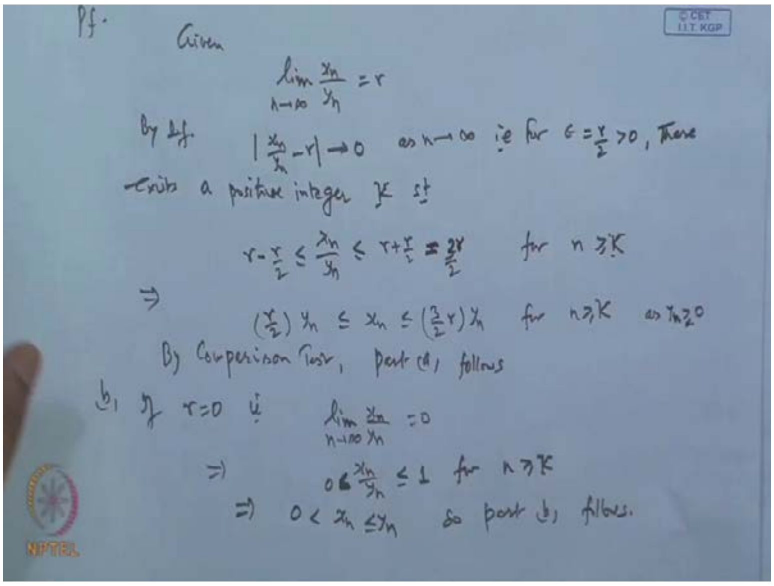
Now, another test is let us first take the test then another test is which we call it as a limit comparison test. In fact, this is the modified version of this. Here, because it is very difficult to get this identity, which is valid for all n, all after a certain stage. It is very, very difficult for a sequence x n and y n 2 arbitrary sequences are there. So, it is not possible always to get such an inequality. Therefore, this stage comparison may not be very much helpful unless you know this inequality.

So, what is a, we have slightly it is modified and a simpler form is given which is known as the limit comparison test. The test says suppose that suppose x which is x n and y which is y n are strictly positive sequences of real numbers. And suppose the limit of this x n divided by y n over n, when n tends to infinity exist. Suppose, this limit exist say is equal to r, then what he says is if r is different from 0 means, when the limit of x n over y n is different from 0. Then the series sigma x n, 1 to infinity converge is convergent if and only if sigma y n is convergent. It means the behavior of the series sigma y n and sigma x n are parallel, that if this limit x n over y n exist and is different from 0.

Then both the series will have same nature. If this series is convergent this has to be convergent. And second part is, if r is equal to 0 of course, the divergence part here is not mentioned. Divergence, we cannot say if one is diverging, other will also be diverging, but; obviously, if we have this sort of relation and that also gives a result for a divergence of the series. But here we can only take the claim about the convergent part of it, that is.

So, for the convergence is concerned, both these series will have the same nature. Same means convergent then and if r is 0 and if sigma y n is convergent, then sigma x n, 1 to infinity is convergent. So, there is certain limitation in this, limitation is that we are unable to test the divergence part. We cannot claim anything about the divergent. But obviously, for the convergent which we are interested more to judge the convergent with the help of the given series.

(Refer Slide Time: 13:55)



So, let us see the proof of this. Now, what the proof is given that, given the limit x n over y n as n tends to infinity is, say r which is different from 0 of course, r. Now, once the limit is given. Then for any epsilon greater than 0, by definition what happened? By definition this mod of x n over y n minus r; this will go to 0 as n is sufficiently large or for epsilon greater than 0. That is for a given epsilon, that is for a given epsilon say, equal to r I am taking r by 2 greater than 0. There exist a integer a positive integer say k, such that, x n over y n. This will lie between what two bound, r minus epsilon, r plus epsilon. So, here the r minus epsilon means r by 2 and then here is also r plus epsilon r by 2. And which is; obviously, less than 2 r, this will be less than or equal to 2 r, this hardly matters, even this 3 by 2 will work.

