A Basic Course in Real Analysis
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 21
Infinite series of Real numbers

So, today we will discuss Infinite Series of Real Numbers, we have already discussed the sequences and various concepts regarding the sequence. So, today we will talk about the infinite series.

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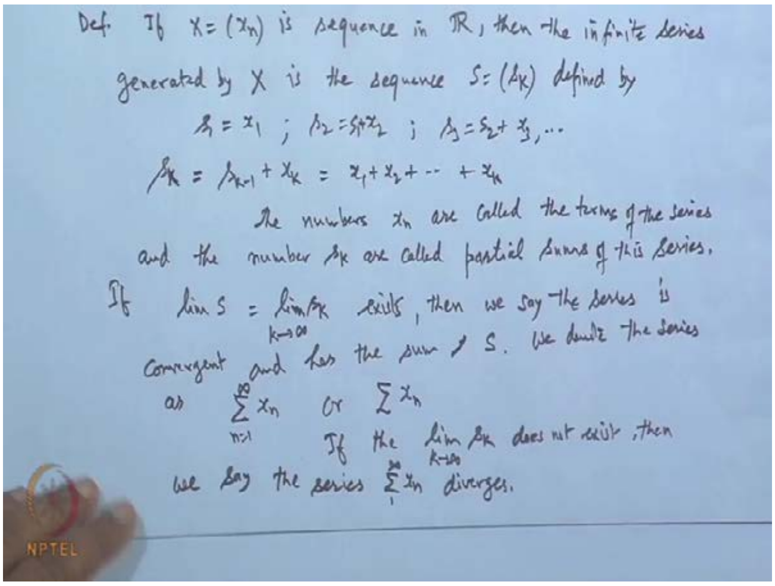

We define normally when we say, if a set of element x 1 x 2 x n are given a sequence, when they are added or subtracted and put it in the formula x 1 plus minus x 2, and like this. Then we call it series, but that is not a authentic way of defining the series, because it does not so what? Anything, whether thus limits convergent or divergent whatever. So, we define the series as follows.

If X which is a sequence of real numbers, x n is a sequence in R; this is set of real numbers, then the infinite series and generated by X is this term sequence, x n is the sequence; is a sequence S of partial sum s k defined by s1 is the first term say x 1, s 2 is the sum of the first term that is s 1 plus x 2, s 3 is s 2 plus x 3 and so on. So, s n will be or s k will be s k minus 1 plus x k, that is x 1 plus x 2 plus x 3 up to x k, the number x k; the numbers s k or x n; this is these are called the terms of the series and the number s k are called partial sums of this series, this s k sum of the first k term of the series and so on.

Now, if this limit, if the limit of s that is the limit of s k, when k tends to infinity. If this limit exists, then we say the series is convergent and has the sum S, we denote the series as sigma x n; n is 1 to infinity, if first term is x 1 or some time we also denote like x n without the suffix, that shows that summation is taken from 1 to infinity. So, this is the way of defining or some of the, if the first term is starting from the 0, then the summation will be taken from a 0 to infinity and like this an if the series whose first term is say x 100, then the series will start from n equal to 1 to infinity. So, that way we can identify now, if this limit does not exist then in that case the, we say the series divergence, if the limit s k, when k tends to infinity does not exist, it means either, it is infinite plus infinity or minus infinity or has a various limits does not exists, then we say the series sigma x n 1 to infinity diverges.

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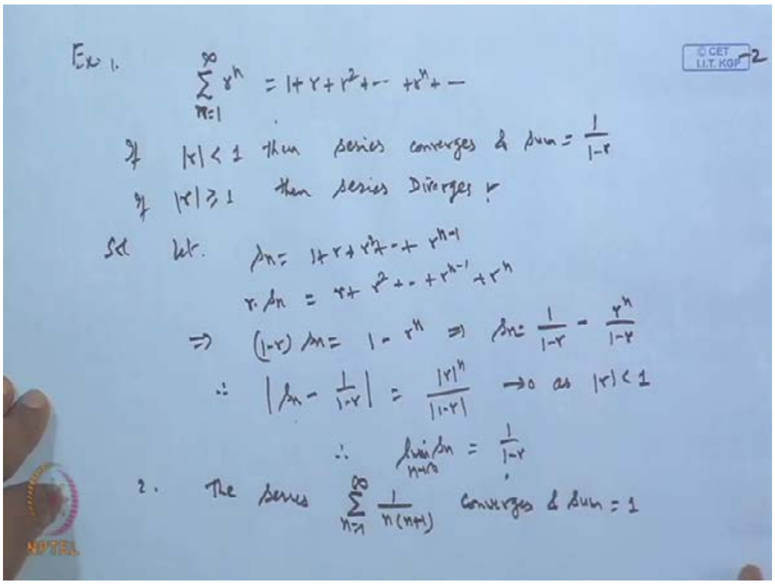

So, that is done, for example, we have a series suppose, we have the series sigma r to the power n, n is 1 to infinity, now we that is the series 1 plus r plus r is square and so on plus r to the power n like this. So, this is the series now, we claim if mod of r is strictly less than 1, then series converges, and the sum of the series and sum is equal to 1 over 1 minus r sum will be this, but if r is equal to 1, if r is mod r is greater than or equal to 1, then the series diverges, the proof is very simple solution, let us find the first sum of the first n terms.

