A Basic Course in Real Analysis
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Lecture - 19
Theorems on Convergent and Divergent sequences

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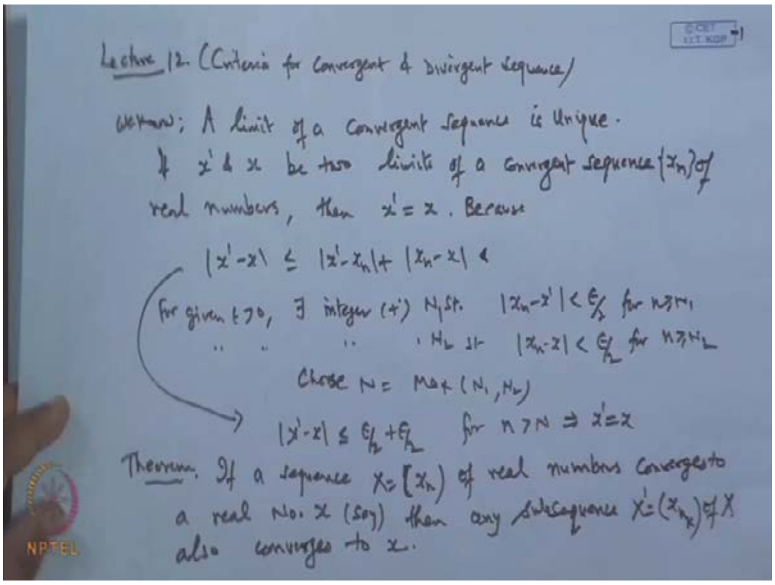

So, today we will discuss the criteria for convergent and diverging sequences. We have already seen that, if x n be a sequence of real numbers, then it is said to be a monotone, if there is a sequence of the positive integer in increasing order or decreasing order; such that, the corresponding terms of the sequence x n 1, x n 2, x n n, then they from the monotonic sequence x n 1 is less than x n 2 less than x n 3 and monotonic increasing or if it is a reverse, then we say monotonic decreasing. And also if a monotone sequence which is bounded above or below, it must be a convergent sequence. So, based on this, we have seen that, sequences, if they are monotone and bounded, we can say that sequence will definitely converge. But every sequence need not be a monotone sequence, because there are the sequences which are not at all monotone sequence; then how to find out whether the sequence is convergent or divergent?

So, for this, we have a certain results, which will directly tell without computing the limit, whether the sequence is convergent or not. And one of them, which is very important result, is given by the Cauchy, which is known as the Cauchy convergence criteria for a sequence of real or complex numbers . So, let us see the first thing, that, if a sequence is convergent, then all of its subsequences will also converge to the limit. We know, if a sequence is convergent, then the limit is unique. This we know, that, as limit of a convergent sequence, we know, a limit of a convergent sequence is unique; that is, if there are two limits, if suppose, if x dash and x be the two limits of a convergent sequence x n of real numbers, then x dash must be equal to x.

The reason is, if we start with x dash minus x, then this can be written as x dash minus x n plus x n minus x. Now, since x n converges to x dash, it is a limit. So, by definition of the limit, the, for a given epsilon greater than 0, for a given epsilon greater than 0, there exists an integer, positive integer, capital N 1 such that, mod of x n minus x dash is less than epsilon by 2, say for all n greater than equal to N 1.

Similarly, for the same epsilon greater than 0, there exists a positive integer, say N 2, such that, mod of x n minus x is less than epsilon by 2, for all n greater than equal to N 2. So if we choose capital N to be the maximum of N 1 and N 2, then this result is also true for n greater than capital N; this is also true for n greater than capital N. Therefore, this thing can be made less than, this thing can be made less than epsilon by 2 plus epsilon by 2 for n greater than N. So, this shows that, x dash must be equal to x; that is, a limit of the convergent sequence is always unique. So, it means, if a sequence which has a different limits, it cannot be a convergent sequence. So, this is the one of the criteria you can say that, if a given sequence is there, find out the limits, if along various path, it has a different limits, then the sequence will not be a convergent sequence; then we call this, such a sequence, of course, a divergent sequence.