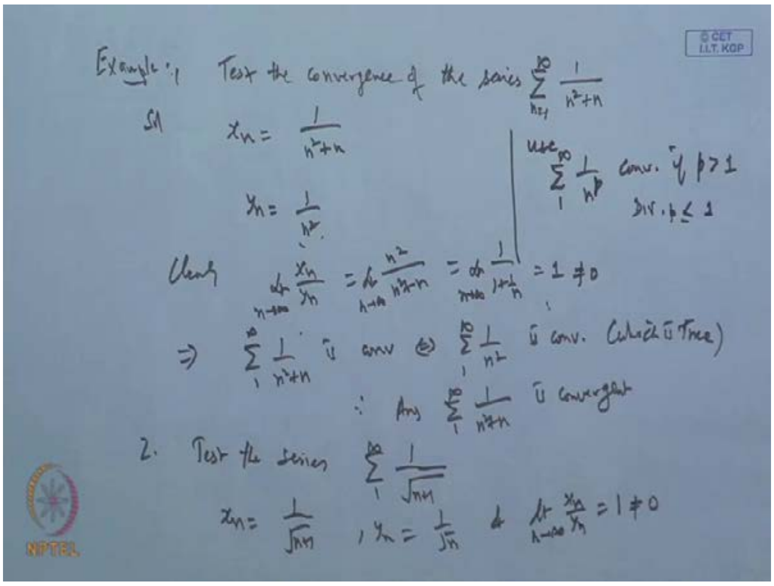
So, we get the 3 by 2 or this is equal to 3 by 2 r. Now, if it let this n, this is 2 for all n greater than equal to K is not this is 2 for all n greater than equal to K so this happen. Now, from here, can we say that r by 2 into y n is less than equal to x n which is less than equal to 3 by 2 r into y n. Now, apply the comparison test, and this is true for n greater than equal to k, because the y n s positive. Therefore, when we are multiplying this it is not going to change the inequality as y n s are greater than or equal to 0 of course, greater than 0 otherwise this will equal to 0 will help problem. So, y n is greater than 0. Now, apply the comparison test. So, by comparison test, we can get, this is comparison o and here also its o not I o comparison test.

So, by comparison test we can say the result as follows: part a follows that is what. Now, second is part b, if r is 0. r is 0 means, limit of this x n over y n as n tends to infinity is 0. Now, since x n and y n both is positive non negative. So, a ratio cannot be negative. So, clearly from here it implies that x n over y n will always be greater than or equal to 0. In fact, it is strictly greater than 0, because they are non negative terms. Strictly positive sequences of real numbers, none is 0 here.

So, x n over y n will be strictly greater than 0. Now, since the limiting value is 0, it means the terms are keep on decreasing and decreases to 0. So, after a certain stage this ratio will remain less than 1. So, once it is less than 1. So, you can say this is less than equal to 1 for n greater than equal to K, this is true because limit is 0 means it keeps on decreasing and decreases to 0. So, after a certain stage the ratio will remain less than or equal to 1 and it is equal to or greater than 0. So, now, what happens? If we apply the, if suppose sigma y n is convergent. So, from here we multiply y by n. So, this implies that x n x 0 is less than x n less than equal to y n. So, if sigma y n convergent, sigma x n will converge. So, part b follows clear.

So, these are the 2 results, which will help you in getting the nature of the series, whether this is convergent or not. Only what we have to do? We have to suitably identify y n and then inequality or may be the limit problem. I advise that limit is a much better way of judging the convergence of the series. So, how to identify y n? So, that limit of x n over y n will exist. And then nature of the y n will decide the nature of the x n.

(Refer Slide Time: 19:39)



Let us see the few examples, which will help in getting this thing clear. Suppose I wanted to test the convergence of the series, sigma n is 1 to infinity, 1 over n square plus n. Now, when we say the convergence, it does not mean that, we have to test only the convergence part. Convergence of the series means, we have to see whether the series is a convergent series or divergent. But the way of writing is test the convergence means it includes both whether the series is converging and diverging.

