A Basic Course in Real Analysis
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Lecture - 2
Irrational numbers, Dedekinds theorem

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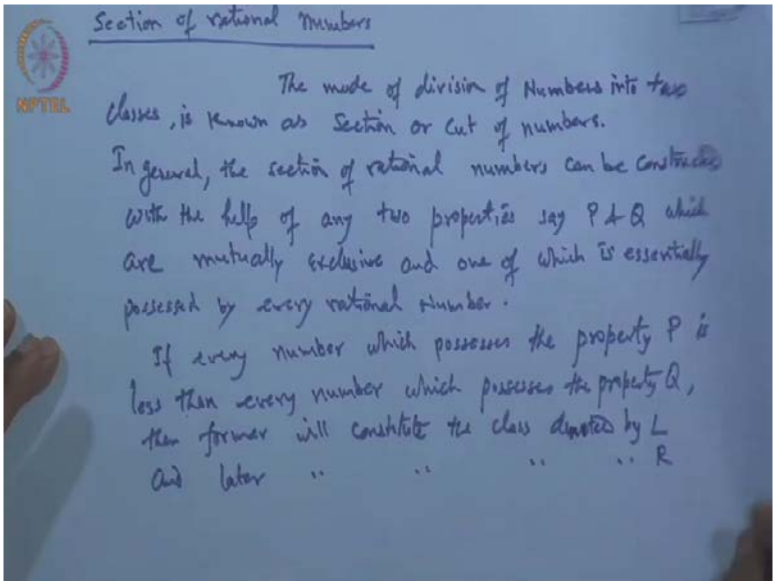

So today, we will start with the section of rational number or cut, section of rational numbers or also called a cut of rational numbers. Now, we already seend that if a rational number is giving, P by Q, where P and Q are integer, where Q is positive and there is no common factor in it; then between any two rational number, one can find to introduce another rational number. In fact, infinite number of way rational number can be introduced in between that, and with the help of this sequence of rational number, one can generate the irrational number. And entire class of the rational number can be divided into two parts, rational lower class and the upper class, is it not? Just like we have seen the set of all rational number which are less than equal to 1 and that set of rational number which greater than 1. So, 1 divides the whole set of rational number in to two parts, those which are less than or equal to 1 and those are greater than 1. Similarly, square root of 2, which is not rational number but it also divides the set of whole those irrational number, whose square, positive rational number whose square in less than 2 and set of all rational number which is positive number is always equal than 2. So this way, we are able to introduce the concept of the rational and irrational also.

So, let us formulized this part, the section of rational number, what you mean by that first? The mode of division of numbers in to two classes which is described just now, is known as the section or cut. So, the mode of division of numbers into two classes, two classes in different way is known, is, is known, is section or cut of numbers, clear? The in general, the section of the rational number, section of a rational number, of rational numbers, the section of rational number can be constructed, can be constructed with the help of, the help of two, any two properties, any two properties say P and Q which are, which are mutually exclusive that it joints exclusive and one of which is essentially possessed by, essentially possessed by every rational number, rational number. The meaning of this is simple; we have a set of rational numbers, a collection of the rational number. I introduce the two properties on it; one is P, another one is Q. So suppose, P is the property, set of all rational number which are less than equal to same 2, and Q is the property se of all rational numbers which are greater than 2. So, what happens that entire rational line number can be divided into the two classes, now, clear? Those number which are less than equal to 2 and the number which are greater than 2, and 2 is the number which is responsible for dividing the entire class into the entire numbers into two sections. So, this 2 becomes the section or cut of the rational. And corresponds, this section (( )) cut corresponding to rational point 2.

Similarly, the rational number square root 2; that is also number which is, which can divide, which can bifurcate the entire sequence, entire collection of rational number into 2 parts, those number which are, whose are positive rational numbers, whose square root is less than 2 and those positive number which square root is greater than 2. So, 2 is greater than greater than, root 2 greater than square is less than 2 and square is greater than 2. So, root 2 will be the number that one.

Now in this section way, we are basically dividing the whole things in 2 classes; the class where the every number in the class, if it is, it is less than the number of, the every number of class are then we say L is the lower class, R is the upper class; or we can say a number which possess the property P is less than the number which posses the property Q, then the property P will generate the class which is denoted by L, L is known in the lower class; and the property Q which generate the class is denoted by capital R and we call it is the upper class, clear? So this way, we are having the two class, lower class and upper class, is it clear now? So, we can say like it, if every number, every number which possess, possess this, possesses the property P, which possesses the property is less than, is less than every number, which possesses the property Q, property Q, which possesses the property Q, then the former constitutes the class, the class denoted by L and the later one will constitute the class denoted by capital R, denoted by capital R..