The another thing is, if a sequence is a convergent sequence, then all of its subsequences will also have the same limit. So, that is very interesting result, which is not true in case of the divergent sequence; divergent sequence, the subsequences have a different limits. So, if a sequence, if a sequence x, which is say x n of real numbers, of real numbers, converges to, converges to a real number, a real number, say x, x, then any subsequence, any subsequence x dash, which is say x n k of x, also converges to x. So, if x n is a convergent sequence of real number, then all of its subsequences will also converge to x. Subsequences, we mean, we have discussed this, that, if x n is a convergent sequence, x n is any sequence, and if we identify this integers n 1, n 2, n n, such that, which are increasing or decreasing order, then this sequence x n k means, increasing order, n 1 is less than n 2 and this is n 3, then this sequence will be a subsequence. So, this will…

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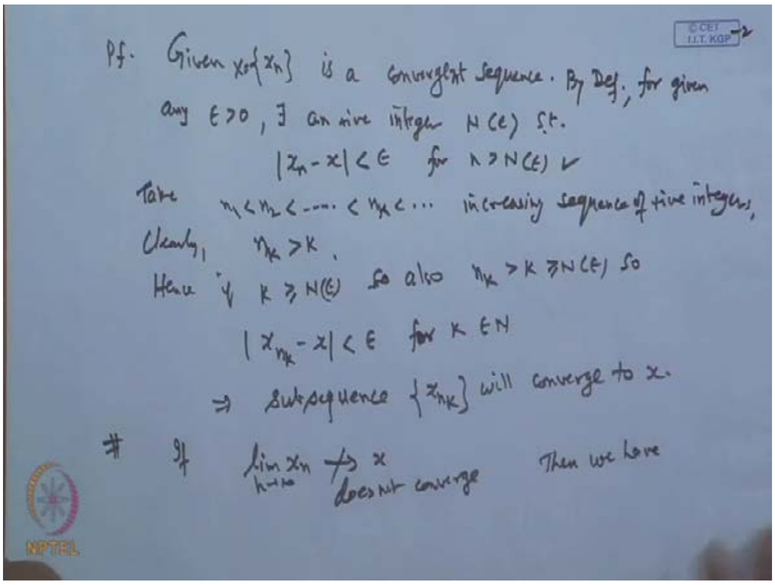

Now, let us see the proof. Proof is very simple. What is given is, the sequence is a convergent sequence. So, it is given, given the sequence x n, which is denoted by say x, is a convergent sequence. So, for the given epsilon, by definition, so by definition for given, for given, any epsilon greater than 0, a positive number, there will exist, there exists a positive integer, say capital N, which depends on, say epsilon, such that, such that, mod of x n minus x is less than, say epsilon, for all n greater than equal to N, depends on epsilon. This is what we get. Now, we are wanted the any sequence, subsequence must converge.

So, let us picked up the terms. Take n 1 less than n 2, less than n 3, less than n k, and like this. This is the sequence of, increasing sequence of natural numbers, increasing sequence of positive integers, positive integers; n 1 is less than n 2, less than n 3, positive integer. Now, obviously, clearly, this n k will be greater than k, for any k, because k is 1; then you can identify n 1, which is greater than 1; then k is 2, you can identify another integer, positive n 2, such that n 1 is less than n 2, which is greater than 2 and like this. So, it is very easy to verify that, this is followed. Now, if this k is greater than this number here, capital K N, if this number is capital N, so what we do? Hence, if this k, which we have taken is greater than equal to capital N, which depends on epsilon, then obviously, n k will also greater than k. So, for all n k s greater… So, n k will also be greater than k, which is greater than equal to n, depends on epsilon. Therefore, for this, these n k s, this result is true. So, from… So, we get mod of x n k minus x is less than epsilon, for all n k, for all k belongs to the natural number N, for all k belongs to N. This is true; but n 1, n 2, n satisfy this condition also, which we have taken. Therefore, the sequence, subsequence x n k will converge to x. So, this implies that, the subsequence x n k, this subsequence will converge to x and this is an arbitrary sequence we are choosing. So, any subsequence of x will definitely converge.

So, if a sequence x n is a convergent sequence, then all of its subsequences will converge. So, this is what. Now, let us suppose, we have a converse part of view. The criteria where the convergent, the limit fails; then it will lead to a sequence, which is a divergent sequence. So, let us find out, what will be the corresponding, or equivalent condition, when the sequence does not converge. So, let us see the criteria for the divergence of the sequence… If the sequence, if sequence x n, or limit of x n, when n tends to infinity, does not converge to x, does not converges to x, or, the limit fails, then what will be the… Then we have, then we have the following equivalent criteria.