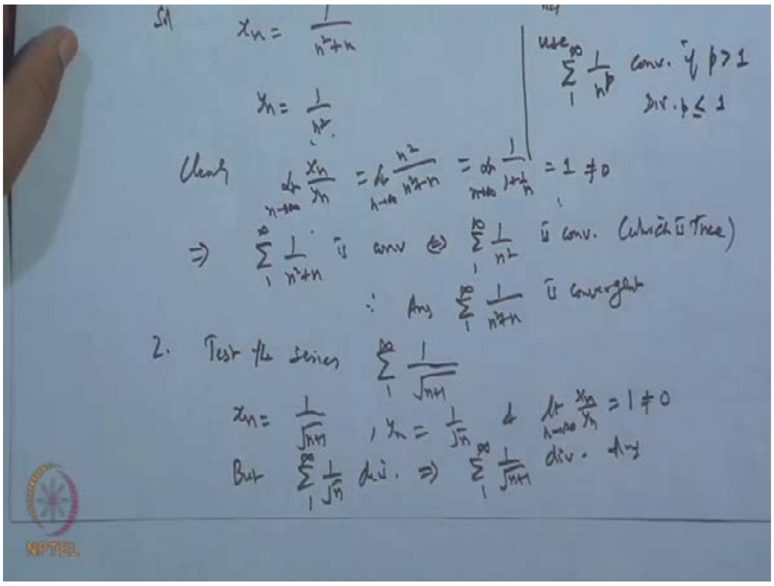
Let us see, this is the sequence of non negative real numbers strictly of course, positive, because n is 1 to infinity. Therefore, we can apply the ratio comparison limit test. But what comparison limit test says, there must be the sequence x n and y n. This sequence, if you want to test this series, sigma n you have to identify y n such a way. So, that the limit of this exists in order to identify this. So, here x n is given to be 1 over n square plus n. Now, normally, when we say y n to be identified, we normally use this result sigma. We know sigma 1 by n to the power p when n is 1 to infinity. This is convergent if p is strictly greater than 1, and diverges if p is strictly less than or equal to 1. So, basically the y n should be chosen in such a way. So, that it will fall in one of this category.

So, here what the trick is, let us take the term n, anything common here outside any square if we take outside common from the denominator take the largest power of n say outside. Then what happens? If we choose y n to be 1 by n square, then when you are choosing n square outside, then this becomes 1 over 1 plus 1 by n. So, limit of this x n over y n will exist. So, clearly limit of x n over y n when n tends to infinity is nothing but what this is equal to n square over n square plus n limit as n tends to infinity divide by n square. So, 1 plus 1 by n, and limit as n tends to infinity, and this limit is 1 which is different from 0.

So, what we see here that if x n and y n both are the strictly positive real sequence of real numbers, such that limit of x n over y n exist which is different from 0. Therefore, this series is convergent if and only if this series is convergent. But this series is of the form sigma 1 upon n to the power p, where p is greater than 1. So, series is convergent. So, this implies that sigma 1 over n square plus n is convergent if and only if sigma 1 by n square is convergent which is true. Because of this, therefore, answer is the series will be convergent.

So, answer is the series 1 to infinity 1 over n square plus n is convergent. Now, similarly if we go for the another example, say let us take this example sigma, test the series sigma 1 over under root n plus 1, 1 to infinity. Again, this is a sequence of non negative real number strictly positive real numbers. So, to test this, we will apply the limit comparison test. So, let us take here choose x n here is 1 over root n plus 1. Then y n, you take it the term which is highest power here is highest power here. So, highest power is 1, there is no problem, n the higher when you take outside it becomes 1 by root n. So, if I take this, and the limit of x n over y n when we choose the limit the limit will come out to be 1 which is different from 0. You just see, take root n, and divide by it means the nature of these two series are identical.

