A Basic Course in Real Analysis

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Lecture - 20
Cauchy sequence and its properties

So, we were discussing about the criteria for the convergence of sequences, and in this process we have seen two criteria; one is the monotonic convergence criteria for monotonic convergence sequence, but that monotonic convergence theorem, which is very wonderful result, which gives the convergence criteria for a sequence, particular type sequence which are monotone, either monotone increasing or monotone decreasing sequence. And another criteria, which is a general in nature, that if we give a sequence is given, one can find the limit of the sequence. And if the limit exists, we say the sequence converges; if the limit does not exist or goes to infinity or minus infinity, we say the sequence diverges.

But you see, in both the cases, there is a… In the first case, particularly there is a drawback in the monotone convergence theorem. The drawback, main drawback is that this result is only applicable to those sequences which are monotonic in nature. But the sequence may not always be monotone sequence. So, in such a case, the monotone convergence theorem is not applicable to judge the convergence of the sequence. And second criteria which we told, though it is valid for any type of the sequence, but in identifying the limit itself is a problem; it is a difficult one.

So, what we want is, will there be any criteria, which can directly say, the sequence, given sequence is a convergent one or divergent one, without going for the limit? And this was given by Cauchy, and is known as the Cauchy convergence criteria. And it is the one of the important result, which relates the convergence of a sequence of real or complex number, with its Cauchyness. So, in fact, if a sequence is Cauchy, then automatically it will be a convergent, and vice versa. Now, this is valid in case of the real, complex numbers, but when we go for a arbitrary metric space, which we have not taken up, then the convergence criteria, obviously does not help there.

So, there it will be defined in a different way, the convergence of a sequence. So, we will go, since we have restricted our study only for the real number or complex number, so therefore this convergence criteria is very much helpful in getting the result following, nature of the sequence, which is convergent or divergent. So, before going with the Cauchy convergence criteria, what is the Cauchy sequence?

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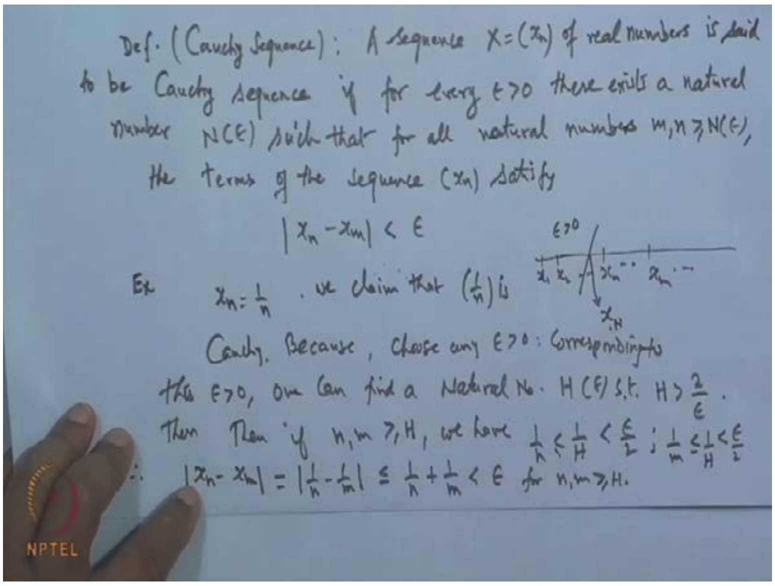

So, we say, first, definition of a Cauchy sequence of real numbers. A sequence x n, X, which is x n of real numbers, is said to be, is said to be, is said to be Cauchy sequence, Cauchy sequence, if, for every epsilon greater than 0, for a given epsilon greater than 0, for every epsilon greater than 0, there exists a natural number or positive integer, you can say, natural number, say n, which depends on epsilon, such that, such that, for all m n, natural number, numbers, natural numbers m, n, greater than equal to capital N, which depends on epsilon, of course, the sequence of the terms satisfy the following condition, the terms of the sequence x n, satisfy the following criteria that, mod of x n minus x m is less than epsilon. It means, a sequence is Cauchy, if a x 1, x 2, x n, this is a sequence of Cauchy numbers, a sequence of real numbers, or complex number, we say, this sequence is a Cauchy sequence, if for every epsilon greater than 0…

So, first, you choose the epsilon. Then corresponding to this epsilon, one can identify a point x n, such that, when we take all the terms after this x n onwards, the difference between any two arbitrary terms after x n, will remain less than epsilon; the distance between x m and x n will remain less than epsilon, or it is in the epsilon neighborhood of one of the, say m I fixed; then one of the point in this. So, they are very close to each other.

The difference, distance between any two arbitrary point after the certain stage capital N one bar, the difference is very, very small, or is less than the given number epsilon. If this happens, then we say the sequence is a Cauchy sequence. For example, if we look the sequence, say x n is, say 1 by n. We claim that, the sequence 1 by n is a Cauchy sequence. So, it means, for any epsilon is greater than 0. So, let… Because if we choose any epsilon, because… Let us picked up, choose any epsilon greater than 0, first; then corresponding to this epsilon, one can find a natural number; then corresponding to this epsilon, to this epsilon greater than 0, one can find a natural number, say capital H, which depends on epsilon such that, such that, let us say, H is such, which is greater than 2 by epsilon. So, H will depend on epsilon.