So, let s n is the sum of the first n term, 1 plus r plus r square plus r to the power n; this is the sum of the first term r n minus 1, because this will be first term, second and up to n terms, then if I multiply this by r, then what happen r times s n; this is equal to what, r plus r square plus r n minus 1 plus r n now, subtract it. So, this implies that 1 minus r s n, when we subtract it get cancel in that, and we get finally, 1 minus r to the power n.

So, what we get it is s n becomes 1 by 1 minus r minus r to the power n over 1 minus r. So, consider mod of s n minus 1 over 1 minus r that is equal to mod of r to the power n over mod 1 minus r now, r I am choosing fixed, but it is less than 1, if it is less than 1 this term will go to 0. So, this tends to 0 as mod r is strictly less than 1 therefore, the limit of this s n as n tends to infinity is adjust n equal to 1 minus r.

So, this series converges and converges to the sound 1 over 1 minus 1, for r is equal to 1; obviously, the terms are 1 plus 1 plus 1 and so on up to s n will be n. So, when n is infinity limit s n will tends to infinity? So, it will divert for r greater than 1, we will see in decodes that it also diverges. In fact, it will limit will not exist and it will go to infinity; so that we will come and obviously, we can find out the sum clear. So, this is very interesting one, similarly another examples and this part we will take up after going for the further test then we save automatically by p h test it will come diverges then let us take this sum this series sigma 1 over n, n plus 1 n is 1 to infinity, we claim this convergence and sum is 1.

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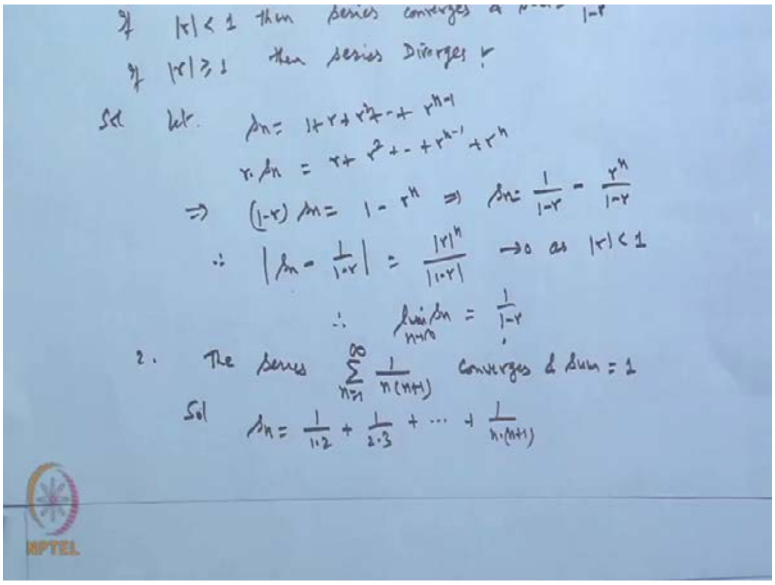

The solution is again simple, if I take the terms s n, what happened this term are s n are 1 over 1 into 2 plus 1 over 2 into 3 plus over up to say n is n in to n n plus 1.

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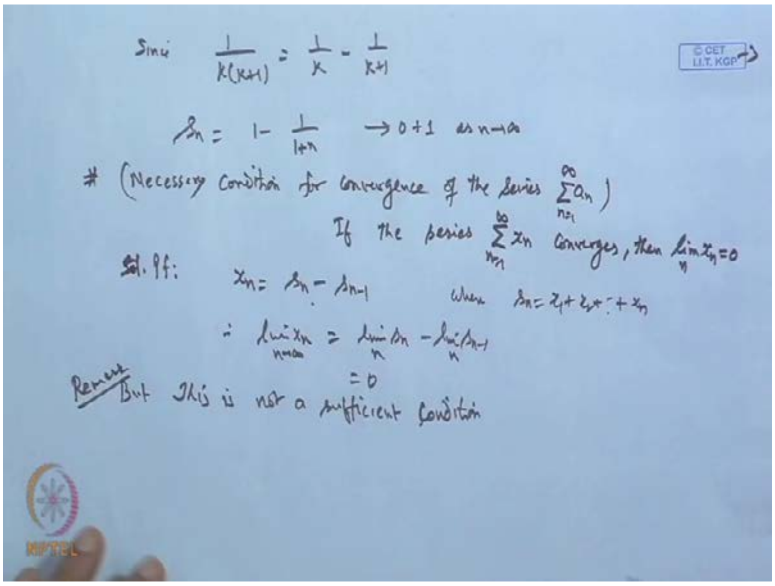

So, this will be, if we write it in the form, since 1 over k k plus 1, we can break up and put it in the form of 1 over k minus 1 over k plus 1. So, using this each term, we can put it in this way, and once you put it this thing, then s n becomes; this is 1 over 1 minus to that is 1 over 1 minus 1 by 2; this will be then 1 by 2 minus 1 by 3 like this 1 by n minus 1 by n plus 1. So, get cancel and finally, you are getting 1 and 1 minus 1 over 1 plus. So, s n becomes 1 over 1 minus 1 over 1 plus n, or n plus 1 now, as n tends to infinity it goes to 1 minus 0 that is goes to 1 as n tends to infinity; this term will go to as n tends with this will go to 0 and this is 1.