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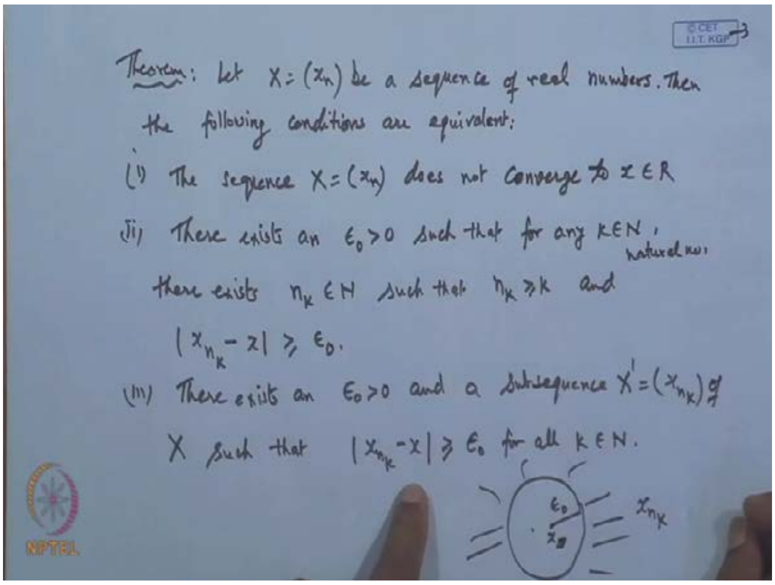

Then, we have…This theorem will help you in identifying the divergent sequence, the criteria for diverging sequence. So, let x, which is x n, be a sequence of real number, real numbers; then the following conditions, then the following conditions are equivalent.

The first condition is, the sequence X, which is denoted by x n, X, x n does not converge to, converge to x in real, means, does not converge to a number x. The second criteria is, there exists an, there exists an positive epsilon, epsilon naught, greater than 0, such that, for any k belongs to N, N is the natural number, this is the set of natural number N; for any k belongs to N, there exists n k, a positive integer belonging to N, such that, such that, n k is greater than equal to k, and mod of x n k minus x is greater than equal to epsilon naught. And third condition is, there exists an epsilon naught greater than 0 and a subsequence, and a subsequence x dash, say x of x n k of X, of X, such that, x n k minus x is greater than equal to epsilon naught, for all k belongs to N.

So, what they shows is, proof, we will see. x n is a sequence of real number, then the following conditions are equivalent. The sequence x n does not converge to x, then it is equivalent to this, equivalent to this. Basically, the difference between second and third is, second says that, n k s, which does not follow the increasing order, may or may not follow the increasing order; that just, there exists some n k, where this is true; but here, there must be a sequence n 1, n 2, n n, which should be a decreasing increasing order; n 1 less than n 2 less than n k, and corresponding to this positive integer, a sequence x n k can be obtained, so that, this satisfy this condition.

Now, what is this condition? When we say the limit of this x n is x naught, or is x, it means, if I draw a neighborhood around the point x, this, with a suitable radius, say epsilon naught, with a suitable radius epsilon naught, then the all the terms of the sequence must fall within this, if the limit exists; but if the limit does not exist, it means, the all the terms of the sequence, after certain stage, will fall somewhere outside of this neighborhood; because this also shows that, if x is, if you draw a neighborhood around the point x with the radius epsilon naught, then all the terms of the sequence after this integer, say n 1 is n 1 x n 2 etcetera, this will not satisfy; this will not fall within this region; it will fall outside of it.

So, this shows the criteria for, this shows the sequence x n does not have a limit. So, we will go to the proof that, how these three conditions are equivalent. Means, we can, if the sequence does not converge, we can immediately say, this condition holds; that is, there exists a epsilon naught and a subsequence such that, the x n, most of the term, infinitely many terms lies outside of the epsilon neighborhood, epsilon naught neighborhood of x.

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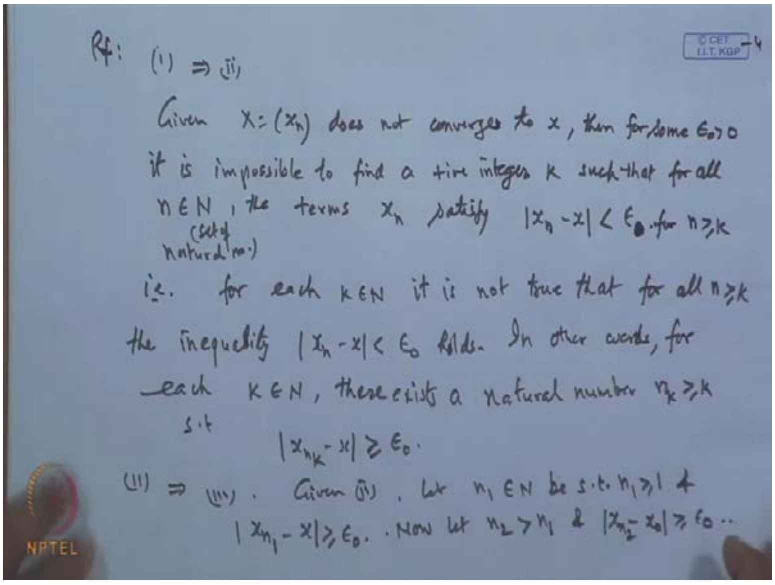