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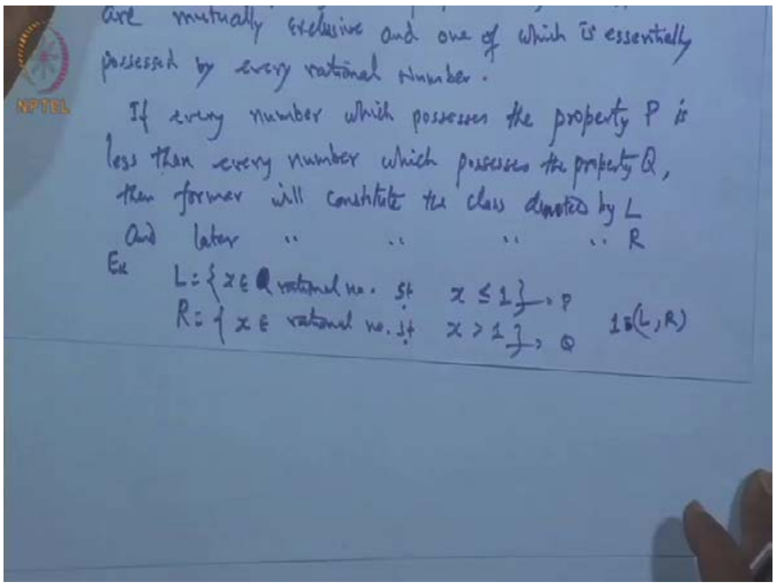

And the examples, we have seen already that the L is set of all those rational number x with Q is rational number, remember, Q is already there, so do not write, the rational numbers, collection of rational number such that x is less than equal to 1, and R is the class of those rational points, rational numbers such that x is greater than 1. So, this is the property P, is a not? This one is the property P, this one is the property Q. Now you see, both the properties are exclusive, mutually exclusive, a point, we cannot get rational number which belongs to P as well as in Q; if it belongs to P, it cannot be in Q; and the second one is every point in P is less than the every point in Q, the property which satisfies the property Q. So, this will be, hence this entire be denoted by L R; and this is this corresponds to one, so we say the one rational number corresponds the section L R, one corresponds to this section LR, is it OK? Like this.

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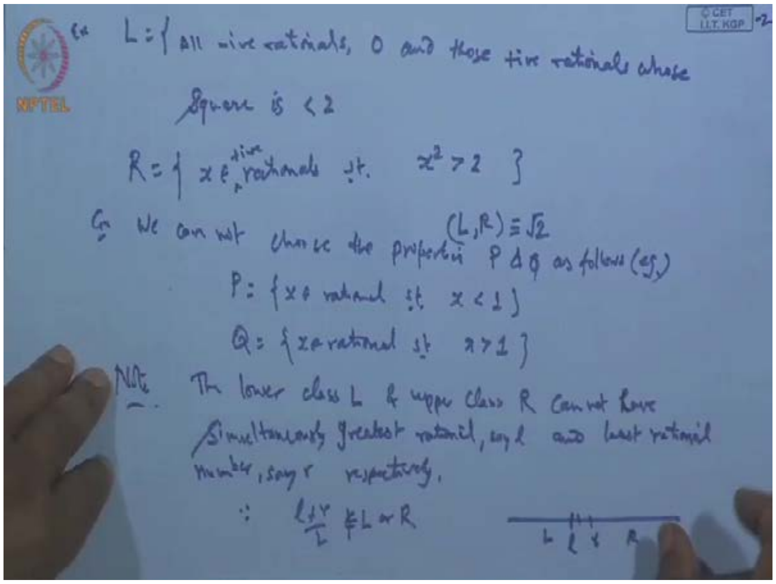

Similarly, if I take L in set of all, all negative rational, rationales 0 and those positive rational, rationales whose is square is strictly less than 2. So, the property I am putting here only, I am heading minus all rational and 0; but property is basically those positive rational number whose is square is less than 2, clear? And in this, in this class, we are introducing the negative as, all negative and 0; and R is the set of all rationales such that square of this is greater than 2 or positive rational sorry, or positive rationales whose square is greater than 2. So now, all these rational number will be filled up here and this class which we denoted by, is nothing but corresponds to square root class, clear? Is it OK? Now, the difference this and this two class is in the earlier case, 1 is a rational number and what is the property which enjoyed by rational number is then, 1 must be or rational number which corresponds to a section a L R must be a point either in L or in R, is it not? Any number which corresponds to section L and R must be a point in either lower class or in upper class. Once, it is in lower class then it becomes the largest number, L lower class will have the largest number equal to say 1 here, and once it is in R then it has the least number in that; but in case of the section, discuss point to the irrational points, here none of these lower and upper class has largest or least number. We cannot have it; this does not belongs to neither L nor R, but it divides the whole rational number into two parts.