We can choose a natural number which is greater than 2 by epsilon. Then according to the Cauchy definition, this difference must be less than epsilon. So, if I choose m, n greater than n…So, if we, then if we take n and m, a natural numbers, which is greater than equal to H, we have, 1 by n is less than 1 by H, or less than equal to 1 by H; but H is greater than this. So, it is less than epsilon by 2. Similarly, 1 by m is also, similarly, 1 by m, which is also less than equal to 1 by H. So, it is strictly less than epsilon by 2. So, this two are less than (( )).

Therefore, when we take the difference between mod of… Therefore, mod of x n minus x m, when n, m are greater than H, this is nothing but the 1 by n minus 1 by m. And by triangular inequality, it is less than equal to 1 by n plus 1 by m. But 1 by n is less than epsilon by 2 and 1 by m is also less than epsilon by 2. So, this is less than epsilon, for all n, m greater than equal to H. It means, the difference, arbitrary difference between two arbitrary terms of the sequence, after this certain stage H onward, is less than epsilon.

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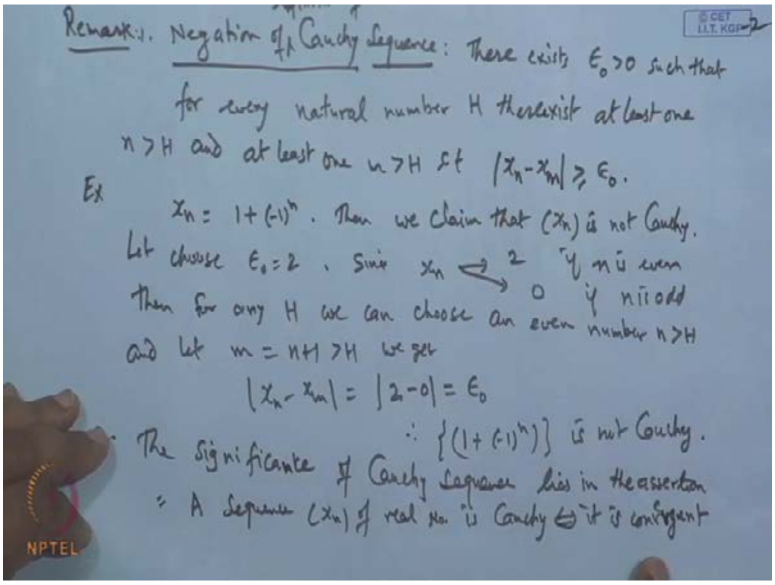

So, sequence 1 by n is a Cauchy sequence, that is what. Now, what is the negation of the Cauchy sequence? The negation of, because that will be required, negation of Cauchy sequence. If a Cauchy sequence does not satisfy, if a, if a sequence x n does not satisfy the Cauchy critetria, Cauchyness… So, what is the negation of Cauchy? Negation of the, definition of Cauchy sequence, negation of definition of Cauchy sequence; sorry definition of Cauchy sequence. The negation is like this. The Cauchy sequence says that, for every epsilon greater than 0. So, negation will be, there exist an epsilon naught and this shows, there exists a natural number. So, we can say that, for every natural number H, then for all natural number H.

So, there will be a natural number m and n, which are greater than H, and satisfy this condition. So, we can say, the negation like this. There exists, there exists epsilon naught greater than 0, a small number epsilon naught, greater than 0, such that such that for every natural number H, for every natural number H, there exists, there exists, for every natural number, there exists, exists say at least one n, which is greater than H, and one m, and one m, at least one m, which is also greater than H, such that, the difference between the terms of the sequence x n and x m is greater than equal to epsilon.

Then this will be a negation of the definition of Cauchy sequence. It means, if you want to show the sequence is not Cauchy, then this criteria must be satisfied. For example, if we take this sequence, say x n is the sequence, like 1 plus minus 1 to the power n, this sequence. So, then we claim that, the sequence x n is not Cauchy. This is a what? It means, this condition is not satisfied. So, let us picked up the epsilon naught. So, let us choose epsilon naught equal to 2. There exists an epsilon naught; you take epsilon naught. Then since the sequence x n is such, which goes to 2 if n is even, and goes to 0, if n is odd. So, if we take the n as even, and n plus 1 becomes odd. So, difference between these two will be 2, which is epsilon naught.

So, in fact, then for every, for any, then for any H, natural number H, we can choose, we can choose an even number n greater than H. And then m, let m is n plus 1, which is also greater than H. We get, mod of x n minus x m is nothing but what? When n is even, the x n comes out to 2, and when m, which is n plus 1 odd, so it is 0, which is exactly same as epsilon naught. So, these satisfy the convergence, negation of the definition of Cauchy sequence. Therefore, sequence x n 1 plus minus 1 to the power n, this sequence is not Cauchy. The significance of the Cauchy sequence is, the significance of Cauchy sequence lies in the assertion, lies in the main theorem, in the assertion. The assertion is that, a sequence is Cauchy, a sequence, a sequence of real number, x n of real numbers is Cauchy, if and only if, it is convergent, convergent; if and only if, it is convergent. So, that is the main idea that, if a sequence is Cauchy, then it has to be convergent. And if the sequence of real number is convergent, then it will be Cauchy. So, this is the main result, which we will show it here and that is why, there exists the definition of the Cauchyness, or Cauchy sequence plays the vital role in the set of, of real numbers, system of the real numbers.