So, the limit of this exist and n equal to 1. So, this way we can identify this. So, these are few examples now, in case of the convergence series there is a necessary condition that, if a series is convergent then is nth term will always go to 0, that is necessary condition for. So, if a series whose nth term does not go to 0, the series cannot be a convergence series. So, that is the criteria which we have the necessary condition for convergence of the series sigma a n, n is 1 to infinity.

So, this we will put it in the form in the way, if the series sigma x n, we are doing the xn. So, let us put it x n, sigma x n, n is 1 to infinity converges, then limit of the xn must go to 0, limit of x n as n tends to infinity will be 0, that any term will always go to 0. So, proof, let us see the solution of proof. So, what is our s n; s n basically x n this is nothing but what; s n minus s n minus 1, because s n is the sum of the first term, where s n is stands for x 1 plus x 2 plus x n. So, s n minus 1 is up to x n minus 1.

So, when you subtract, you are getting as a now, this series is convergence. So, limit of the s n will exist. So, limit of s n and limit of s n minus 1 will be same. So, therefore, limit of x n, when n tends to infinity; this will be the limit of s n minus limit of s n minus 1, but both the limit will be the same. So, it will come off to 0. So, this is the necessary condition for a series to be convergent, but this is not a sufficient condition, but this is not remark, but this is not a sufficient condition sufficient condition that is, if it any term of the series convergent to 0, then it may or may not be a converges series.

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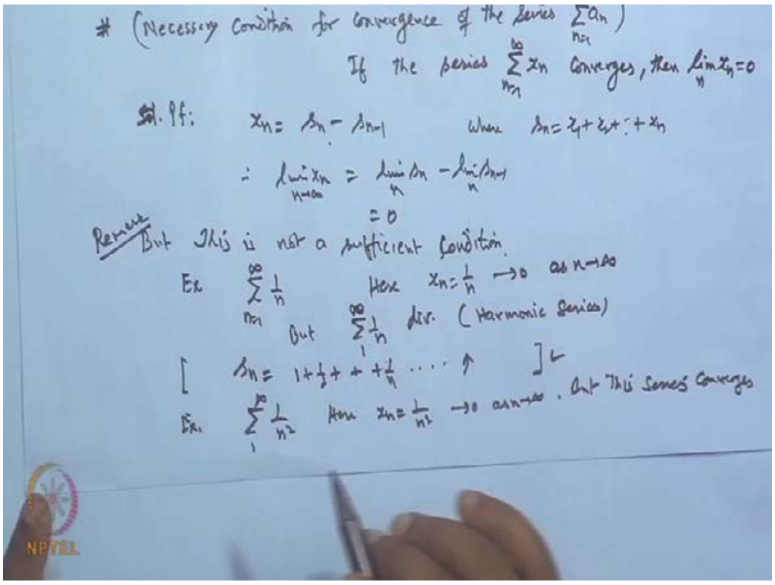

For example, suppose I take this series, sigma 1 by n, n is 1 to infinity, here x n is what one by n, which tends to 0 as n tends to infinity, and this we have seen that this series and, but the series s n; the series sigma 1 by n n 1 to infinity diverges, as we have seen earlier, because it is a harmonic series. In fact, the proof we have gone already seen because the, if you remember, we got the s n to be 1 plus 1 by 2 plus 1 by 3 up to 1 by n is a not, and then it keeps on increasing, when we take the n then it keeps on increasing.

In fact, limit of s n will go to this keeps on increasing diverges. So, this, we have already increasing functions like this and keep shown unbounded one. So, this we are not already shown in the first lecture were going for this series sequences, in case of sequence, this example we have taken and so on, the series diverges. So, here this divergent, if we take another example, say sigma of 1 over n square 1 to infinity now, here thus x n which is 1 by n is square goes to 0 as an tends to infinity, but this series convergence, that will also be shown in the next few after few article to be covered, and test for the series there, then this is 1 upon sigma n to the power p type, where p is greater than 1 series is converge.

So, that we will prove it, hence from there you can say this is convergent, but what the way, you see this two series, in both the series the nth terms goes to 0, here also will goes to 0, but once series diverges other converges. So, taking the sequence x n the nth term checking the whether it is tending to 0 or not this will not give a conclusion, just tending to 0 will not implies the series is convergent, because it may be divergent also; however, it will take the any sequences x n, any series x n whose nth terms does not go to 0, then series has to be a divergent series, because if it is convergent then correspondingly the nth term must go to 0.