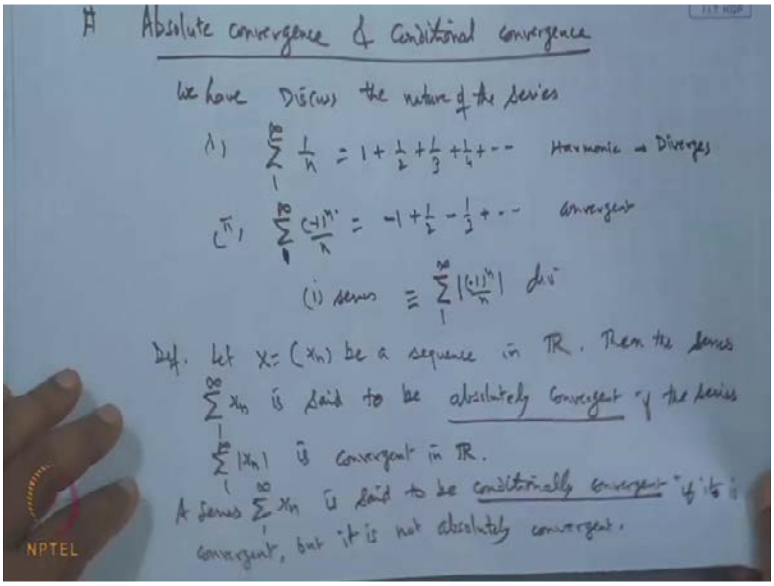
(Refer Slide Time: 24:43)



But the series, sigma 1 by root n 1 to infinity. This is of the form sigma n to the power 1 upon n to the power p where p is less than 1. So, diverges therefore, the series sigma 1 over under root n plus 1 will also diverge and that is the answer. So, we can get it this way. So, main idea is that you have to pick it up the suitable the terms y n. So, that we can compare it with our given sequence, given series x n, the terms of the series x n and hence, the one can identify the nature.

Now, this is the one which we very important these test are very testing important, because it gives immediately the series nature of the series without going for the sum. Because sometimes we are not interested in getting exactly the sum of the series or the limit of the sequence of partial sum, because it will not help as we are not interested in finding the sum of the series. We are interested only finding the series whether it is convergent or divergent and for these this test is very much helpful. Now, there are some more tests which we will discuss after the concept of the absolute convergence and conditional convergent.

(Refer Slide Time: 26:14)



So, let us take this absolute convergence and conditional convergence. Now, we have seen the 2 types of series, we have discussed the nature of this 2 series; one is sigma 1 by n, n is 1 to infinity, and second one is sigma 1 to infinity minus 1 to the power n. Now, what is the difference between these two? If we look at the first series all the terms are positive. And in fact strictly greater than 0, they are non negative is strictly positive strictly positive real numbers. And this is a harmonic series which we have already shown is a diverging one diverges, while in the second case the terms are alternately positive negative.

So, if we take we start with this 1 to infinity or may be 0, I will take 0 no 1 to infinity. Let us take here we can say minus 1 plus half, minus 1 by 3 and so on like this. So, the terms are alternately positive negatives. And this is an alternating series, and we have also seen that this series is a convergent one. Because if you remember, we have find out the even number of terms and the odd, sum of the odd number of terms and then the difference of this two is bounded by 1 upon 2 to the power 2 n plus 1 which goes to 0. So, limit of even number terms and odd number terms are coming to be the same and then limit exists. And in fact, the sum we can identify the sum for this series. So, it is a convergent.

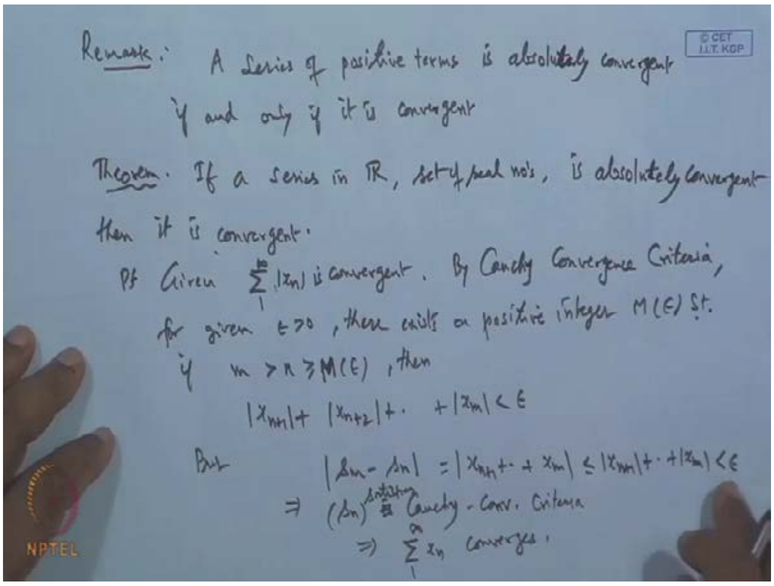
Now, this 2 series gives an idea of the 2 different types of convergence; one is whether when the original series is convergent, but when you replace the series terms of the series by its absolute value, because basically 1 by n is the absolute value of this, because this first part is nothing but what? This first part 1 series, first series that is the sigma is nothing but the absolute term of this thing equal to this. So, the original series is convergent, but when you take its absolute sum of its absolute term it diverges. So, this shows that the series the convergence can be break up into 2 types; one is the odd convergence which is we call it as a conditional convergence, another one is the absolute convergence. So, we define the absolute convergence of the series as follows: let x which is x n be a sequence in r in the real numbers set of real numbers, we say, the series is convergent.