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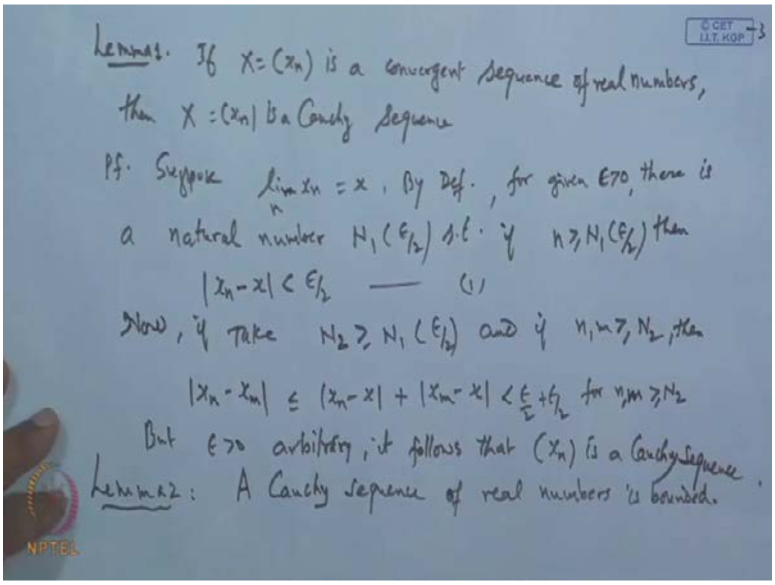

Let us see the first Lemma which is a, which is used, say Lemma 1. If X, which is x n, is a convergent sequence, is a convergent sequence of real numbers, then the sequence X is Cauchy, is a Cauchy sequence. So, this is the first result, which we want to show that, every convergent sequence is a Cauchy sequence. The proof of this is very simple. Suppose, suppose, limit of the x n, as n (( )) x, because it is a convergent sequence, limit will exist and suppose, this limit is x. Then by definition of the limit, for given epsilon greater than 0, there exists, there is a natural number, there is a natural number, say n 1, which depends on, say epsilon 2, such that, if we take n greater than equal to n 1 depends on epsilon 2, then mod of x n minus x is less than epsilon by 2. Let it be 1. Now, if we take…Now…So, now, if we take another number n 2, we take n 2, which is greater than or equal to say n 1, depending on, say epsilon 2, and if we choose n and m greater than or equal to n 2, then obviously, for this particular n 2, this result is also valid, because this is true for all n greater than n 1.

So, the, we take n n, which is greater than n 2, that will also be satisfied, because it is greater than n 1. So, it will be satisfied. Then we have mod of x n minus x m, apply the, this is less than equal to x n minus x plus x m minus x. Now, this is less than epsilon by 2. This is less than epsilon by 2, for all n, m, for all n, m, greater than n 2. Then in that case, the difference between x n minus x m is less than epsilon. So, this shows that… But epsilon is arbitrary; any, choose, any number, we can choose. Therefore, it follows that, the sequence x n is a Cauchy sequence, is a Cauchy sequence. So, that is what we wanted to prove.

Now, as we have seen that, a convergent sequence is a bounded sequence and we have seen that, every convergent sequence is a Cauchy sequence. So, basically, that, we are going to show this also, that, every Cauchy sequence is also bounded sequence. So, that will be the next result is, Lemma 2 says that, a Cauchy sequence of real numbers, of real numbers is bounded. So, the proof is just based on the similar lines as we have done it in case of real. So, what we do is, we take the sequence x n, which is a Cauchy sequence, since Cauchy criteria is satisfied. So, applying the Cauchy criteria, you can write the sequence x n is dominated by a certain number of results in a space and for the remaining one, the finite sum, take the maximum value. So, that is what.

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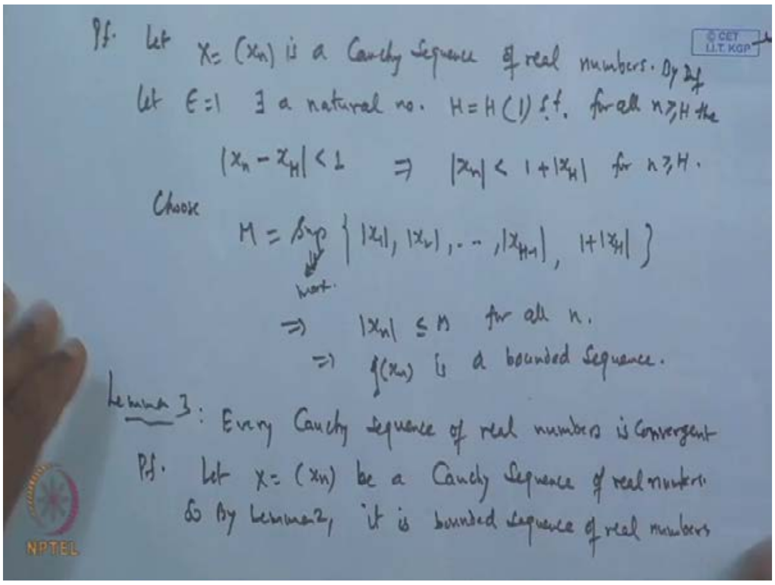

