A Basic Course in Real Analysis
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Lecture - 18
Fundamental theorems on limits,
Bolzano Weierstrass Theorem

So, in the last lecture we have discuss the few theorems like cauchy's theorem of first kind and Cauchy's theorem of the second kind, where we have developed some results and which will help in getting the limit of the sequence here.

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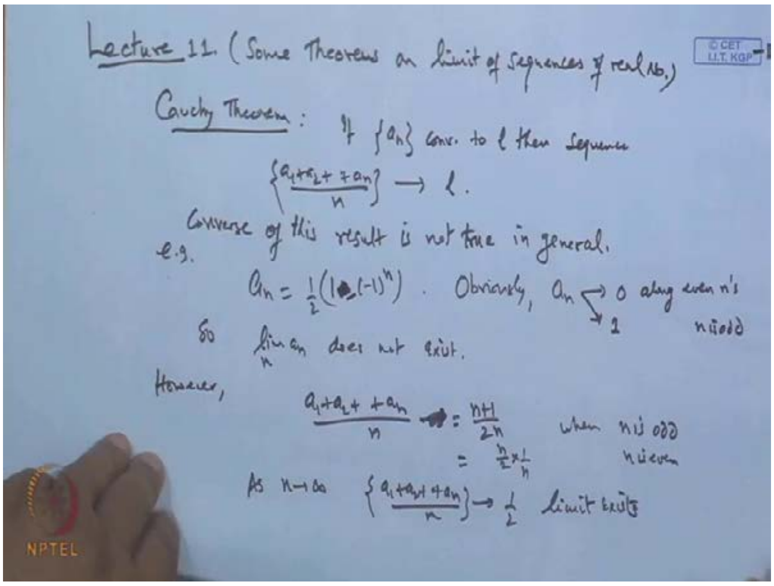

And remember that Cauchy's theorem cauchy theorems that is, if the sequence a n if the sequence a n of real numbers converges to l, then the sequence a 1 plus a 2 plus a n by n, this sequence will also converge to the same point l. And, we have seen the proof for it that is a sequence convergence to l will imply the convergence of this sequence of their arithmetic means, the converse of this is not true. Converse of this result is not true in general that is a sequence a n for which this a 1 plus a 2 plus a n by n will go to l, but sequence a n may not go to l. For example, if we consider the sequence a n as 1 plus minus 1 to the power n divide by two, take this sequence; obviously, the sequence a n tends to when n is, let us take this minus 1 sorry. So, let us take this as minus sign or minus take the minus sign 1 minus minus 1 to the power n. So, if the sequence if we choose the sequence a n as 1 minus minus 1 to the power n then as n is even then this term becomes positive, and this minus so a n will go to 0 along even n's and n tends to infinity.

Then along the even direction and when n is even and tends to infinity then it will go to 0, while when n is odd then this becomes minus plus two and this will go to one, so limit of this sequence does not exists. So, limit of a n as n tends to infinity does not exist; we do not have a single limit here. However, if we consider a one plus a two plus a n divided by n, then this comes out to be say when n is odd, so when you take n is odd then a one becomes, this is plus one, so a three becomes again one, because a three is this minus minus plus one so, one plus one this will go and total values when n is odd are n plus one.

So, it will go to the n plus one divided by two and then divided by n, so it will go to tends to this as when n is odd. When n is odd you are getting this value, and when n is even then it will be n divided by two into one by n. So, in both the cases as n tends to infinity this sequence a one plus a two plus a n divided by n will go to half, because here it is clearly half and when you divide by n then you are getting one plus one by n by two which n tends to infinity it will go to half, so limit exists. So, here in cauchy theorem one side is true, that if n converges to l then the sequence of their mean values a one plus a two plus a n sequence of the means will go to the same limit l.