Then the series sigma x n say 1 to infinity is said to be absolutely convergent if the series of its absolute terms that is sigma of mod x n n is 1 to infinity is convergent is convergent in r. So, one more thing when we say the series is convergent in r it means the sum of this series must be a real number or the limit of the sequence of x n must exist, and should be a real number, the point must be in real number. There are this, if suppose, I take the set say minus 1, say 0 to 1 open interval. And if I say the sequence 1 by n, then this sequence is not convergent in 0 1 why because the limiting value is coming to be 0 which is away from the set.

So, for the convergence when we say the limit of the sequence of partial sum must exist, and it should be the point in the set, where the domain, where we are considering. So, when we say it is convergent in R means this sum must be a real. So, a series sigma x n is said to be absolute convergent if the series of its corresponding series of its absolute term is convergent in R. And a series is said to be a series is said to be conditionally convergent, a series sigma x n 1 to infinity is said to be conditionally convergent if it is convergent, but it is not absolutely convergent. So, that is what in.

So, such type of basically case occurs when the series is having at some positive, some negative terms, because in that case only we can talk about the conditional convergence and the absolute convergence. If all the terms of the series are non negative, and if the series is convergent then; obviously, absolute convergence, conditional convergence is the same, because there is no difference at all. So, we say the results, which is valid for the non negative series of the non negative terms.

(Refer Slide Time: 33:19)



So, as a remark you can say a series of positive terms, is absolutely convergent if and only if it is convergent. Because once the non negative terms are there, no question over the conditional case circles. Now, in this sequence we have few results which will help in further study. First result is if a series in R in set of real numbers we would denote by R real numbers if a series in R is absolutely convergent then convergent then it is convergent.

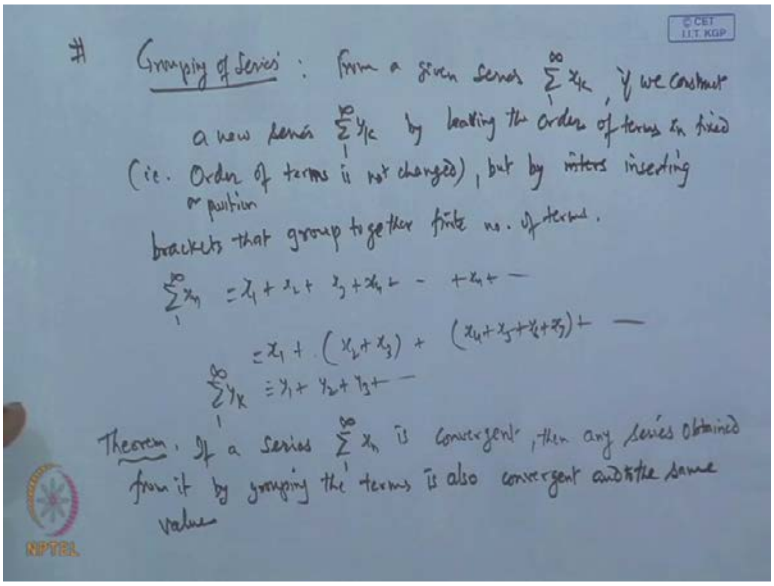
I think proof is simple when the series in R is absolutely convergent. So, what is the meaning of this is? The series of its non negative terms is convergent that is when we replace the term y is a corresponding absolute terms, then the corresponding series is convergent. So, given that sigma of mod x n 1 to infinity convergent even this series is convergent.