So, let X be a, proof, let us see, the proof is…So, let our sequence X, which is x n, is a Cauchy sequence of real numbers. So, by definition, so by definition, for epsilon, any epsilon greater than 0, so if we take epsilon, let epsilon is equal to 1. So, corresponding to this epsilon, there exists a natural number, this is sign for there exists, a natural number H, which depends on epsilon, epsilon means 1, such that, such that, for all n greater than equal to H, the difference between x n minus x H is less than 1; because n, we can choose. So, n is H. This implies that, the sequence x n is less than 1 plus mod x H, for all n greater than or equal to H, by modulus; because if this is greater than, we can take it, or less than, greater than equal to mod x n and then... So, basically, this is mod x n, mod x n is less than this. Let it be.

And then further, you choose capital M as the maximum value or supremum value of mod x 1, mod x 2 mod x H minus 1 and then 1 plus mod x H; we can also use the maximum. In fact, supremum, I have not defined, because these are finite number. So, we can change this by a maximum also, maximum term, in a set of supremum; that is, we can also take this maximum. So, when we take the maximum of these two, suppose M, then what happens, all the terms of sequence x n will be dominated by M, and this is true for all n. So, this shows, the sequence x n, this sequence is a boundary sequence, boundary sequence of real numbers. So, every Cauchy sequence of real numbers is bounded.

Let us see the converse part and converse part will prove the Cauchy convergence criteria, that is complete. So, what the Lemma 3 says, Lemma 3, or you can say, every Cauchy sequence, every Cauchy sequence of real number, of real number, every Cauchy sequence of real number is convergent. So, combining first and second, Cauchy sequence of real number is bounded and sorry, Cauchy sequence of the real number is, already we have seen the convergent sequence of real number is Cauchy, and we have proved the Cauchy sequence, every Cauchy sequence is convergent.

Then, we can go for the Cauchy convergence criteria. So, let us see the proof of it. Suppose we have a sequence X, which is x n, is a Cauchy sequence, be a, let it be a Cauchy sequence of real numbers, of real numbers. Now, this Cauchy sequence, by previous result, a Cauchy sequence of real number is bounded by Lemma 2, if we see, every Cauchy sequence of real number is bounded. So, by Lemma 2, it is a bounded sequence. So, by Lemma 2, it is a bounded sequence of real numbers and by Bolzano-Weierstrass theorem says, if a sequence is bounded, then it must have a convergent subsequence.

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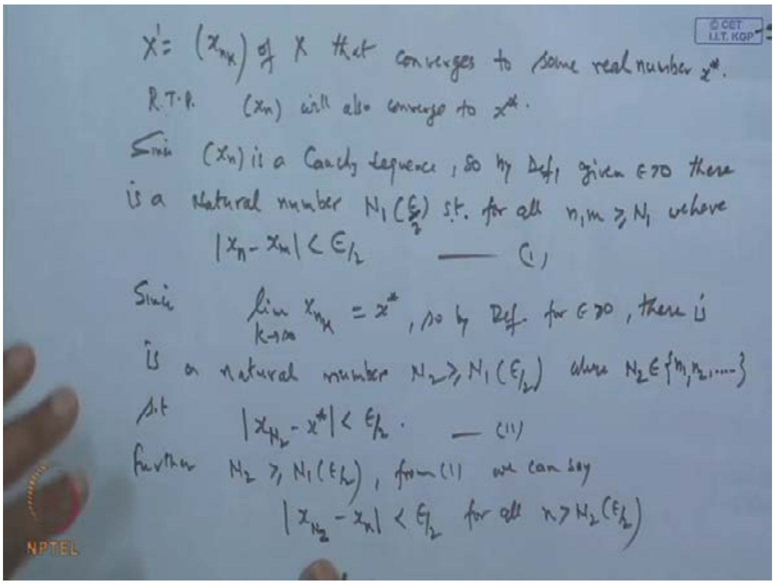

So, by Bolzano Weierstrass theorem, by Bolzano Weierstrass theorem says that, there is a, there is a subsequence X dash, subsequence X dash, say, equal to x n k, x suffix n k of X, x that by Bolzano, there is a subsequence x n k that converges to the some real number x star, real number x star, some real number x star. Now, if I prove that, our sequence x n also converges to the same number x star, then result is complete. So, what is required to prove is that, the sequence x n will converge, will also converge to x star; this is our problem to solve. Let us see how. Now, what is given, the sequence x n is a Cauchy sequence; this is given. So, since sequence x n is a Cauchy sequence, so by definition of Cauchyness, for given epsilon greater than 0, there is a natural number, there is a natural number, say N 1, depends on epsilon such that, for all n and m greater than or equal to N 1, say epsilon by 2, let us take the epsilon by 2, because we will need further, epsilon by 2, N 1, we have, such that, for (( )), we have the difference between x n minus x m is, is less than epsilon by 2. Let it be 1. Now, the subsequence x n k is a convergent subsequence, converges to x star.