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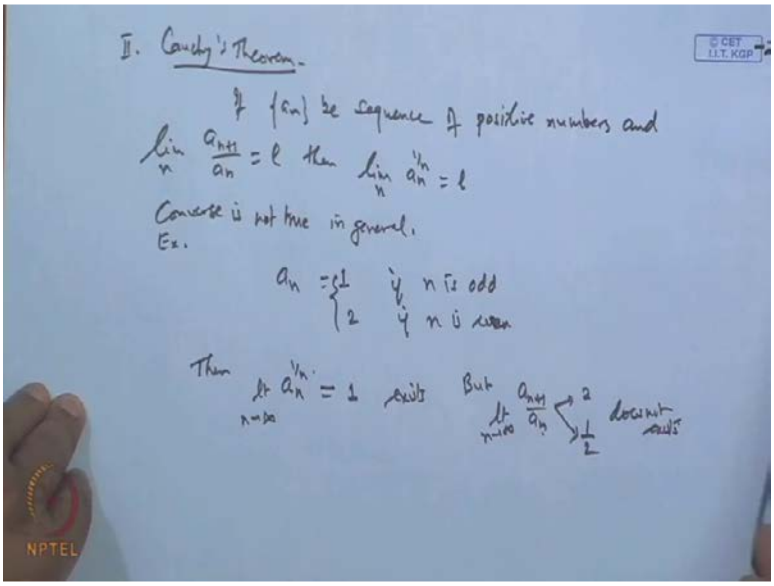

Similarly, when we go for the second result of the Cauchy, the second result of the cauchy say that if a one a two a n this is sequence, if n be the sequence of positive number where a n ‘s be a sequence of positive numbers and limit of and limit of a n plus one by a n as n tends to infinity exist and equal to l, then limit of the a n to the power one by n when n is sufficiently large will also be l. The converse is again converse is not true in general for example, suppose I take the sequence a n’s, a n’s equal to one if n is odd and equal to two if n is even n is even, then a n to the power one by n this limit is as n tends to infinity; basically a constant to the power one by n, and hence this is large it will go to one, so limit will go to one; exist. But the limit of this a n plus one by a n, if I picked up these thing then what happen is, when n is odd the value this will be even, so this will go to two by one, and if n is here even then it is odd, so this is even, so it will go to half. So, basically the limit will go to as n tends to infinity; it is the ratio, limit does not exist because its keeps on jumping does not exist, and this shows the converse is not true. So, here these are two examples which are which we have discussed, let us now, we will give few more results which are useful for that.

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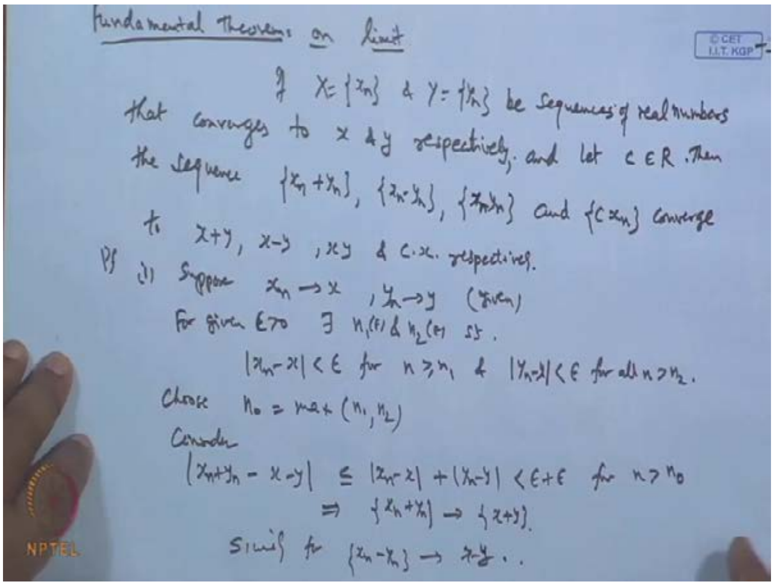

So, let see the first fundamental theorems on limit on limit limit of the sequence of real numbers. So what this theorem says, if x n and y n with the sequences of real numbers. if X which is a sequence of x n and Y which is sequence of y n, if x n, y n be the sequences of real numbers real numbers that converges to the real point x and y respectively, and let C be any arbitary constant R, C belongs to R; real number, then the sequence x n plus y n, x n minus y n, x n into y n, and c x c x n and c of x n will converge or converge to x plus y, x minus y, x into y, and c into x respectively.

So, let us see the proof; proof is very simple, we have already discussed the proof where considering the sequence of rational numbers and in terms of the cant[or' s] cantor's and dedekind's case. So, let us see one of the proof of this, suppose x n and y n are convergent given suppose x n converges to x, y n converges to y this is given, so for a given epsilon greater than 0 there exist an n one and n two which depends on epsilon such that mod of x n minus x is less than epsilon for all n greater than equal to n one and as well as mod of y n minus y is less than epsilon for all n greater than n two, that is it. Now choose n, choose. say n not which is the maximum of n one and n two, so there is n one and n two, so consider mod of x n plus y n minus x minus y, now this is less than equal to x n minus x plus y n minus y. Now, when n is sufficiently large, greater than n one then this part will be less than epsilon, when n is sufficiently large from n 2 this part is less than epsilon. So, when you choose n to be greater than n not then this is less than epsilon plus epsilon for all n greater than n not. Therefore, the sequence x n plus y n will converge to x plus y, and that proves the things. Now if we take other parts say second, similarly x minus similarly for x n minus y n this sequence will go to x minus y, proof will be the same.