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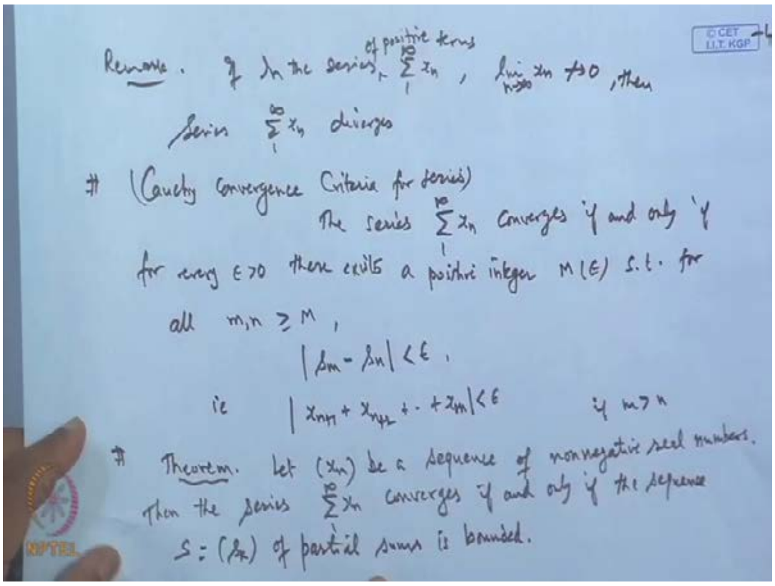

So, that is why here, this will third point remark, if in the series sigma x n 1 to infinity, the nth term of positive term, sigma x n a series, if the x n is a sequence of terms, if the limit of x n, when n tends to infinity does not tends to 0, then the series diverges, then the series sigma x n 1 to infinity diverges, and the reason is very simple, the reason is why because if it is convergent then it according to necessary result earlier, the necessary condition for a convergence of a series any term must go to 0

So, this is not happening therefore, it will diverge. So, if the series of positive term let us take the positive record, the series of positive terms, we are discussed. So, this is part of it now, there is another result just like a Cauchy convergence criteria, we have a Cauchy convergence criteria for this sequence, any sequence of real number is convergent, if and only Cauchy, that is satisfies the cover Cauchy convergence criteria that after a certain in stage the difference between any two arbitrary terms of the sequence is less than f signer, even f signer, then we say the sequence is convergent. So, correspondingly, we also have result for a series that is known as Cauchy convergence criteria for series.

So, next is Cauchy convergence criteria for series, the result is the series sigma x n 1 to infinity converges, if and only if for every f signer greater than 0 they are exist a positive integer n there exist sequence of there exist a positive integer capital M with depends on f signer, such that for all m n greater than or equal to capital M, the following condition holds the mode of s m minus s n, the partial sum of the series m eth term of m eth term and up to nth term is less than f signer for all, that is equivalent say is the mod of x n plus 1. If I choose m is greater than n, if m is greater than n then x n plus 1 x n plus 2 up to x m is less than f sign, that is clear this is known as the Cauchy convergence criteria and this part is if and only.

That is if the series converges, then sequence of its partial sum will satisfy this condition is clear and; obviously, we have define the convergence of the sequence s, when limit of the s m goes to limit exists. So, when the limit of sequence of partial sum exists, then only we say the series converges. So, suppose the series convergence. So, limit of s m exist, it means the limit of s m is exist, it means it must satisfy the Cauchy convergence criteria. So, this is nothing but Cauchy convergence criteria, and conversely, if this is true, then the limit of the sequence s m must exist, the limit of s m exist means this series must be convergent one.

So, that is shows the connectivity now, based on this we have a very interesting results, the result is in the form of theorem, let x n be a sequence of non negative real numbers, then the series sigma x n 1 to infinity converges, if and only if the sequence of partial sums s sequence s of s k of partial sums is sums is bounded, and in fact when it is bounded the limit superior of this limit of x n will come out to the sum of this. So, let us see the proof of it, what is given is the sequence x n is sequence of non negative real number, positive numbers; non negative may 0, the series is convergent, if and only the sequence is of the partial sum if done.

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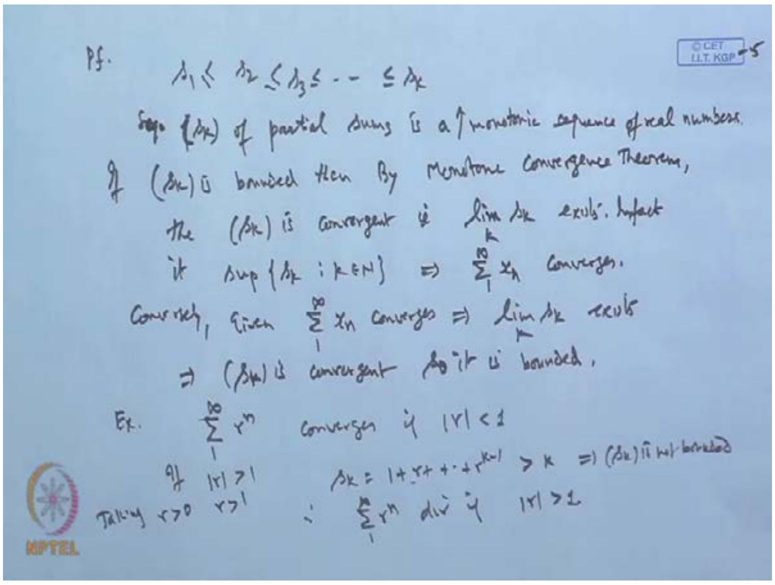