So, by definition by Cauchy criteria so Cauchy convergence criteria, we can say that for a given epsilon greater than 0 there exist a positive integer capital M say which depends on epsilon such that, if we take m and n both are greater than or equal capital M, because we have chosen capital M which depends on epsilon capital M, then s m minus s n should be less than epsilon. Then mod of x n plus 1 plus mod x n plus 2 up to mod x m is less than epsilon. But basically, what? This is nothing but and this will be, but what is the mod of s m minus s n? This mod of s m minus s n is mod of x n plus 1 up to x m which is less than equal to mod of x n plus 1 and so on up to x m by (( )), and but this is less than epsilon.

So this is less than epsilon. So, this implies shows the sequence s n satisfy the Cauchy convergence criteria is Cauchy. Hence, it is convergent Cauchy is satisfying, or Cauchy sequence s n satisfies Cauchy convergence criteria. Therefore, the limit of s n will exist therefore; the series sigma x n 1 to infinity converges. That is what is because epsilon is n arbitrary thing. So, we can see this is convergent. Now, when we have the series sigma x n then a term of the series is fixed up now, that is first term, second term, third term onwards.

Now, without changing the position of the terms if I regroup the terms, then the new series obtained. The question is whether this series will retain the same character as the earlier one. If the earlier one is convergent, whether by regrouping the terms of the series without changing their order, the new series whether it remain convergent or not the answer is yes. If a series is convergent, and if we do not change the order of these terms that is the first term remain on the first position, third term remains at the third position. But we regroup it, may be first three terms we are combining, then another 5 terms we are combining, like this way Then the new series so obtained will be convergent if the original series is convergent, and not only will this it will have the same sum.

(Refer Slide Time: 39:10)

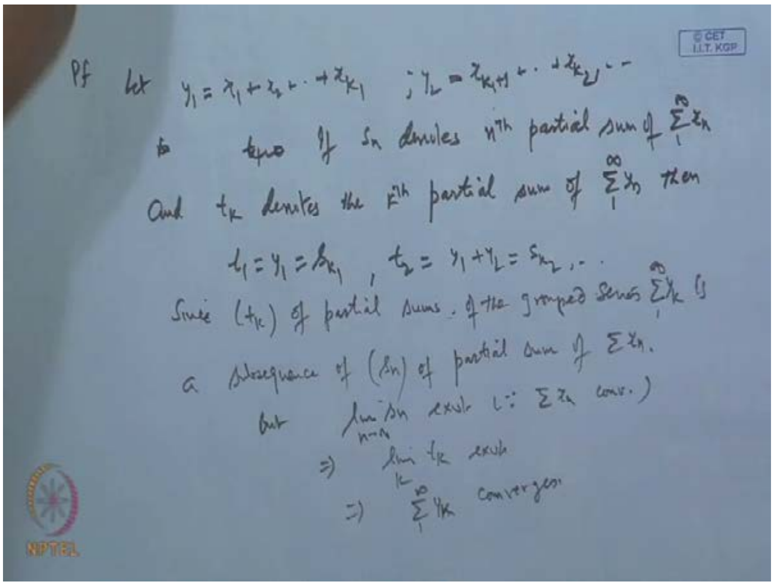


So, this is called the grouping of the series. So, what is the grouping of series? Grouping of series we mean, suppose a series is given from a given series sigma x k, k is 1 to infinity. If we construct a new series sigma y k, k is 1 to infinity by leaving the order of the terms x n fixed that is the order of the terms or the position of the terms is not changed.

So, if a new series is obtained by leaving the order of these terms x n fixed, but by inserting the brackets by combining, by inserting brackets, that group together finite number of terms. Then such a series we call it as grouping. Suppose I have a series x 1 plus x 2 plus x 3 plus x 4 plus x n and so on. This is our original series 1 to infinity, and now what we do is? We construct a series like this x 1 say plus x 2 plus x 3 say x 4 x 5 x 6 x 7 like this way. So, this new series which obtained, say equivalent to the y 1 plus y 2 which is equivalent to y 1 plus y 2 y 3 and so on.