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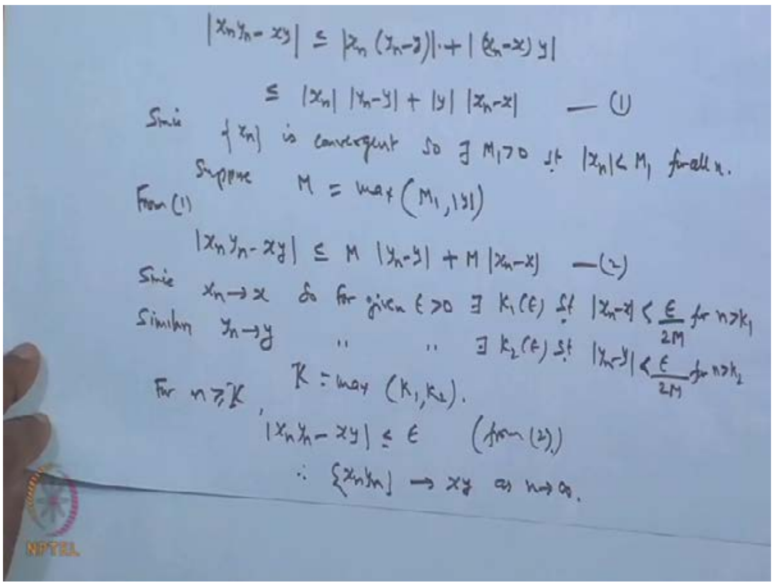

Then for x n into y n let us take this sequence, let x n converges to x and y n converges to y x n converges to x and y n converges to y., so we wanted to show that this is less than. So, consider x n y n minus x y; mod of this, now this will be less than equal to x n y n minus y; mod of this plus mod of x n minus x into y; just by adding and subtracting and applying the triangular inequality, which is less than equal to mod x n mod y n minus y plus mod y plus mod x n minus x; let it be one. Now, since our x 1 since the sequence x n is convergent sequence it is bounded, so there exist M one greater than 0 such that all the terms of the sequence is less than equal M one for all n, because it is a convergent sequence So, it must be bounded for all n.

Let us suppose capital M is the maximum of M one and mod y, so we can choose these thing like this. Now, further this part from one we get mod of x n y n minus x y is less than equal to this is less than M into mod y n minus y plus M into mod x n minus x. Now, y n and x n both are convergent sequence, so for given epsilon, since x n converges to x, so for given epsilon greater than say 0 there exists some k one depends on epsilon such that mod of x n minus x can be made as small as we please, so suppose I take choose the smaller number as epsilon by two M. Similarly y n converges to y, so for given epsilon there exists k two which also depends on epsilon such that y n minus y is less than epsilon by two M. Now, once you have this then you choose the k as the maximum of k one and k two, let us take k to be k one. then if x 1. So, for n greater than equal to k, for this number what happen use the form two, we get mod x n y n minus xy is less than equal to, now this is less than equal to epsilon by two M.

So, basically this is epsilon by 2M, this is less than epsilon by two M, but this is true for n there exists k one such that this is true for all n for all n greater than k one, and this is true for all n greater than k two. So, if I choose the k as a maximum of k one and k two then both the results are true for n greater than k. So, when choose n greater than k then this is less than epsilon by two M this is also less than. So, this is less than epsilon from two. So, this shows that x n y n converges to this, so this implies x n y n, this sequence goes to x y as n tends as n tends to infinity. So, this will be then similarly, when you take the third say c of x n in a similar way you can do otherwise, by n you can consider as the constant c. So, y n a sequence c c c c and it will converge c x. Now, if the sequence, another result is if x n and y n in this results let it be a; this is the fundamental a.