So, suppose the series convergent it means limit of the x n will be exist and what is our s n, s n is basically is the partial sum s 1 is first term, s 2 is the sum of the 2 terms, 1 and 2 and since x 1 x 2 x n are non negative number real numbers. So, few may be 0 or may be is strictly greater than 0 also positive number. So, s 1 is definitely less than equal to s 2 which is less than equal to s 3 and so on like this. So, what we are getting is the sequence of the partial sums.

So, the sequence of partial sums is a monotonic sequence of increasing numbers, a increasing monotonic sequence of real numbers now, what is the monotonic convergence theorem? monotonic convergence theorem says any monotonic sequence which is either increasing or decreasing and if it is bounded, a monotonic sequence which is increasing and bounded above, then it must be convergent, or monotonic sequence which is decreasing bounded below must be convergent, here this is a monotonic sequence, increasing sequence of real numbers, and what is given is a series is convergent, if this sequence is given to be bounded is suppose I take this part if sequence is bounded, then it means this monotonic sequence is bounded. So, once it is bounded it has to be convergent.

So, if the sequence s k is bounded, then by monotone convergence theorem, the sequence s k is convergent, that is limit of s k exist in fact it is the upper bound for this, it is the supreme value of all s k, when the k belongs to integer supremably its upper bound for this, we are not discuss upper bound, we will take it after in fact this limit. So, if the s k is bound partial sum is bounded, then we can immediately say by monotone convergence theorem this limit will exist, hence the series is convergent.

So, this implies the series sigma x n 1 to infinity converges, conversely if given the series sigma x n 1 to infinity converges, it means what; that means, the limit of the sequence is s k over k exist, fine now, every convergence sequence is a bounded sequence. So, this means that sequence s k is convergent. So, it is bounded. So, this proves both ways that if this series is convergent, then the sequence of partial sum will be bonded sequence and vice a versa, if sequence of partial sum is bonded, then the corresponding series will be convergent. So, that is the convergent criteria.

Now, if you look that series that what we are talking about, we have seen this example sigma r to the power n 1 to infinity, and we have seen that this series converges, if mod r is less than 1, but if mod r is greater than or equal to 1 than what happen; what is our s k, s k become 1 plus r plus r, r to the power k minus 1 now this is 1 is greater than 1.

So, it is greater than k is it not, because it is greater than 1. So, limit of s k does not exist unbounded therefore, s k sequence is not bounded. So, this series cannot be convergent, because if it convergent because if it is convergent the sequence of the partial sum must be bounded as per to his result.

So, this shows the series are to the r to the power n 1 to infinity diverges, if mod r is greater than 1, here we have taken one more thing is here, we are taking all the terms of the sequence to be non negative, we cannot apply this result for the series whose terms are some are positive, some are negative like this, we cannot apply this result or we cannot do for the non negative terms, if all the terms are negative, what we can do we can take the minus sign outside. And make it all the term to be positive, but if the series is alternatively positive, negative, then off course this result is not helpful, this result is only valid in the terms of series are all non negative.

So, here we are taking mod r is greater than 1, basically we are choosing all r to be greater than 1 positive; this is positive. So, r is greater than 1 taking r to be greater than 0, or positive then this diverges, and r is equal to 1; obviously, again we take 1 plus 1 plus n. So, s n diverges therefore, unbounded therefore, it is not convergent similarly, now we have another result which is also very that result is yes.

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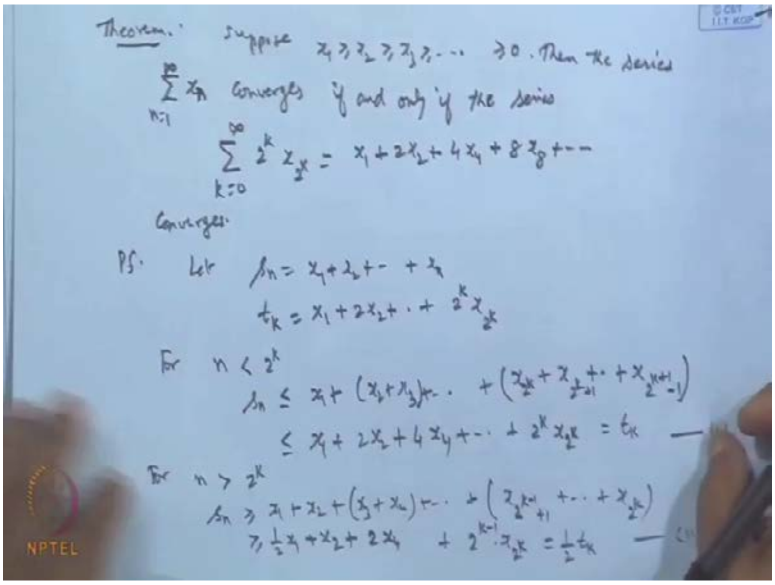