The third case is we cannot choose the property P and Q as follows, for example, for example suppose I take the property P, if the set of those rational x such that, x is strictly less than 1; and I if a choose property of Q, those rational such that x is strictly greater than 1. Why we cannot choose this? The reason is by taking this type of property, it gives the lower class and upper class fine, all the L number rational numbers which are less than 1 will form a lower class, all the number which are less than grater than will full form the upper one. But what about the number 1? Is neither belongs to lower class nor the upper class. So, we are looser, we are not including all these, is it correct? So, all the rational numbers can not be covered.

Similarly, a rational number, note, in case of the, that if the lowest, the lower class L, the lower class L and the upper class capital R cannot have simultaneously greatest rational number say L and least rational number, rational number say R, respectively, what is the meaning of this? Let us see, first expand this part; what I mean is, suppose we have a this is our lower class, this a upper class, what I am saying is that it i not possible to have the largest number, upper lower class will have a largest number as well as upper class will a lower number, lowest number, it is not possible. One of the class, if lower class have a largest number, upper class cannot have a lowest number and vice versa, if lower class does not have a upper largest number then lower all will have a least number, lower number, least number, why? If suppose it is true. It means L number which is largest belongs to L, R number which is the least which belongs to R, it means the number lower than R will not available capital R, the number which is greater than R will not be available in L.

So, what will happened between L and R? Between L and R, because L plus R by 2 is a number neither belongs to R or nor R. so, there are infinitely many number will be left out in between L and R, which does not occupy the position neither in L nor in R. So, it means our section of all rational number is not complete, we are not able to adjust all the rational number in one of the class, is it not? Some rational numbers are out. So this way, the division is not comfortable, is not permissible. So that is why we have to say, we cannot have a property like this which can leave many of the rational number without putting either in R or in L, is it clear? So this about.

Now, it may possible that there may be only one class; other class may not exists, there may be possible. Suppose I take all the rational numbers, all the rational number get, say positive negative and 0, they all belongs to one class, that is all. The rational number which are less than infinity is from lower class approach, we cannot have it upper class. So, there is a possibility that we can have, out of these two, lower and upper, we have only one. But that is not the case which we are dealing, in fact it is only the case when minus infinity or plus infinity taking in consideration, is it not? So, if I want to introduce infinity or minus infinity and then this type of possibility is there. But what we dealing here is we are taking only the finite rational numbers, we are, we always exactly have the lower class and the upper class, and this section L R, the cut is noted by an alpha which is a either rational number or may be the irrational number, is this clear? So this may be yours…

So now, we have the combined with this then what we get finally is that we are having section of three numbers; one is three type of the classes, that is, one is the lower class has the largest number, upper has the lowest number; second, class is the largest lower class does not have a largest number, upper class has a lowest number; and the third case, none of the class has neither upper nor lower. So first two class correspond to a rational, rational number, the third class correspond to a irrational number, and this the way the Dedekind has introduced the concept of the irrational number in the number system.

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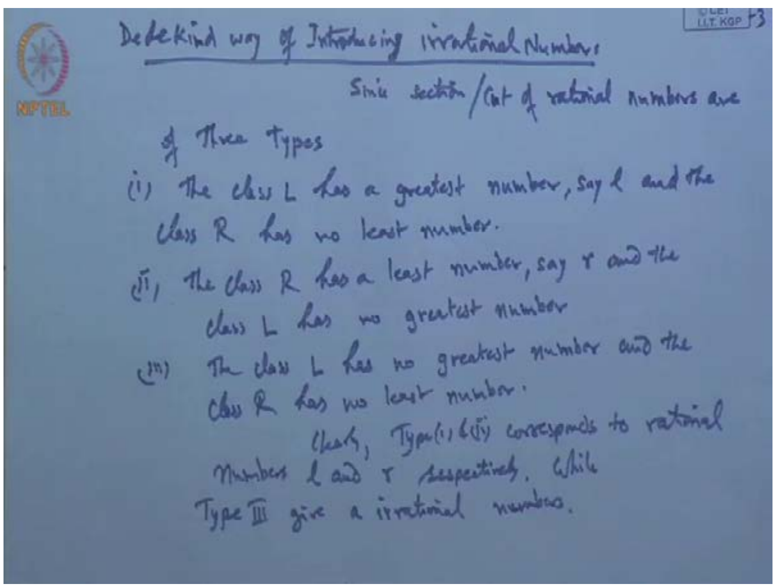

So that is way Dedekinds introduction of rational, Dedekinds way of introducing, introducing irrational number. So, what this is? Section real number, since sections or cut of rational numbers are of 3 types; 3 type, the first type is the class L, the class L has a greatest number say L, and the class, and the class R, and the class R has no least number this is the one type; second type is the class L, the class R has a least number say R and the class L has no greatest number, greatest number; and third type of these is the class L, the class L has no greatest number, greatest number and the class R, class R has no least number, least number. So, first two types, clearly type one and two corresponds to rational number numbers L and R respectively, while type three gives, gives of, gives an irrational number, or corresponds to a irrational. And this way, this is the way dade kind set introduce a concept of irrational number, is it clear?