So, by definition, again, apply the definition of convergent sequence, which has limit as x star. It will limit of x n k, when k tends to infinity, will is nothing but x star. So, since x n, since the limit of this x n k, when k tends to infinity, is given to be x star, this is given, we have got it, is it not? So, by definition of the limit, we can say, for given, for the same epsilon, for epsilon greater than 0, which is the same epsilon, there exists, there is a natural number, say N 2, which is suppose, greater than or equal to N 1, which depends on epsilon by 2, such that, N 2 and such that, this N 2 where, where N 2 greater, such that, this N 2 belongs to this set N 1, N 2, N etcetera; this set means, we are choosing the positive integer from a sequence n k s.

So, there exists a natural number N 2 from this set, which is greater than equal to N 1, such that, mod of x N 2 minus x star is less than epsilon by 2; because this is, difference of this can be made as small we please. So, when k is sufficiently large, it goes. So, we can choose the N 2, which is in this, is greater than N 1, so that, this condition is satisfied. Now, consider, and this is true for what? This is, this is less than…Now, further, further, this N 2 is greater than equal to N 1, which depends on epsilon by 2. So, from 1, the result is also valid, when you replace m by N 2. So, from 1, from 1, we can say, we can say, the result mod of x N 2 minus x n is less than epsilon by 2, for all n greater than N 2, depends on epsilon by 2, for all n; because m, I can choose to be N 2.

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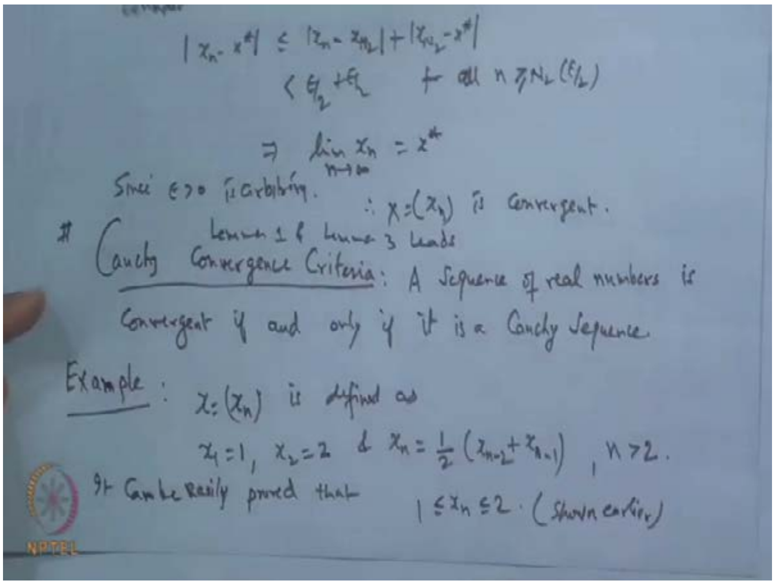

Now, consider, now, consider, mod of x n minus x star. Now, this is less than equal to mod of x n minus x N 2 plus mod of x N 2 minus x star, and then this will remain less than epsilon by 2 plus epsilon by 2, for all n, which are greater than or equal to N 2, depending on epsilon; because both conditions 1 and 2 are satisfied by this. So, what is that, the limit of x n is also n, limit of x n as n tends to infinity is also x star. This shows, the Cauchy sequence is convergent, because epsilon is arbitrary, since epsilon is arbitrary. So, once it is arbitrary, we say limit of this. Therefore, the sequence x n, which is X, is convergent. Now, this Lemma 1 and Lemma 2 and 3, if we combine this, sorry, 1 and 3, if we combine, we get a result, which is known as the Cauchy convergence criteria, Cauchy convergence criteria and that is very important. What this criteria says? A sequence of real numbers, a sequence of real numbers is convergent, is convergent, if and only if, if and only if, it is Cauchy, it is a Cauchy sequence. And that proves the, this one. So, combine the Lemma; say, this Lemma 3. Every Cauchy sequence, real numbers is convergent and then Lemma 1, that, x is convergent sequence of real numbers. Then Lemma 1 and Lemma 3 will give, so Lemma 1 proves, Lemma 1 and Lemma 3 leads the following Cauchy convergence criteria.

Now, let us see few examples where we can use this Cauchy convergence criteria to test whether the given sequence is a convergent one or not, where it is difficult to identify the limits. For example, if we take, define the sequence, x n is, suppose, a sequence x n is defined as x 1, first term is 1, second term is 2 and there is a relation between the term after 2 onward; after second term onward, the relation is like this, that, x n can be obtained as a average of x n minus 1 and x n minus 2 by 2, average of this. So, after second term onward, third term will be obtained as the sum of the, third term x 3 can be obtained as a sum of what, x 1 plus x 2 by 2; fourth term like this way. So, this sequence, this is true for, this is defined for n greater than 2.