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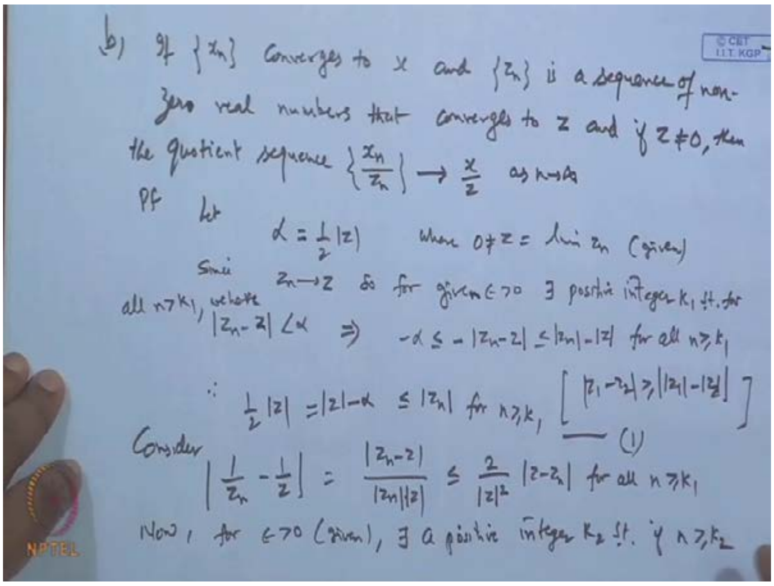

So let, next result is b, if sequence x n converges to x if sequence x n converges to x and z n is a sequence of non 0 and z n is a sequence of non 0 real numbers that converges to z, and if z is not equal to 0 and if z is not then the quotient sequence x n over z n will go to x over z as n tends to infinity; that are a limit of the sequence say one to n. Again the proof can be followed with the epsilon delta definition and the proof done is as follows, let us suppose z n is a sequence of number which is given non 0, let us suppose alpha is half of mod z where z is the limit of z n which is given non 0; this is given. So let us pick up alpha like this, now since z n converges to z, so for a given epsilon greater than 0 there exists a natural number or integer there exists a natural number positive integer K one such that mod of z n minus z mod of z n minus z is less than alpha. Now, this implies that minus alpha is less than equal to minus z n minus z which is less than equal to mod z n minus mod z for all n greater than equal to k one why, because this is true such that for all n greater than k one we have this thing.

Now, if I take minus sign then minus alpha is less than this, fine, apply the triangular in equality, so what you get is mod of z minus, because z one minus z two mod of this is greater than equal to mod z one minus mod z two, this result is true, so using this result we get this part, half of this z. Therefore, what will be the mod z minus alpha, bring it here this is greater than equal to mod z and less than equal to mod z n for n greater than equal to k one, but z minus alpha; alpha is mod z by two, so it is basically half of mod z. So, when it is a sufficiently large the sequence of the term mod z n is greater than equal to half of mod z. Now use this, say this is one, now consider mod of one by z n minus one by z, now this can be written as mod z n minus z over mod z n mod z, but mod z n is greater than this number, so it is less than equal to two over mod z square into mod z minus z n, and this true for all n greater than equal to k one. Now, as z minus z n goes to 0, because z n sequence converges to z, so for a given epsilon now for given epsilon greater than 0 we can find there exists a natural number or positive integer k two such that if n is greater than equal to k 2.

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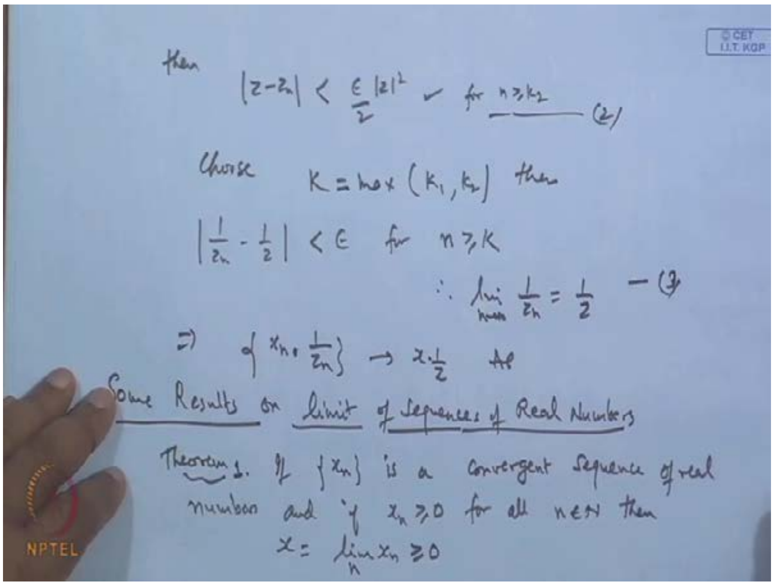