So, what one more thing we can observe on these type that if you pick up any real number, it is always represented by means of the section L R. So, any real number can be represented as a section L R lower class and upper class. Now, this section, if it falls into first two category then real number becomes the rational, it will represent the rational point, if it falls in the third category then that real number will be the irrational number point.

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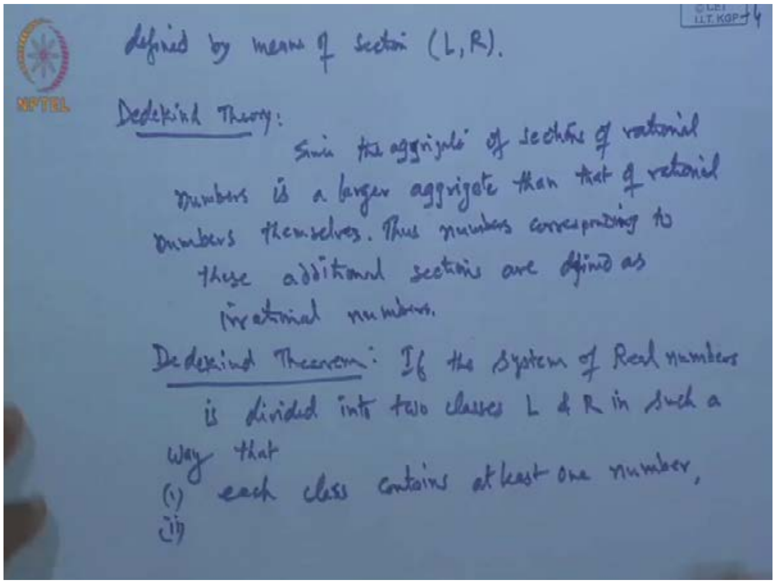

So now, we can say every, thus any real number, any number, any real number alpha can be, can be defined, can be defined by means of section L R, is it OK? So that is what here into. We have started with, here we started with a rational numbers only; and then by introducing the section, we have comes across would irrational point. So, thus, this way, we have develop the sections which represents the rational number and some time it also represent the irrational number, is it not? So, if we consider the collection of all these sections, aggregates, sets of all these sections then this set of all these section will be the bigger class then the set of rational number, why? Because this section collection of these section represents either a rational number or real number, and every real number, rational number can be represented by means of this section or every section represents some rational points; but as well as there are this sections which also represents a irrational points; but here, right hand side we are taking only the set of rational numbers. So, it means the aggregate of these sections is much bigger than the aggregate of the set of rational number, is it not? And that gives the concept of the real numbers. So, Dedekinds, this way has extended the system of the real, extended the rational numbers into a real number by introducing the irrational point in between, is this clear? So, this is the known as Dedekinds theory. So, the let us see what is the Dedekinds theory is, Dedekinds theory, this is the Dedekinds theory.

So, since, since the aggregate, aggregate, agg, aggregate all the, set of all the rational points, aggregate, aggregate of sections of rational numbers is a larger aggregate, larger aggregate then that of, then that of rational numbers, rational numbers themselves, themselves. So, it includes the (( )). And thus, we defined the number corresponds to the additional section. And thus, the numbers corresponding to these additional section, corresponding to these additional sections, additional sections are defined, are defined is rational, irrational number that is what is… So, we have now the Dedekind theorem. Now, before going something else, we just write down what is dade kind theorem; if the system of real number, real numbers is divided into two classes two classes L and R in such a way, in such a way that one is class contains each class contains at least one number, at least one number. Second, at least one number.

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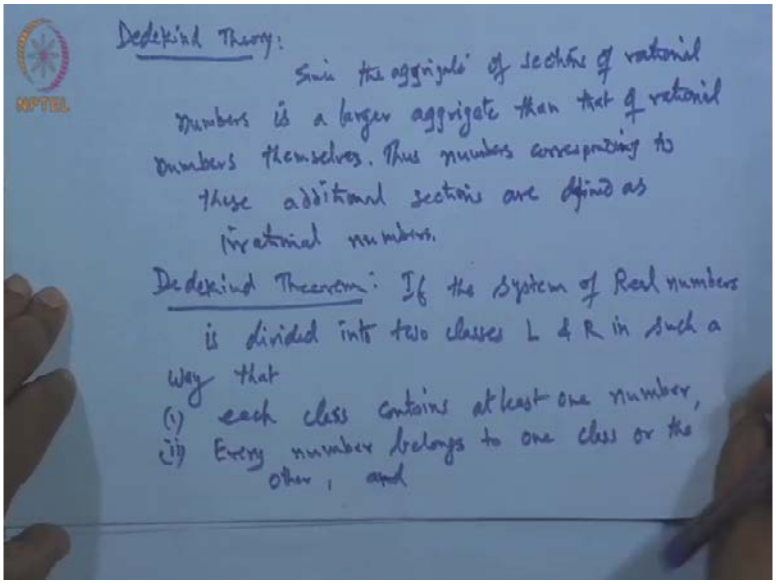

Second is every number, every number, every number belongs to one class or the other, or the other.