Let us see the proof. Proof of this. So, first is, very, first implies to, one implies to, let us see. What is the first? Sequence does not converge. Then we have to find out this thing. So, if the sequence does not converge…Given, the sequence x n does not converge, does not converge to x. It means what? It means, for the convergence, for any epsilon greater than 0, there will be an n, such k s, such that, all n greater than n, difference is less, but it does not converge. So, we can identify a epsilon naught and corresponding to the epsilon naught, we can identify the sequence, where it violates the condition of the neighborhood, that. So, then for some epsilon naught greater than 0, for some epsilon naught greater than 0, it is impossible, it is impossible to find a natural number, find a positive integer, find a positive integer, say k, such that, for all n belongs to a natural number capital N, say, this is set of natural number, set of natural number belongs to capital N, such that, for all k, k, such that, for all n greater than n, the term, the terms x n, the terms x n, satisfy the condition, satisfy, mod of x n minus x is less than epsilon naught.

So, what he says is, if suppose it does not converges, it means, for at least, for some epsilon naught, it is impossible to find an integer k, such that, when you choose all n, for all n greater than equal to k, this condition will not satisfy. It is impossible to satisfy this condition. That is to say, that is, this means that, for each k belongs to, for each k n belongs to N, it is not true that, for all n, for all n greater than equal to k, for all n greater than equal to k, the inequality, inequality mod x n minus x less than epsilon naught hold, for this holds; that is the same thing for each k, it is not…What do you mean? It means, we can identify a sequence, we can identify a natural number n k, such that, which n k is greater than k, and this condition violates; that is, in other words, in other words, we can say that, for each k belongs to capital N, there exists a natural number, a natural number or positive integer you can say n k, which is greater than or equal to k, such that, mod of x n k minus x is greater than or equal to epsilon naught, that is what here; because it does not converge means, this condition will satisfy, right? It is impossible to find out n k, which will follow this condition.

So, there must be a n k, some integer can be obtained, which will violate this condition. So, that is what we get. So, 1 implies 2; now, 2 implies 3. It follows immediately. Second condition say that, there exists an epsilon naught such that, for any k, there exists n k, such that, n k is greater than equal to k with. Now, you can find out n 1, which is greater than 1 and satisfying this condition. Then n 2 which is greater than n 1, greater than this, will satisfy. So, using this, given 2, then let n 1 is a integer, is a natural number be such that, n 1 is greater than equal to 1, and it violates, it satisfy this x n 1 minus x is greater than equal to epsilon naught. Then you choose n 2, which is greater than n 1 and again, it satisfy the condition, is greater than equal to epsilon. So, continue this way. So, continue; once you continue, you will get a sequence n k.

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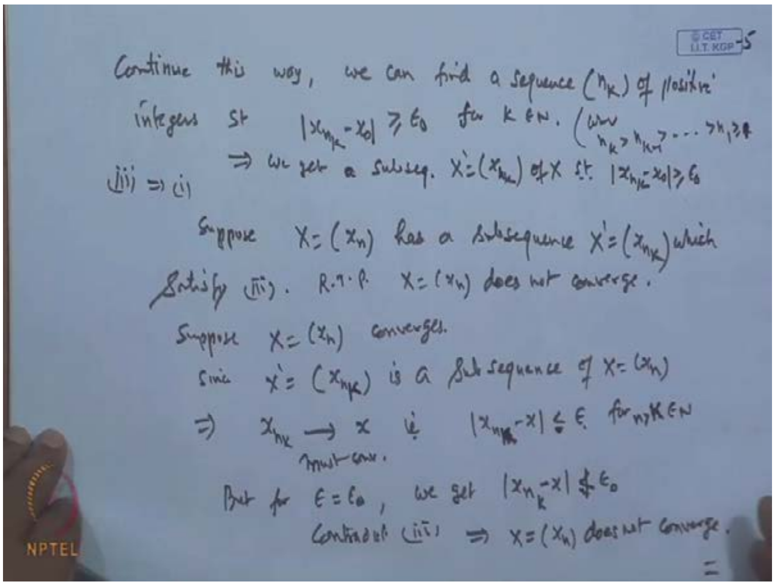

So, continue this, continue this way, we can find a sequence n k of positive integers, of positive integer, such that, such that, x n k minus x naught is greater than equal to epsilon naught, for all k belongs to N. Where, what is n k, where this condition is satisfied, where n k is greater than n k minus 1 and so on, is greater than n 1, which is greater than equal to k 1, like this. So, we can identify this sequence, clear? And this shows what? This shows that, the condition third is valid, equivalent. So, third, now, third implies first; given, this condition holds, that is, there exists an epsilon naught and a subsequence x dash n, so and a subsequence, say, x dash. So, we can say, subsequence x, should I write this way? Yes, this was the third condition. There exists a epsilon naught and a subsequence x dash n, such that, this condition holds. So, here, this is given; we want the x n does not converge.