So, this series sigma y k k is 1 to infinity this is called the grouped series of corresponding to x n, a new series is obtained. Now, the nature of these two series will remain the same. So that is the result say, if a series sigma x n 1 to infinity is convergent. Then any series obtained from it by grouping the terms is also convergent, and to the same value it means the sum will not change.

(Refer Slide Time: 43:28)



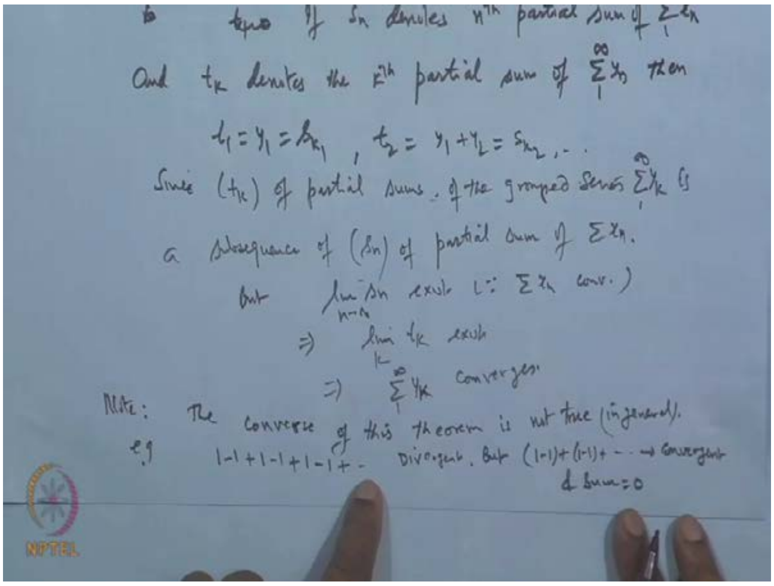
Let us see the proof of it. Proof is also straight forward, simple. What is given is a series is given to be convergent. Let us construct, let y 1 is the one term which is grouped up to say k 1 terms y 2 is another term which of the new series, which is grouped from the original series. By choosing these terms up to k 2 k 1 start from k 1 plus 1 to k 2 and like this continues.

So, what happens is that basically. So, we get the t 1the original series term. Let us suppose, if s n denotes the nth partial sum of the series sigma x n 1 to infinity. And t k denotes the kth partial sum of the series, sigma y n, 1 to infinity. Then what we see here is t 1 that is the first term. First term is what? Up to here is here y 1 that is the y 1. So, y 1 is nothing but what? This the first k 1 sum of the original series. So, this is the s k 1 then t 2, t 2 is the second term means this. Now, what is this term? This is starting from x k plus 1 and so on. So, basically when you are choosing this t 2 partial sum then this is equal to what? y 1 plus y 2 this is first term. So, y 1 minus y 2 that is equal to y 1, y 1 is this term or up to k 1, and then this is up to k 2 k 1 is say another term.

So, we can say this is equal to y 1 plus y 2. If I take this set of first two terms sum, then what we get is y 1 plus y 2. This is equal to s k 2 up to a k, y 1. This term plus second term of this new series and continue. So, what is the first term is the partial sum of the original series, second term is also the partial sum of the original series, and the original series is convergent. So, this sequence of the partial sum will converge. Therefore, this sequence will also converge. So, since the sequence t k of partial sums of the grouped series sigma y k, 1 to infinity is a subsequence of the sequence s n of s n, s n is the first interval, is a subsequence of this series. And this series partial sum of s n of partial sum of the series sigma x k x n and, but this series is convergent.