Now, this sequence, if we find out the x n s by simply choosing x 1 is this, x 2 is this, then x 3, x 4, and so on, then what we see, that, this sequence is basically, all the terms of the sequence are bounded and bounded by 2, because that, I think, this result, we have already seen in the first, earlier one, because this is a monotonic decreasing, when n is odd, or, the increasing, when n is even, like that way, so but if all the terms cannot exceed by 2, and below it will always be greater than 1. So, it can be easily, it can be easily proved that, all the terms of the sequence x n are lying between 1 and 2. In fact, it was shown earlier, shown earlier, by choosing decreasing and increasing, odd terms and even terms separately, and they are dominated by 2.

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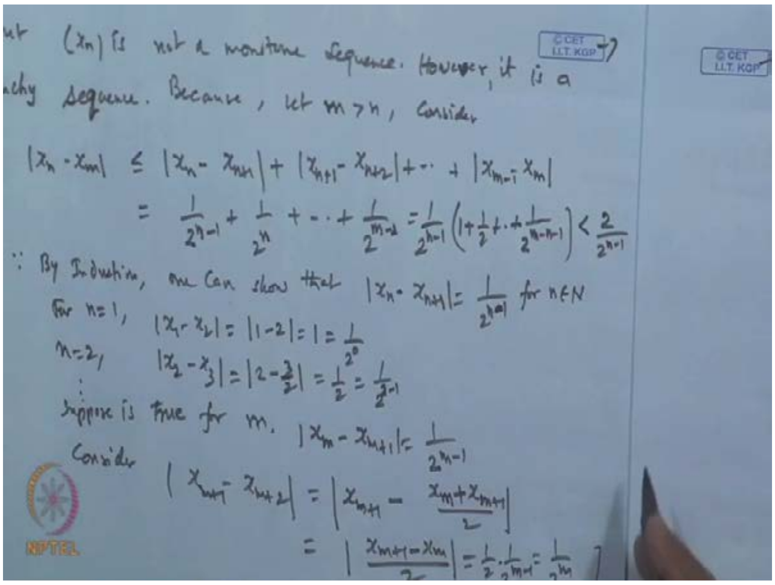

But it is not a monotone sequence; but the sequence is not monotonic; but sequence x n is not a monotone sequence; it is even and odd terms are monotonic; it is even and odd terms; but as a sequence, it is not monotonic; some terms are coming down, then up, and like this. So, as a whole, that monotonic sequence is not there. So, you cannot say the limit exists, because when you say it is monotonic, then only the limit will be, you can say, dominated by 2, bounded. But here, it is not a monotonic sequence. So, you cannot choose anything; (( )). However, it is a Cauchy sequence; however, however, it is a Cauchy sequence. The reason is, because if we start with this x n minus x m, say, where m is greater than n, let m is greater than n, consider, mod x n minus x m. Now, let us see, this is less than equal to what, x n, x m, x n minus x n plus 1, because n is less than m; then x n plus 1 minus x n plus 2, like this, and then our last term will be this.

Now, this will be equal to 1 by 2 to the power n minus 1, plus 1 by 2 to the power n, up to 1 by 2 to the power m minus 2; why? The reason is, because if we look this term x 1 by induction, induction, one can show that, mod of x n minus x n plus 1, this is exactly same as 1 over 2 n minus 1, 2 n minus 1, this can be find. And this is true for all natural number n, belongs to natural number. It can be tested for, for n equal to 1, for n is equal to, say 1. What happen, mod of x 1 minus x 2; x 1 is given to be 1; x 2 is given to be 2. So, it is basically 1, which can be written as 1 to the power 0. For n is equal to 2, if we take, then you take x 2 minus x 3; x 2 is 2 and what is x 3? x 3, from here, is the average of, when n is 3, it is x 1 plus x 2 by 2, x 1 plus x 2, 1 plus 2, 2, that is, 3 by 2. So, this will be equal to 4 1 by 2, which is 1 over 2 to the power 2 minus 1. So, suppose this is true for n.

So, suppose, it is true for, it is true for n, true for n, then m, then what we get is, we are taking mod of, mod of x n m minus x m plus 1; this is given to be 1 by 2 to the power m minus 1. So, consider now, mod of x m plus 1 minus m plus 2. So, that will be equal to what? Now, x m, x m plus 1, this is equal to x m plus 1 minus x m plus 2, if I write, then this is equal to what, x m plus 2, so x n, x m plus x m plus 2. So, m plus 1 by 2 and that is nothing but equal to x m plus 1 minus x m by 2; and that will be equal to, according to this, it will be half of this, half, 2 to the power m minus 1; that is, half of 2 to the power m, like this. So, it can be proved this way, like this.