So, suppose the sequence x… Here, you can say in this, the subsequence, we can say that, there exists a, we obtain, we get a subsequence x dash, which is x n k, x n k of x, such that, this condition violates; that is all, so this condition. Suppose, a sequence x n converges; suppose, the sequence x, which is x n, has a subsequence, has a subsequence x dash n, x n k, satisfies, which satisfies, which satisfy condition three. So, once it is there, then we have to show, required to show is, required to prove, the sequence x, which is x n, does not converge; this we wanted to show.

Suppose, it is not true; suppose, the sequence x, which is x n, converges; suppose, it converges. Now, x dash, since x dash, which is x n k, is a subsequence, is a subsequence of x, and this sequence, we have assumed to be convergent. So, all of its subsequence must be convergent, by the just previous shown. So, this shows, this implies, the sequence x n k, this sequence must converge to x, must converge to x, or converges to x. It means that, that is, mod of x n k minus x is less than equal to epsilon for all n, for all k belongs to N, if it is convergent, or for all n, you just say, x n minus x, for all n greater than equal to capital K, belongs to N. But that will violate this condition, because the condition said, x n k minus x naught is greater than epsilon, now. So, for this particular, epsilon equal to epsilon naught, but for epsilon equal to epsilon naught, we get x n minus x is not equal to less than epsilon naught; it is greater than or equal to epsilon naught. So, this is what violates that condition.

Therefore, this is a contradiction of our given condition three. So, contradicts condition three; because if it, suppose, you assume convergent, then it must satisfy this condition that, x minus x n k for all k belongs to N, this must satisfy condition. But for this particular epsilon naught, this condition is violated, because this is, this condition is not satisfied, because it is greater than epsilon. Therefore, contradicts the condition three; and contradiction is because our wrong assumption, sequence converges. Therefore, this implies, the sequence x, which is x n, does not converge and that is proves the result, clear. So, this is. So, now, there are some criteria for the divergence.

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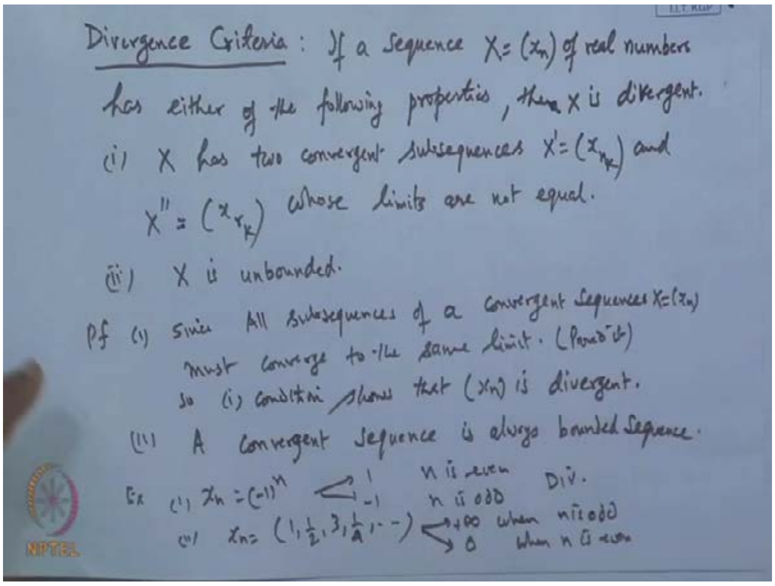

So, this, we call it as a divergence criteria, divergence criteria. If a sequence, sequence X, say, x n of real numbers, of real numbers, has either of the following, of the following, either of the following properties, then X is divergent, divergent. What is the first is, X has two convergent, two convergent subsequences, two convergent subsequences, X dash, say x n k, and x double dash, say x r k, x r k, whose limits are not equal. And second one is, x is un-bounded.