Next result which will help in drive more results; this result is in the form of theorem; what this result says is suppose x 1 is greater than equal to x 2 is greater than equal to x 3 is greater than equal to all terms are greater than equal to 0 means non-negative terms, then the series sigma x k or x n, n is 1 to infinity converges, if and only the series sigma k is equal to 0 to infinity, 2 to the power k, x to the power k x 2 to the power k, x suffix 2 to the power k that is a series x 1 plus 2 times of x 2 plus 4 times of x 4 plus 8 times of x 8 and so on this series converges.

So, this is very result, which will help in driving the few more results with the help of this. So, this is the proof lets, what he says is suppose a series sigma x n is given whose all terms are non negative, then the nature of this series that is a series will be convergent, if the corresponding series will converge and vice versa. So, in order to test this series to be convergent, you convert this series in to this form, which will be more comfortable in a suitable form, which can easily we put to be a convergence or divergence.

So, corresponding nature of this, if it convergence this will converge that is what he says. So, let us consider the partial sum, let s n is strands for the partial sum of the first series, say x 1 plus x 2 plus x n, and let t k is stands for the partial sum of second series x 1 plus 2 times of x 2 plus 2 to the power k x suffix 2 to the power k suppose, we are taking this term up to say 2 to the power k 2 k, we are there are few gets is not, because is it not continue sum x 1 x 2 then that after x 3 is missing x 4 like this.

So, we are choosing this term now, let us take the different cases for n which is strictly less than say 2 to the power k, then what happen the s n, this will be less than or equal to x 1 plus x 2 plus x 3, if we combine this, and then let us write this form x 2 k 2 to the power k plus x 2 to the power k plus 1 k plus 1 up to say 2 to the power x 2 to the power k plus 1 minus 1 say I am taking x n to be less than this number; obviously, when I am taking n is less than 2 to the power k so; obviously, this terms which we are taking is much higher than this, and total term in this case will be what now, this will be x now, x 1 is greater than 2 x 2 is greater than x 3.

So, basically this will be x 2 is greater than x 3. So, we can say this is x 3 is lower than x 2. So, it is less than 2 of x 2. So, this will be less than x 1 plus 2 times of x 2 similarly, when we go for x 4 x 5 x 6 x 7 and x 8, then we will get next term will be 4 less than x 4 that term, and continue this what will happen to this, there are only 2 to the power k terms.

So, 2 to the power k into x 2 to the power 2 k, this will be, because this is largest term and rest is decreasing is it not. So, it will be less than this, but this term is nothing but what t k. So, what we get is that s n is less than equal to t k, this is one thing now, for n, let us take n to be strictly greater than 2 to the power k, and then again we will write suitably s n can be written as greater than equal to x 1 plus x 2, and then combine this term x 3 plus x 4 like this, and last term is start with the x 2 to the power k minus 1 plus 1 this term, and up to go x to the power x 2 to the power k up to this.

So, what happen is this is greater than equal to now, let us write x 1 is; obviously, is greater than half of x 1 is it not, because it is positive term than plus x 2 remain as it is, then x 3 now x 3 x 4 x 3 is less than x 4 or x 4 is greater than. So, if I replace x 3 by x 4 then it is greater. So, it is greater than equal to two times of x 4 this will be yes, is it not two times of x 4 and like this up to 2 to the power k minus 1 into x 2 to the power k like this.

So, x 3 is greater, x 3 is greater like this now, will check it, this will be what if I take that t k; t k is coming to x 1 plus 2 x 2 plus 4 x 4 like this. So, it is half of this. So, basically it is the half of t k. So, what we get from 1 and 2 we get from this.

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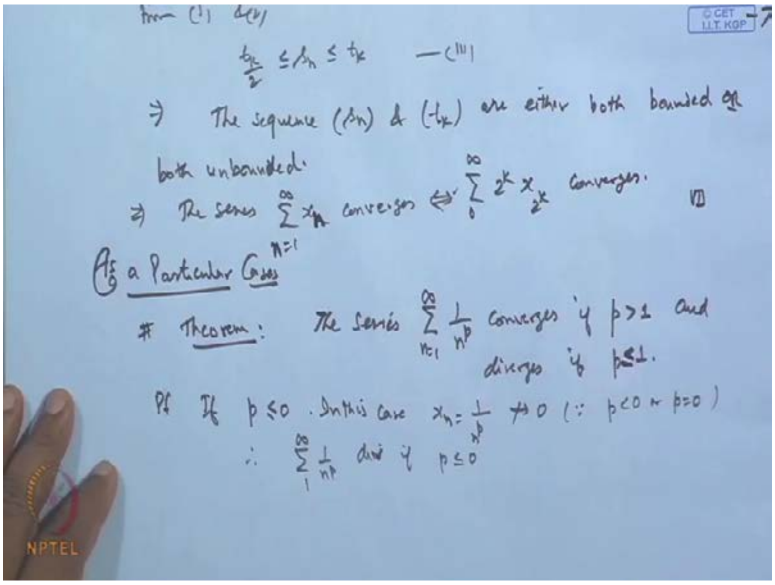