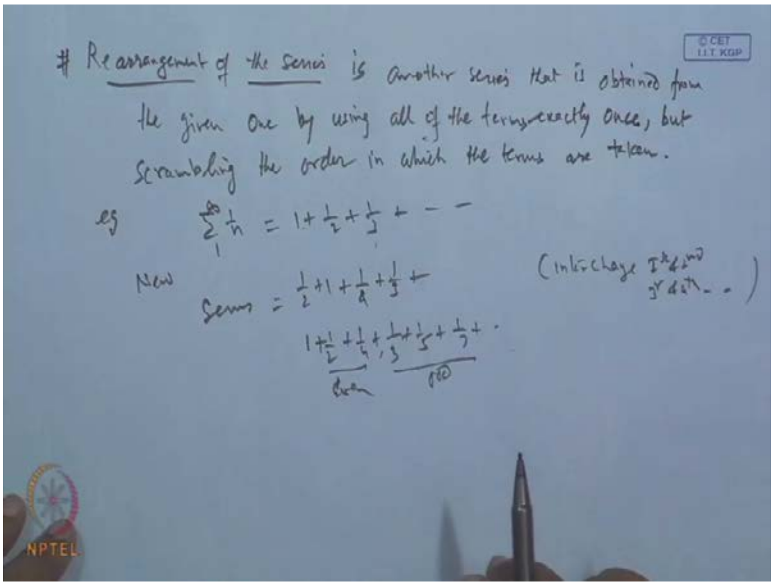
So, this partial sum is convergent. Therefore, limit of s n as n tends to exist, because the series is convergent. Therefore, this implies the limit of t k over k will exist. And hence implies sigma y k n y k 1 to infinity converges and that is the proof for it. So, what we have seen is here that if a series is given which is a convergence series. And if I regroup the terms of the series without changing their sum then the new series obtained will also be convergent. Now, let us think a converse part of it. Suppose a series is given whose nature is not known, but we are regrouping the series, and getting a new series which is suppose say convergent. Now, the question is whether the original series is convergent or not. The answer is not necessary to be true.

(Refer Slide Time: 48:28)



So, the converse is the note or as a remark, you say the converse of this theorem is not true in general. For example, if I take this series, for example; if we take this series 1 minus 1 plus 1 minus 1 plus 1 minus 1 and so on. And if I regroup this which is a diverging series; this is a diverging, but if I regroup the series 1 minus 1 plus 1 minus 1 and so on. Then it is convergent it is convergent and converges to what? And the sum is 0. So, what we see here? The original series is, when we regroup from the original we are getting a convergent series, but the original series basically is not a convergent series. So, the converse of this part is not true in general. So, that is the very interesting in general.

(Refer Slide Time: 49:56)



Now, another concept is also rearrangement of the series rearrangement of series. Now, in case of the grouping of the series we are not changing the order of the terms that is the place of the term is fixed. But in case of the rearrangement of the series we are free to take up to shift the element from first place to ninth place, seventh place to hundredth place like this.

So, if I keep on shifting the position of the element we are getting infinitely new sequences, new series from the original one. Now in this process, the series new series so obtained, whether that series will also retain the same character as the original one if the original series is convergent, and by rearranging the series if we are getting a new series, whether the new series is also convergent. So, the answer is not true that a (( )) series which is given to be a convergent series. And if I rearrange the terms of the series then in general you cannot say the new series.

So, obtained will be a convergent and will have the same limit. And in fact, this is a very good result, and very important result which is given, say, by one of the famous mathematician, Riemann; what he said; in case of a alternating convergent series if I regroup the terms of the series then we can get any real number which is the sum of that series. That is, for any given real number, one can have a rearrangement of the series; one can have a rearrangement according to that given real number. So, that the series will converge and converge to that real number.

So, rearrangement of the term is so clear. What is rearrangement? Let me just talk about another series. Rearrangement of a series is another series that is obtained from the given one by using all of the terms exactly once, but a scrambling the order in which the terms are taken. For example, if we take a harmonic series this is our harmonic series, sigma 1 by n 1 to infinity 1 plus 1 by 2 1 by 3 and so on. The new series, suppose I take the rearrangement of this harmonic series, interchange is for first and second terms. So, half plus 1 third and fourth term, one forth and one third like this. So, interchanging the first and second term, first and second, third and forth and so on we get new series.

So, this is the new series which is the rearrangement of which is obtained by rearranging the term or may be another series. If I take this 1 plus half 1 by 4 1 by 3 1 by 5 1 by 7 and so on, means first the even terms we are choosing, first even two even terms 1 then 1 by 2 then again after 1, we are two even terms, then three odd terms and like this, odd terms, even terms and so on. Now, this will give a different series, but the arranged series will have a different nature that we will discuss later on.

Thank you.