The proof is very simple. If a sequence x n of real numbers has either of the following properties, then x is divergent. So, what is this property? First property says, if the sequence x has a two convergent subsequences, whose limits are not equal, then the sequence will be considered as a divergent sequence. Or second one is, if x is unbounded, that is, the limit of x n will go to either plus infinity or minus infinity, then also, sequence is considered to be a divergent sequence. The proof proves immediately from the fact that, if suppose, x is a convergent sequence, then in case of the convergent sequence, any subsequence of the convergent sequence must be convergent, and converge to the same limit, because the limit is unique.

So, if x, any sequence have a different limit along the different subsequences, different path, then the sequence cannot be a convergent one; so it has to be divergent. Similarly, every convergent sequence is a bounded sequence. So, if a sequence is unbounded, it cannot be convergent. So, proof, first is, follows, since, since all subsequences of a convergent sequence, of convergent sequence x equal to x n, say all subsequences of convergent sequence must converge to the same limit. This is the criteria for the convergence, is it not? That is what we have proved also; this is proved earlier; proved it. So, if a sequence has a subsequence which does not converge, then obviously it will be divergent.

So, first criteria, first condition shows, first condition shows that, the sequence x n is divergent, divergent. Second follows, since a convergent sequence is bounded, is always bounded sequence. So, if a sequence is unbounded, then it must be a, it must not be a convergent; it will not be a convergent sequence. Now, here we are not saying boundedness, because even a divergent sequence may be a bounded sequence.

So, we are not saying. What we are saying is, a convergent sequence will always be bounded sequence. So, once it is unbounded, definitely that sequence will be a divergent sequence. So, that proof follows. So, these criteria, I think, we have given already examples taken. For example, if we take a sequence x n, which is minus 1 to the power n, this is sequence, then this has a two limits, 1 and minus 1, when n is even and when n is odd. So, it is a divergent sequence. And second, if I take the sequence x n to be, say, like this, 1, half, 3, 1 by 4 and so on. So, this sequence, when there is a odd numbers, it will go to infinity, when n is odd; and when n is even, then it will go to 0, because half, 1 by 4, 1 by 6, 1 by 8 and so on; so when n is even. So, it has the two limits. Therefore, it will not be a convergent; divergent, yes.

Now, we have for the monotonic sequences, we have a criteria that, every bounded monotone sequence is convergent, and that is called the sequence, monotonic convergent, bounded sequence of monotonic sequence are always convergent. And based on this, we have a monotone subsequence theorem. That theorem says, if x n is a real number, then there is a subsequence of it, that is monotone. In fact, if a sequence is given, then it is not necessary that sequence be a monotone sequence. But always, we can identify at least a subsequence, which has a, which will be a monotone. So, every sequence of the real number have a monotone subsequence.

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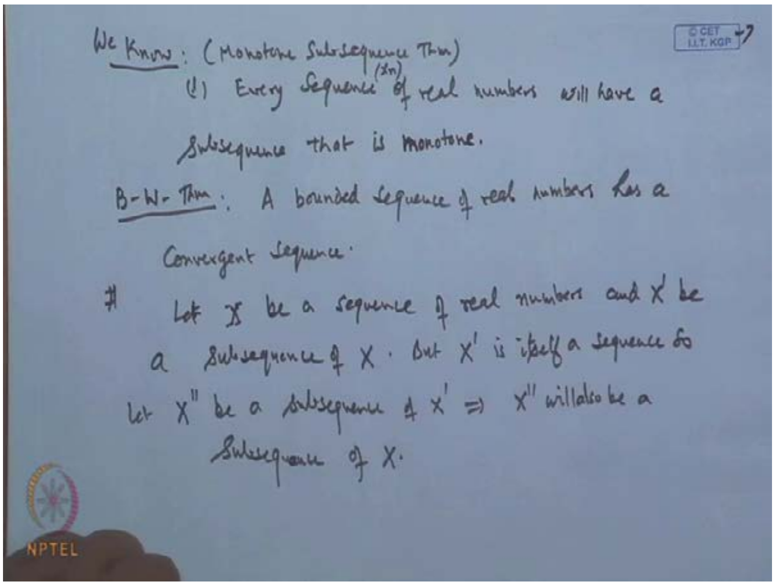

So, that, we know that, every sequence, every sequence, every sequence of real number, of real numbers, will have a subsequence, will have a subsequence, that is monotone; either monotone increasing, or monotone decreasing, so on. So, once it is monotone, and if it is bounded, then that particular sequence will be, monotone sequence will be convergent. But it does not mean that, the sequence x n will be convergent, because the sequence x n does not have all the subsequences which are monotone. So, you cannot say, the sequence itself is a convergent one; only one particular subsequence, which comes out to be monotone sequence, and if it is bounded sequence, it will converge; if it is unbounded, it will diverge; like that. So, again, the criteria for the monotone will not help you much. Unless the sequence is monotone, we are unable to identify, whether the sequence is convergent or not. And there is one result, which also we have seen, the Bolzano-Weierstrass, Bolzano-Weierstrass theorem. This theorem, we have shown, a bounded sequence of real number, sequence of real numbers has a convergent subsequence.