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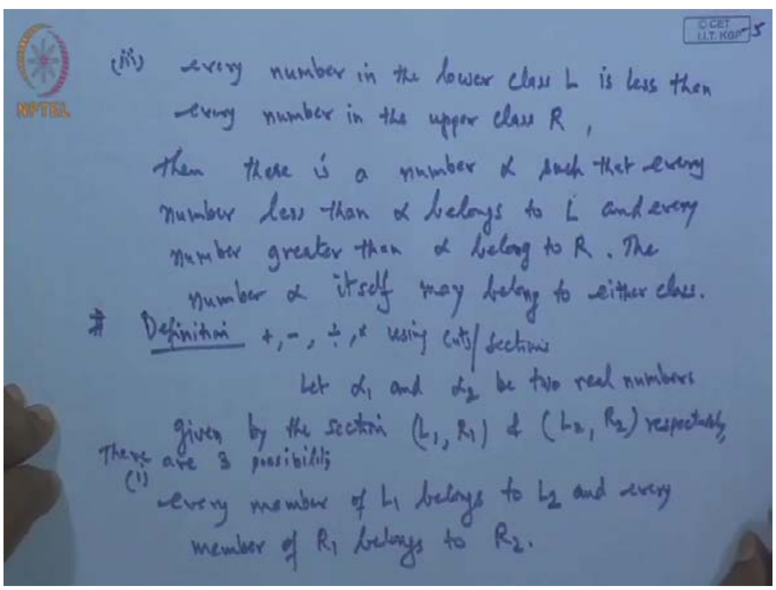

And third condition is, and third condition is, and third part is every number in the lower class, lower class L is less than, is less than alpha, is less than every number in the upper class, every number in the upper class R. Then there is a number, is a number, there is a number alpha such that, such that every number, every number less than alpha belongs to L, belongs to L and every number greater than alpha, there is a number alpha such that every number less than alpha belongs, every number greater than alpha belongs to R, then alpha belongs to R, R, the number alpha itself, the number alpha itself may belong to either class, either class, same thing which we have written; means we have a set of real numbers. Now, what we did is that this set of numbers divided into two parts, lower class and the upper class and which has the property that every member is, each class contains at least one real number.

Second is every member in the lower class belongs to 1, each contains at least 1, and every number belonging to one class or the other; and third is every number in the lower class is less than the every number in the upper class. Then the number alpha is such that every number is less than alpha belongs to L, well the every number greater than alpha belongs to R. Now, what happened to alpha? What he says is the alpha itself may belongs to either class, so when is may belongs to alpha, it means there is hidden thing; if alpha is a rational number then it will belongs to one of the class, it has to belongs to one of the class as we are seen. But alpha is the irrational number then it will neither belongs to L nor belongs to R. So that is, that is what he developed the… Now, the question is when he started with these cuts and with the help of cut, he introduced concept of the generalized set of rational number and introduce the concept of rational number, then with the help of cut, how did he justify the addition, subtraction, multiplication, division? Because every, we will need take two real numbers the addition of the two real is again a real number; so, it should represent some by a cut, subtraction is also real number, division is also real number, provide it is non-zero, like this reverse. So, he has also introduced the way the addition of the two cuts is, how to defined; subtraction of the two cuts to how to define; the reciprocal of cut, how it is defined. And i. So, we will take a first those what are these definitions. So the ration of (( )). So, we can define, introduce or definition of, definition of addition, subtraction, division, multiplication in terms of cuts, using cuts, or sections; how to define? How to define this thing? Definitions. Let alpha 1, let alpha 1 and alpha 2 be two real numbers, two real numbers given by the section, given by the section, section or cut you can say section L 1, R 1 and L 2, R 2 respectively, respectively.

Now we say number one, if every member of, there are four possibility, three possibility, there are three possibility, three possibility; number one, first possibility is every member of L 1, every member of L 1 belongs, every member of L 1 belongs to L 2 and every member, every member of R 1 belongs to belongs to R 2, this is the first possibility; the second possibility is, the second possibility is every member of L 1, L 1 belongs to L 2, belongs to L 2 and every member of R 1 and belongs to, but every member of R 1, but every member L 1 belongs to L 2, but every member of R 1, R 1, this not belongs to R 2. This is second possibility.