So, from 1 and 2 what we get is that s n is less than equal to t k, and s n is greater than equal to t k by 2, but what is t k and s n, let us see, let t k and s n; the t s n is the partial sum and it partial sum of this series sigma of x n, and t k is the partial sum of the second series sigma 2 to the power k x 2 to the power k now, this partial sums satisfy this condition in equality. So, if t k is bounded s n has to be bounded now, s n is unbounded t k has be unbounded. So, this implies the third criteria implies that the sequence s n and the sequence t k are either both, bounded or unbounded are both unbounded that it.

So, once they are bound they unbounded, the result follows it means the nature of this series converges. So, if they are bounded then they will be. So, this shows, this implies the series sigma x n converges, if and only if sigma 2 to the power k 0 to infinity here 1 k is equal to 1 to infinity, it is in should I write is sigma 2 to the power k x x 2 to the power k converges. So, this part we get it, this should be writing n because otherwise that will be different. So, n is 1 to converges, if and only this one and this completes the proof. So, now, Corary to this or as particular cases, we have this result theorem in the form the series sigma 1 by n to the power p n is 1 to infinity converges, if p is strictly greater than 1 and diverges, if p is less than or equal to 1.

So, this we get see the proof of this, it will follow from the previous result now; obviously, if p is less than or equal to 0, suppose p is 0, then what happened this each term becomes 1. So, each term becomes 1 means, thus any term will not go to 0. So, when p is 0. So, in this case the x n which is 1 by n p n to the power p will not got to 0, because when p is less than is equal to 0, this will come in the up and it will not go to 0 it will diverge, because p is negative or 0. So, by necessary condition does not satisfy therefore, the series sigma 1 by n to the power p 1 to infinity diverges, if p is less than or equal to 0 now, take that case when p is strictly greater than 0. So, if p is strictly greater than 0 then what we claim is, when p is lying between 0 and 1 even is diverge, and when p is greater than 1 it will converge.

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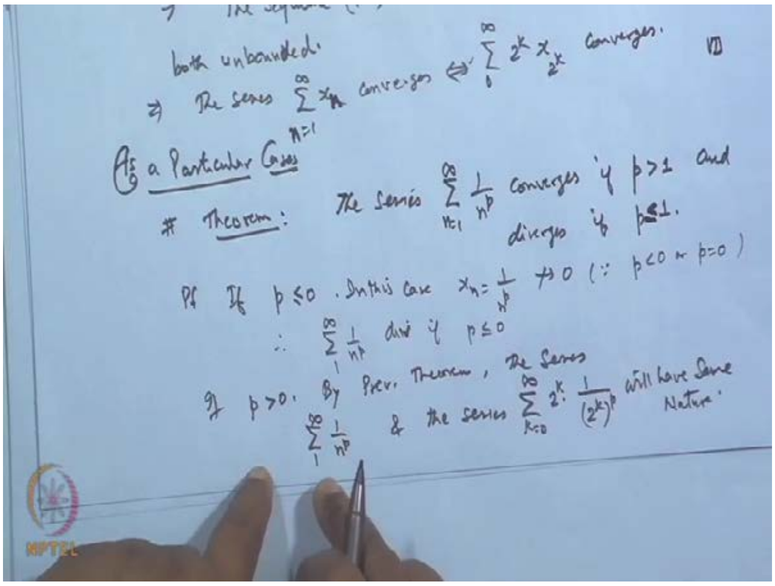

So, now, apply this result by the previous result by previous theorem, the 2 series will have then the 2 series sigma 1 by n to the power p 1 to infinity, and the series sigma k equal to 0 to infinity 2 to the power k 1 by replace n by 2 to the power k power p the n to the power p I am replacing n x n by this x n 1 upon n to the power p. So, replace n by 2 to the power k n to the power p, this 2 series. So, the series will have same nature that, if this is convergent it has to be converge, and vice versa this is convergent and this and similarly divergent one will have the same nature.

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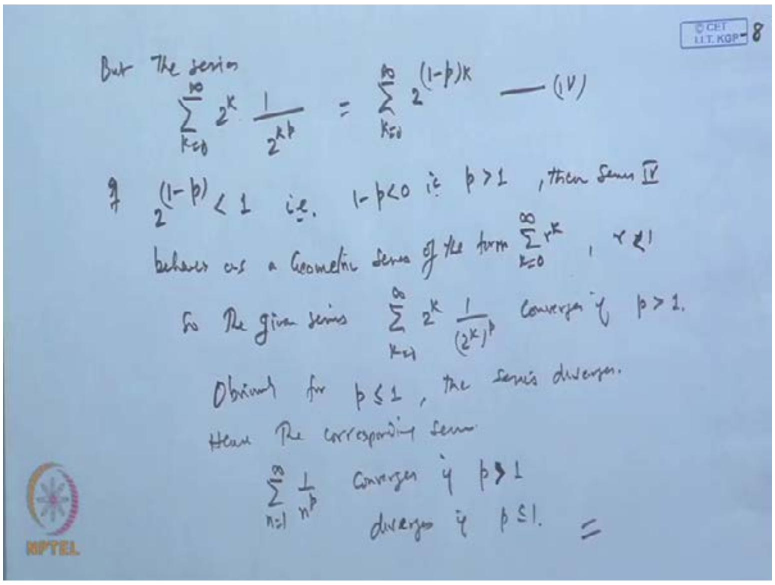