So, by induction, we can show this result. Now, use this result here, I use this result here and then once I use there, then you can take common. So, 1 upon 2 to the power n minus 1, if I take common, then you are getting 1 plus half, up to 1 over 2 to the power m minus n minus 1. Now, this is the part of the geometric series. In fact, geometric series is 1 plus half plus half square and so on, and infinite terms are there, where the sum is, if you make the sum a over 1 minus r, that is 2. So, this is strictly less than 2 over 2 n minus 1; that is equal to, equal to 2 over 2 to the power n minus 2.

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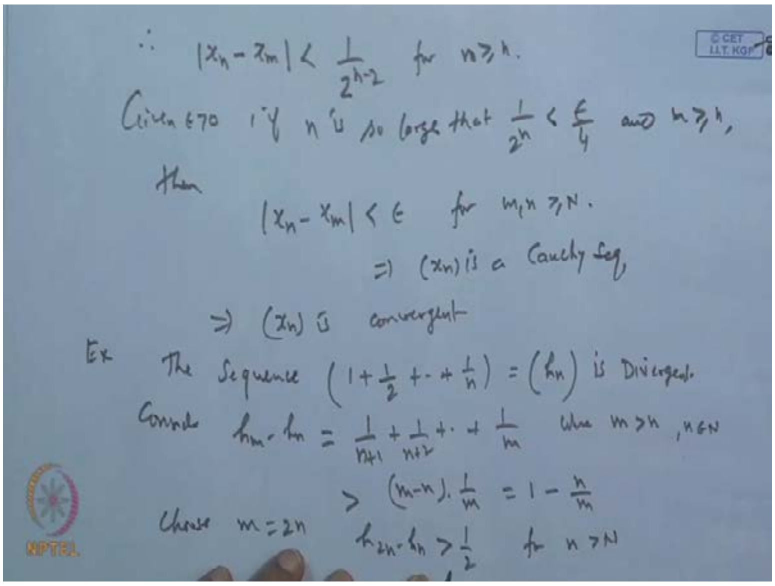

So, what we get from here is, therefore, mod of x n minus x m is less than 2 to the power, 1 upon 2 to the power n minus 2. And this is true for all n, for all n, for large n, when m is greater than or equal to n, this one. Now, if I choose any epsilon, given any epsilon greater than 0, a given any epsilon, if I choose n, if n is so large, so large, that, 1 by 2 to the power n is less than epsilon by 4, and if we take m to be greater than or equal to n, then what happens, the mod of x n minus x m is less than this entire thing, we want less than epsilon by 4. So, basically, it is less than epsilon. And this is less than epsilon, for all m, n, greater than equal to capital N, some integer N. Therefore, this is a Cauchy sequence. So, this implies, sequence x n is a Cauchy sequence. Once it is Cauchy, therefore, the sequence x n is convergent. Hence, without calculating the limit, one can go, the sequence is convergent.

Similarly, if it is not Cauchy, then one can show it is a diverging sequence. For example, if we take the sequence, say, the sequence 1 plus 1 by 2 plus 1 by n, this sequence, that is the sequence h n, where h n is the sum of this. Now, we claim that, this sequence is diverging one. So, obviously, if it is not Cauchy, then it may be divergent, because the every Cauchy sequence has to be convergent. So, consider this thing, now, consider h m minus h n. When you take h m minus h n, then what you get? You are getting from here is, 1 over n plus 1, 1 over n plus 2 and then 1 over m, where m and n, m is greater than n, and this result is true for all n belongs to capital N.

Now, this one is, the lowest term is 1 by m, is it not, 1 by m. So, it is greater than, and total number term is, m minus n is greater than this number, which is greater than, say, equal to 1 minus…This is equal to 1 minus n by m, 1 by n. Now, m and n are our arbitrary number, because one can choose m, n, any integers. So, I choose m, n, choose m to be equal to 2 n. Then what happens, this is, this h, 2 n minus h n, this difference is greater than half. And this is true for all n greater than capital N, after a certain stage. So, this sequence cannot be Cauchy sequence. So, the sequence is not Cauchy; h n is not Cauchy. So, once it is not Cauchy, it means, the sequence h n is not convergent. So, it is divergent, that is what. And this is known as the harmonic series, which will come later on in the discussion, for a point.

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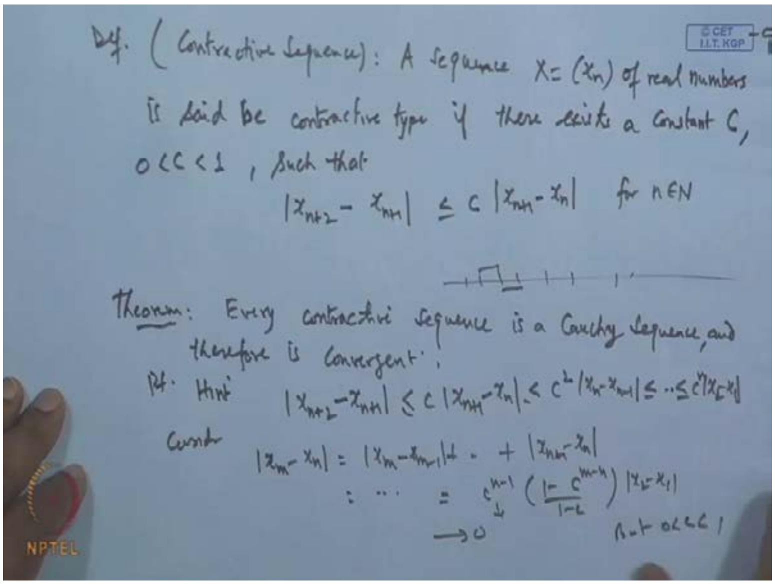