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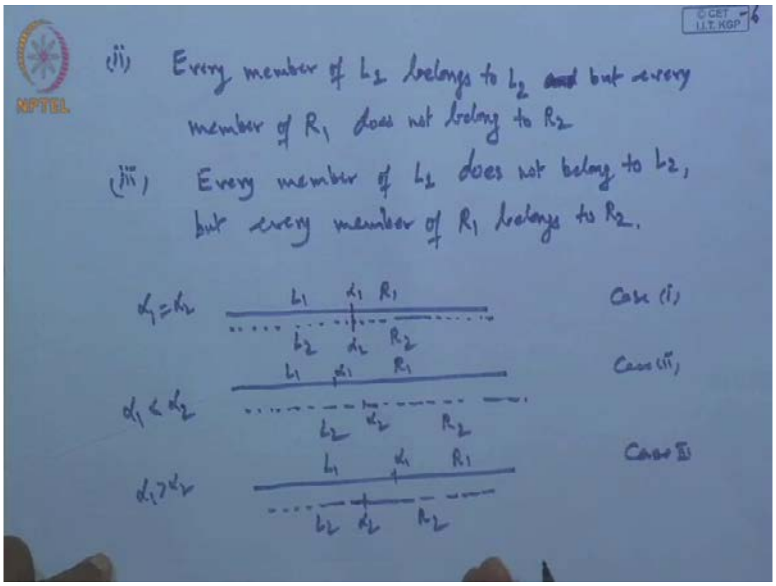

And third possibility is every member of L 1, of L 1 does not belong to belong to L 2, but, but every member of R 1, R 1 belongs to R 2. Let us see, this is first case; suppose this is, these are two classes, this 2, this our L 1, R 1; this is L 2, R 2. Now, what he says first (( )). If first possibility is every member of L 1 belongs to L 2 and every member of R 1 belongs R 2. So, that is only possible when basically L 1 coincide to L 2 R 1 coincide with R 2. So, this is alpha 1, here is alpha 2, so here is the case alpha 1 and alpha 2, is it not?

Then second case is, case two, here say this is our L 1, here is R 1; and say, here is this L 2 and this is R 2. So, here this is alpha 1, here this alpha 2. So, second case says that if every, this one, every member of L 1 belongs to L 2. So, every member of L 1 belongs to L 2, it means the cut alpha 1, all these entire line must be line on part of L 2; but every member of R 1 does not belongs to R 2, it means the R 1 class super the R 2, is it not? So that is only possible when the position alpha 1 and alpha 2 is such as alpha 1 is strictly less than alpha 2. And then third case is if we take say L 1, R 1, this is alpha 1 and this is alpha 2, L 2, R 2. So, every member of L 1 does not belong to L 2. So, it means L 2 is subset of L 1; but every member R 1 is containing, so the position is alpha 1 is greater than alpha 2. So, if the two numbers alpha 1 and 2 are giving, then one can order them that they are either equal or one is less than the other. It means set of real number is basically and ordered set, field. One can easily identified the ordering between the two and with the help of cuts also, is it OK? So that is what…

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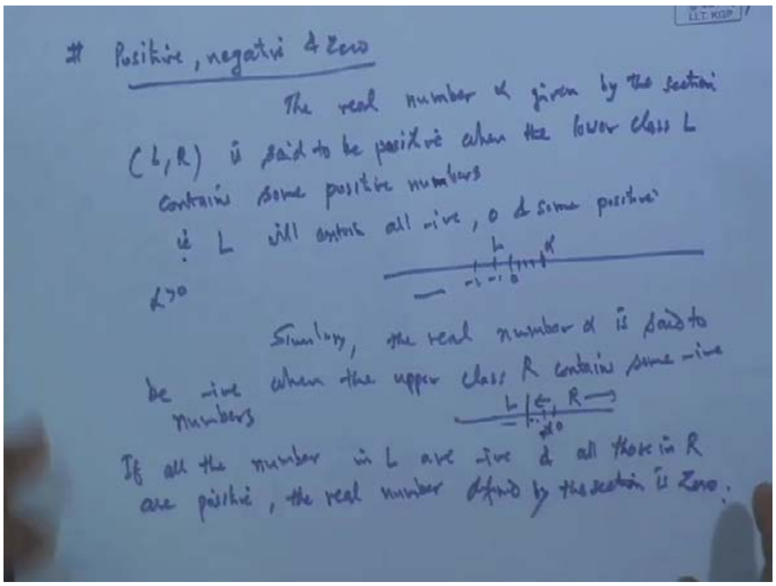