But what is this the series, but the series sigma k 0 to infinity 2 to the power k 1 by 2 to the power k p k into p this is nothing but what this is equivalent to sigma k equal to 0 to infinity 2 to the power 1 minus p into k now, this is basically a geometric series this is a geometric series.

So, if we take, if 1 minus p is 1 minus p 2 to the power 1 minus p, if this part is less than 1 is strictly less then it will behave as a geometric series sigma 1 r to the power k, here r mod r is less than 1. So, this will be this series, then it means that is what; when it is less than 1, when 1 minus p is negative that is p is strictly greater than 1. So, in this case, the series say fourth, if then the series fourth behaves as a geometric series of the type of the form sigma r to the power say k, where k 0 to infinity and r is less than 1 because r is this term. So, in that case this converges. So, the given series given series, sigma k equal to 0 to infinity 2 to the power k 1 by 2 to the power k power p converges, if p is strictly greater than 1, and if p is less than, if this thing is greater than 1; obviously, then greater than 1.

So, it is the limit of any term does not go to 0 therefore, it diverges when equal to 1 also it will diverge. So, this shows otherwise and; obviously, for p less than or equal to 1, the series diverges, hence the corresponding series sigma 1 by n to the power p n is 1 to infinity converges if p is strictly greater than 1, and diverges if p is less than equal to 1 that proves the theorem now, when p is equal to 1, it is a harmonic series and it diverges that what we shown, when p is equal to 2, it is convergent and the example which we have chosen that necessary condition x n tends to 0 is not a sufficient condition is justify from here that is all.

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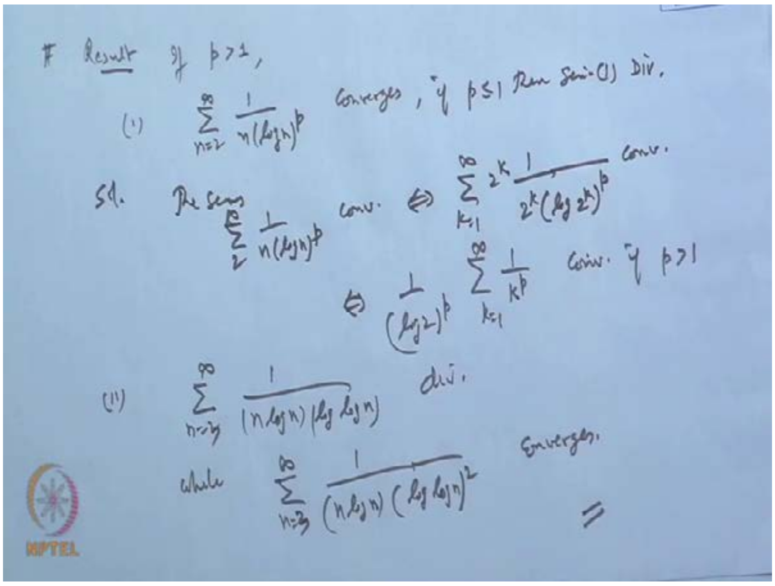

Now, based on this we can also say another example, let us say suppose, what result, because this will also be useful, if p is greater than 1, then the following series converge then number 1; sigma n is equal 2 to infinity 1 by n log n to the power p converges, log n to the power p converges, when p is greater than 1 converges, and if and otherwise diverge and if p is less than equal to 1, then the series 1 diverges. Again it follows from the same thing, if you go through that one, then that follows the same thing, how the solution will come to test the sigma x n convergent, what we do is we find out the 2 to the power k.

So, the series sigma 1 by n log n to the power p 2 to infinity converges, if and only if sigma k equal to 1 to infinity 2 to the power k log 1 by 2 to the power k log 2 to the power k power p; obviously, when k is 0 we cannot choose because log 1 is 0, that is why we are dividing that one this is. So, this converges means if this converges, but this converges equivalent to what? Is it not same as the 1 by log 2 power p sigma of 1 by k to the power p, when k is 1 to infinity, and this series converges, this is convergent, if p is greater than 1 because this is the any power test, so this series converge. Similarly, if I take the second point, suppose I take this series sigma n is equal to 2 to infinity 1 over n log n and log n into log log n into log log n log log n this diverges, while the series sigma n is 2 to infinity 3 to infinity, because this is and if I think three because otherwise 2 will not n; this 3 to infinity n is 3 to infinity 1 by n log n then log log n 2 power 2 converges, and the reason is the same as above this one.

Thank you very much.

Thanks.