So, both these results, that is, monotonic subsequence theorem, this is the monotone subsequence theorem, and Bolzano-Weierstrass theorem, though it gives a rough idea about the some of the subsequences, but it does not give the total idea about the entire sequence. So, we are unable to get this, whether the sequence is convergent. But the Cauchy has given his idea, which is known as the Cauchy convergence criteria, which, without calculating the limit of the sequence, which can tell, whether sequence is convergent or not. So, for this, we will develop, first few results and then go. Let X be a sequence of real number, real numbers and let X dash be a subsequence; X dash is a, and let X dash is a subsequence, be a, be a subsequence of X. Now, if we consider X dash as independent sequence, then basically, it is also a sequence. So, again, we can identify a subsequence of X dash again. So, we can get, let X double dash…But X double dash is itself a sequence. So, we can identify the... So, let X double dash be a subsequence of X dash. Then obviously, the elements of X double dash are also element of X; so obviously, X double dash will also be, will also be a subsequence of X, is it ok? So, and continue like this, till we get this. Now, we have a very interesting result. The result is theorem.

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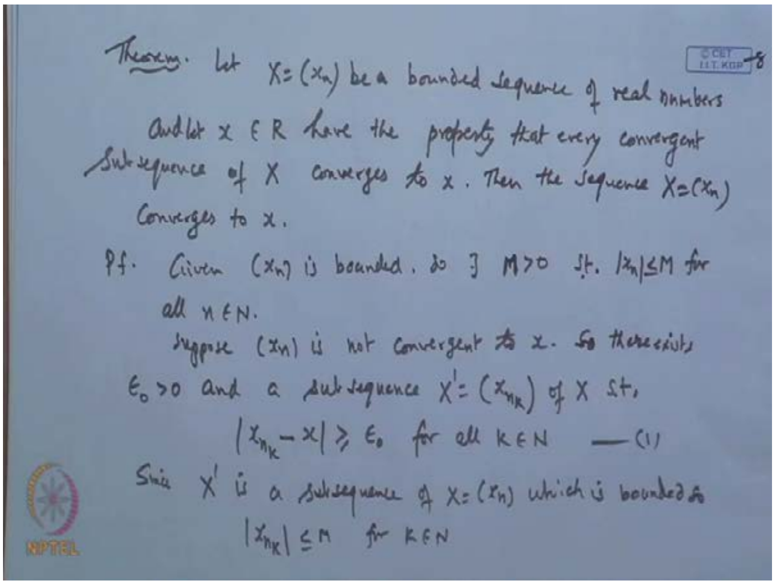

The theorem says that, let X, which is x n, be a bounded sequence of real numbers, sequence of real numbers, bounded sequence of real numbers and let X and x is a belongs to R, and let x belongs to R have the property that, every convergent subsequence, convergent subsequence, every convergent subsequence of X converges, converges to x. And let x belongs to, have the property, as every convergent subsequence of x converges to x. Then the sequence X, then the sequence capital X converges, which is x n, converges to x. Now, this is the converse part of the previous results. In the previous result, we have shown that, every, if a sequence is convergent, then all of its subsequences will be convergent and converge to the same limit. Now, this shows the converse part. Suppose, a sequence is bounded sequence and if all of its subsequences converges to the same point x, then the sequence must be a convergent one.

So, proof is… Now, given that, x n is a bounded sequence, given, the sequence x n is bounded. So, it means, all the terms of the sequence are dominated, or less than equal to some number. So, there exists an M greater than 0, such that, all the terms of the sequence is less than equal to M, for all n, for all n belongs to N, because it is a bounded. Now, we wanted the x n is convergent. So, suppose, this sequence x n is not convergent, is not convergent to x, is not convergent to x, not convergent and to the x point, convergent and converges to x, not convergent to x, then what you… Once the sequence is not convergent, does not converge to x, so we will apply the criteria which we have discussed for the diverging sequence. Then by the criteria we can say… So, there exists, so there exists, there exists an epsilon naught greater than 0, and a subsequence, and a subsequence X dash, which is x n k of X, such that, such that, x n k minus x is greater than equal to epsilon naught, for all k belongs to a set of natural number N, belongs to N, is it not. So, let it be 1.