Then in a similar way here, introduced the positive negative and 0 number, positive, negative and 0 number, in the, in terms of the cuts. The real number alpha, the real number alpha given by the section, section L R, given by the section L R is said to be positive, is said to be positive if or when the lower class L contains some positive numbers; it means that is that is lower class L will contain all negative. Zero, and some positive, is it OK or not? Some positive that is if this your line, then a number alpha will be a positive number, if basically it is greater than 0. So, here it is 0 and rest of minus 1, minus 2, etcetera. So, this lower class must contain some positive number, then only alpha will treated as positive, greater than 0. Similarly, the alpha is negative means, what? The real number alpha, real number alpha is said to be negative, is said to be negative when the upper class upper class R contains, upper class R contains some negative numbers, negative numbers that is officially true; here is L, this is 0 here is L, this is R, if R contains some negative number, then only the number correspond to this will be negative number, is it not? The real number are said to be negative, when the upper class are contain some negative numbers. So, L is totally negative and R contain some negative numbers. So, that corresponding to say minus 1, this will the cut, like this. And 0 means what? And if the, if all the numbers in R, if all the numbers, if all the numbers in L are negative and all the numbers, and all those in R are positive, then the real number, real number defined by the section, section is 0, officially, clear? So, there is nothing is very simple, clear? (( )) This is clear. So this alpha summed here, like this; then addition, subtraction is certain can be done.

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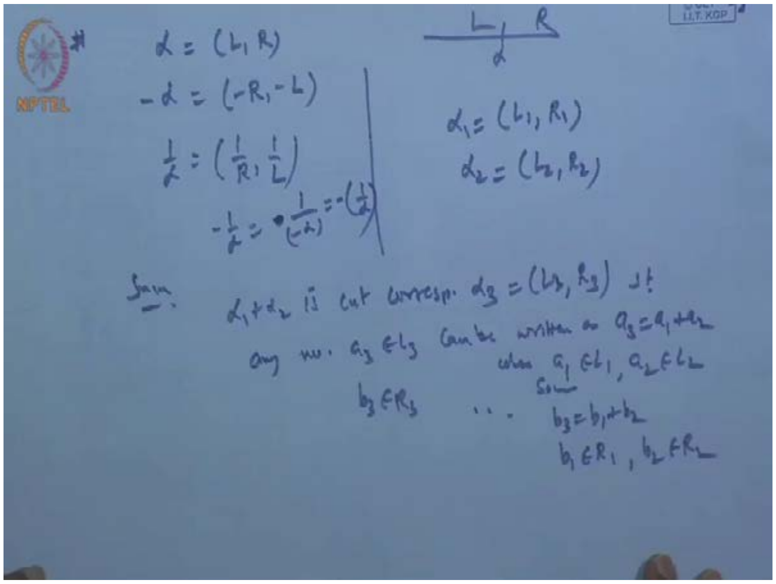

If suppose alpha is a lower class, then how to define this number? If alpha correspond to the lower class L R, then minus alpha corresponds to minus R minus L that is all, clear? This is the class alpha, lower and upper. If minus alpha means change the sign of elements of R and make it take the lower class and change the sign of L be minus and make the upper class. Now, what the difference will come? If L has a greatest number, alpha is a greatest number in lower class as a cut alpha, where the lower class L is greatest number, then minus alpha, here upper class will have the least number that is all. Suppose, I take L to be less than be equal to 1 then 1 is the greatest number in the lower class; when you find minus 1, what happens to this? Minus 1 becomes here the least number, is it not? For the upper class. So similarly, if alpha is there, then 1 by alpha will be denoted by 1 by R, 1 by L, all the property makes the same, similarly for other. For positive, for negative, for positive and for negative also, we can write as a minus 1 by alpha is minus 1 upon minus alpha like this, clear? Take minus and then reverse it. So all this means (( )) algebra expressions can be also justified by using the our cuts system. So and that is what is a…

Then some in difference, if suppose we get the alpha 1 is 1 cut, L 1 R 1, alpha 2 another cut L 2 R 2, then sum of this alpha 1 plus alpha 2 is a cut correspond to alpha 3, that is L 3 R 3, such that any number alpha 3, any number a 3 belongs to L 3 can be written as a 1 plus a 2, where a 1 belongs to L 1, a 2 belongs to L 2 for some. Similarly, R 3, b 3 belongs to R 3 can be written as b 3 and b 1 plus b 2, where the b 1 is element in R 1, b 2 is an element in R 2. Sir, in upper class, minus 1 by alpha is equal to 1 by minus alpha. So, to get the minus alpha, first you take this one and reverse it. When we will reverse it, again it will come minus 1 by alpha. So, it will reverse again the things. So that will come, is it Ok or not? Minus 1 by alpha, in fact, it is same as minus 1 by alpha, is it not? Minus 1 by alpha and reverse it 1 by alpha will be this, then minus 1 by R; now, if you take this minus, this means this, reverse it. So, minus 1 by R L, same thing will come. Now, here also addition can be done like this, is it not?