Now, there is a another type of the sequence, which we call it as a contractive type sequence, so contractive sequence. A sequence x n, a sequence x is x n of real numbers, sequence x n of real numbers is said to be contractive, is said to be contractive type, contractive sequence, if there exists, there exists a constant, say C, where C, lying between 0 and 1, such that, strictly less than, greater than 0, is strictly less than 1, such that, such that, mod of x n plus 2 minus x n plus 1, this mod of x n plus 2 is less than equal to C times of mod x n plus 1 minus x n, and this is true for all natural number N; this one. It means, that sequence is like this, that, if we take the difference between two consecutive terms of the sequence, say, this difference, then this difference cannot exceed by the difference of the previous one; that is, a C constant times, is always be less than equal to some number, which is less than this; because C is less, lying between 0 and 1; if I take C equal to 1, then only it is equal to. So, it is less than equal to. So, it keep difference is keep on reducing, keep on reducing that.

Now, what this result says, which is very, it is…The result is, every contractive sequence, contractive sequence is a Cauchy sequence, every contractive sequence is a Cauchy sequence, and therefore, and therefore, is convergent. Now, again, we will use the similar type of trick, as we did earlier; that first, you make this a difference x n minus 2 and so on, like this; prove it. In fact, to prove this, I just give the hint.

Hint say that, first, you show this part, x n plus 2 minus x n plus 1, this will be less than or equal to C times mod x n plus 1 minus x n; again apply this. So, it is less than C square mod x n minus x n, and continue this. So, finally, what we are getting is, finally, you are getting less than equal to C to the power n x 2 minus x 1. So, this is coming like…Now, consider mod of x n, x m minus x n and when you break up into the parts, mod x m minus x m minus 1, up to mod x n plus 1 minus x n, then substituting these values, and finally, you will get, this thing will come out to be C to the power n minus 1, 1 minus C, m minus n divided by 1 minus C, and then x 2 minus x 1; but C is lying between 0 and 1. So, this part will tends to 0, as n tends to infinity. So, it is a Cauchy sequence; so Cauchyness proved.

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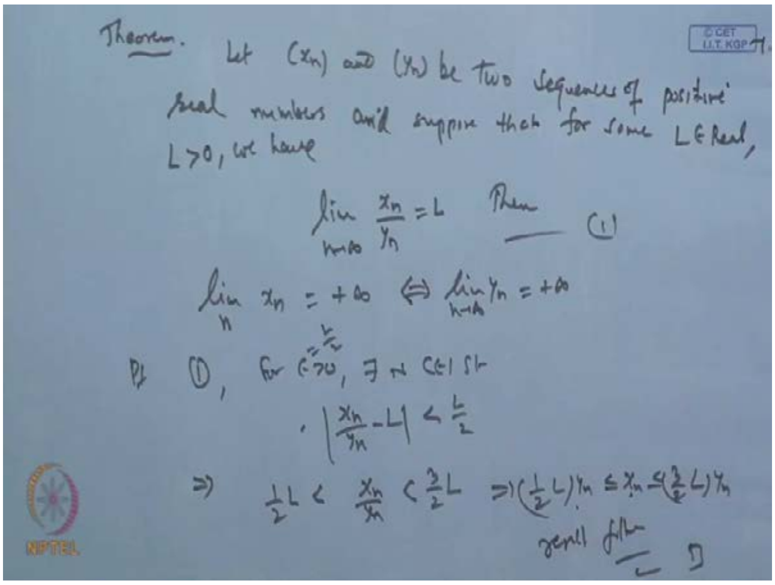

Now, there are only one result, which I will show for the divergent sequence, this result is very useful for the divergent. Let x n and y n be two sequences of, sequences of positive real numbers, positive real numbers, and suppose that, for some L, which is real, belongs to real, and positive, positive, we have limit of this x n over y n, as n tends to infinity, is L. Then limit of this x n over n is infinity, if and only if, if and only if, limit of y n is infinity. The proof is very simple. This is given. So, from 1, we say, for given epsilon greater than 0, there exists an n, such that, depends on epsilon, such that, mod x n y n minus L is less than epsilon; but epsilon, I choose to be L by 2; so is less than L by 2. And if I open this means, we get from here is, half L is less than x n by y n is less than 3 by 2 L; when positive, you get this. Now, if I multiply by y n, then what happen? This implies that, half L into sequence y n is less than equal to x n less than equal to 3 by 2 L into y n. Now, these are finite. So, if x n, y n goes to infinity, x n will go to infinity; if x n goes to infinity, y n. So, the result follows; that is all.

Thank you very much.

Thanks.