Now, X dash is a subsequence and element of X dash is also the element of X, and X is given to be bounded. So, the element of the subsequence is also bounded. So, since the X dash is a subsequence of X, which is bounded, given, so all the terms of the sequence X n dash is also less than equal to M, for all k belongs to N. This criteria will have… Now, this subsequence is a bounded sequence. So, X dash itself is a sequence. Now, this sequence is a bounded sequence. Once it is a bounded sequence, then by Bolzano-Weierstrass theorem, a bounded sequence of real number has a converging subsequences, convergent subsequence, has a convergent subsequence. So, by Bolzano-Weierstrass theorem this convergent subsequence…

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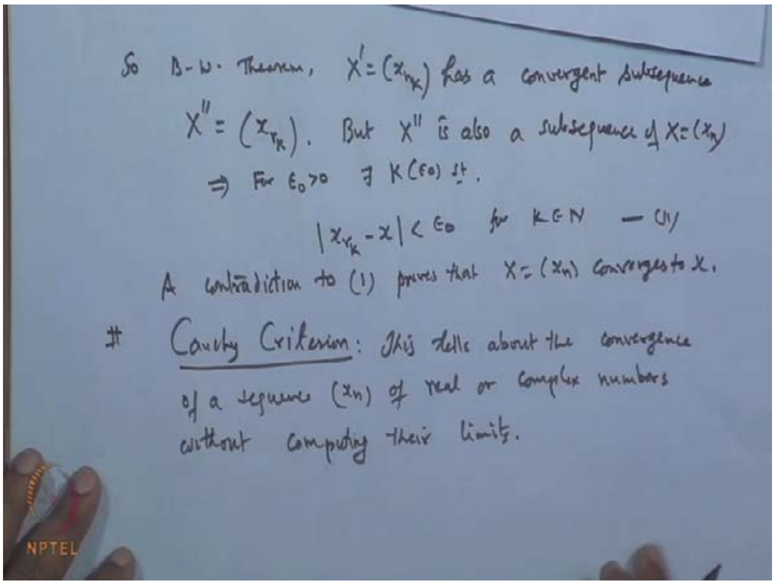

So, by Bolzano Weierstrass theorem, Bolzano Weierstrass theorem, we can say that, X dash has a convergent, has a convergent, convergent subsequence X double dash, X double dash, say, x r k, convergent subsequence X double dash. But this X double dash, is also a subset; the elements of this, are the elements of X. But X double dash is also a subsequence of X and this sequence has a subsequence which is convergent, is it not?

So, this subsequence converges. Therefore, what we get? So, by this…So, this implies that, for a given epsilon naught greater than 0, there exists an integer, capital say K, depends on epsilon naught, such that, mod of x r k minus x is less than epsilon naught, is it not? But for all k belongs to capital N, let it be 2. Now, first and 2 are contradictory. This first is, x n k minus x k is greater than epsilon for all n; now, this n k covers r k; because r k is one of the n k s, because if this is r, this x r k is a subsequence of x n k.

So, these are the points belongs to X dash. So, they are also satisfy the condition 1, but these, they also satisfy this; they are also satisfying 2, so a contradiction to 1. So, a but… So, a contradiction, contradiction to 1, proves that…So, why it is contradiction, because our assumption is wrong that, x n is not convergent, proves that, sequence x n is convergent; x n converges to X; this is what we have to…So, this proof. Now, let us come to Cauchy convergence criteria, criteria. Now, this Cauchy convergence criteria tells, this is, this tells about the convergence of a sequence x n of real or complex numbers, without computing their limit, their limit, limits. Because what happens is, if the sequence x n is given, then one can easily identify by taking the limit. If I consider the limit of the sequence, and if the limit comes out to a finite quantity, limit exists, comes out to a finite quantity, then we say, the sequence is convergent.

But if the limit goes to infinity, or does not exist, means, along different subsequence, different limits, then we say, the sequence diverges. But what the Cauchy convergence criterion says that, you need not to compute the limits of a sequence x n. Just simply apply that criteria, which is given by Cauchy, one can identify, whether the given sequence is a convergent on, or divergent, or divergent one. That is a very advantage of this Cauchy convergence criterion. Because the previous criteria which we have seen, whether it is a monotonic convergence theorem, or may be a Bolzano-Weierstrass theorem, all these theorem depends on certain particular cases. Say, monotone convergence, unless it is, sequence is monotone and bounded, you cannot say it is a convergent sequence. Bolzano-Weierstrass theorem, a bounded sequence has a convergent subsequence, which does not say about the sequence itself, whether the sequence is convergent or not. So, this Cauchy convergence criteria is very interesting and important, and it directly relates to the convergence part of the sequence. So, we will go in detail in the next class for this.

Thank you very much.