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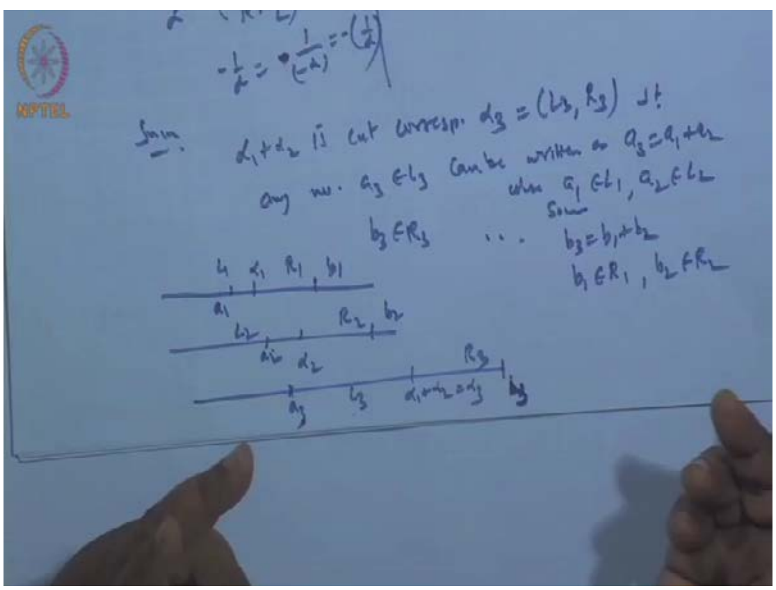

What is the meaning of this? This addition is, suppose we have this alpha 1 cut L 1 R 1, alpha 2 is this cut L 2 R 2. When we add this thing, this is alpha 1 plus alpha 2, we denote by alpha 3. So, this is L 3 R 3. Now, if I take any point here a 3 then corresponding we get a point a 1 here, a

2 here such that a 1 plus a 2 becomes a 3. Similarly if we take b 3 here somewhere, is it not? Here b 3, then b k, we can find out b 1 here, b 2 here such that b 3 can be written as b1 plus b2 here, clear? Similarly, subtraction, multiplication, and another. So, we are not going this thing, if we found the multiplication; in a similar way, you can write the multiplication and division, is it OK? So, we are not worrying for the details. So, then between any two real number and finite number of real number one can easily prove; just like in between rational number, there are infinite rational number is there. So, between any two real number, one can always find the name.

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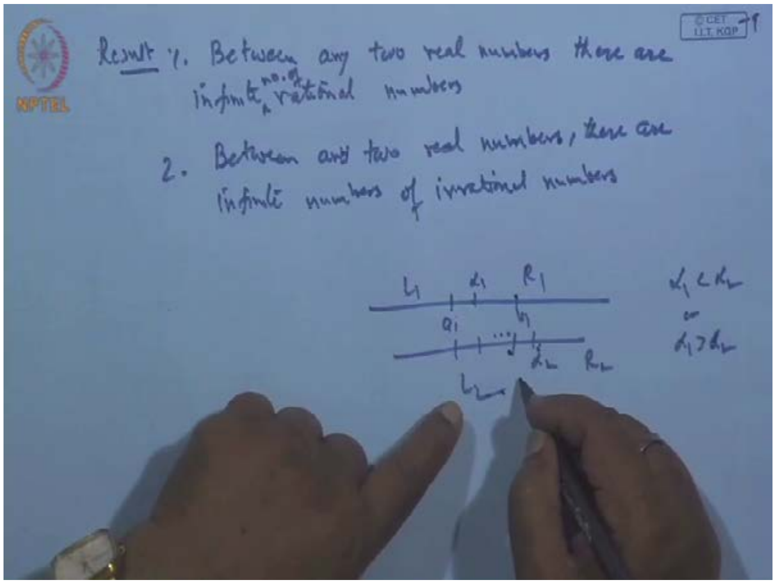

So, this between any two some, result, the result is between any two real number, two real number it will take two real numbers, there are infinite, there are infinite number rational numbers; second result is between any two real numbers, between any two real number there are infinite number of irrational point, number of irrational numbers, infinite number of irrational point, infinite number of irrational point, numbers. In fact, this I will prove it, but just I will give a hint, how to, suppose there are two numbers, real numbers alpha 1 and alpha 2. They are two numbers, this is alpha 1 this is alpha 2, L 1 R 1, L 2 R 2. So, between these two number alpha 1 alpha 2, there are infinite number of rational numbers, why? Because L 1 alpha 1 is less than, alpha 1 and alpha 2 are given; one can easily identify alpha 1 is less than alpha 2 or alpha 1 is greater than alpha 2 or may be equal. Suppose less than alpha 1, then every element of L1 is the element of L 2, but every element of R 1 is not this; it means some of the point R 1 is also in L 2. So, if I take any point a 1 here and here it is a b 1, then these are also the point here L 2, and L 2 is collection of all the number which are less than alpha 2. So, in between those we can still find that rational number. So, there are infinite rational number can be introduced. Similarly here. So, we will do this (( )).

Thank you very much. Thanks.