Then mod z minus z n, this sequence is less than the number epsilon mod z square by two, this is a smaller number, since z n converges to z, so we can identify this thing for all n greater than k 1. Now, this is true for all n greater than k 1, this is true for all n, sorry all n greater than k two, this is true for all n greater than k 1. So, if I pick this is 2. So, let it be two, so if we want to combine one and two then, what happens is that this k, if I take K a integer which is maximum of k one and k two then this difference can be made less than epsilon for all n greater than equal to K, because both these are true and this shows the sequence one by z n goes to therefore, limit of the sequence one by z n as n tends to infinity is one by z, let it be three.

Now we are interested in x n by z n, so it is as good as x n into one by z n. So, the sequence x n into one by z n will go to x into one by z, that is the answer because of the product of the two sequences, this product of two sequence will go to this. Then next result is now, some results on limits of sequences of real numbers, few more results on this. The first result which we called it as a theorem the theorem one says if x n is a convergent sequence is a convergent sequence of real numbers, of real numbers and if all the terms of the sequence x n are positive; non negative greater than equal to 0, for all n belongs to capital N, if all the terms of the sequence are non negative then the limit cannot be negative, then the limit of this x which is the limit of x n over n will also be greater than or equal to 0, it cannot be a negative limit, the limit of the sequence of non negative term will always be a non negative.

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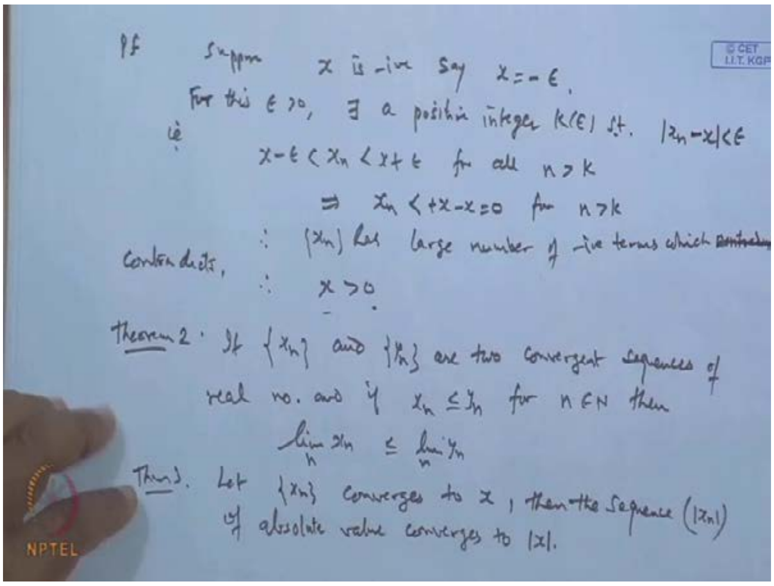

The proof of this let us prove by contradiction, suppose the limit is negative and then apply the condition for this, so let us suppose suppose x is negative, say x is equal to minus epsilon then for this epsilon for this epsilon which is greater than 0 there exists for this epsilon there exists a positive integer say k depends on epsilon such that mod x n, minus x n because this sequence is a convergent sequence, so basically x n will lie between the two term, x n minus x remain less than epsilon that is x n will lie between these two terms less than x plus epsilon and greater than x minus epsilon for all n greater than k is it not. Now, epsilon is given is already chosen minus k minus x, so from here this shows that x n is less than minus x plus x minus x that is 0, and this is true for all n greater than k, it means the large number of the terms x n are negative.

This shows the sequence X n has large number of terms has large number of negative terms which contradiction which is not which contradicts contradicts why contradicts because x n already are positive therefore, our assumption is wrong so x must be positive that is it. Then another results two says, if x n and y n are two convergent sequences are two convergent sequences of real numbers of real numbers and if x n is less than equal to y n for all n belongs to N; natural number, then the limit will also follow the same in equality, limit of this is also less than equal to limit of y n. And similarly, for the theorem which we have shown earlier that if x n and y n are the two sequences and z n is a sequence which lies between x n y n and if the limit of x n y n converges to the same limit then z n will also converge, so that is the theorem which also we have shown earlier. Now, next result is divergence sequence, we have discussed now theorem third. let the sequence x n converges to x then the sequence mod of x n, this sequence of absolute values absolute values converges to mod x, that is if the limit of sequence is x then limit of the mod x n will be mod of x.

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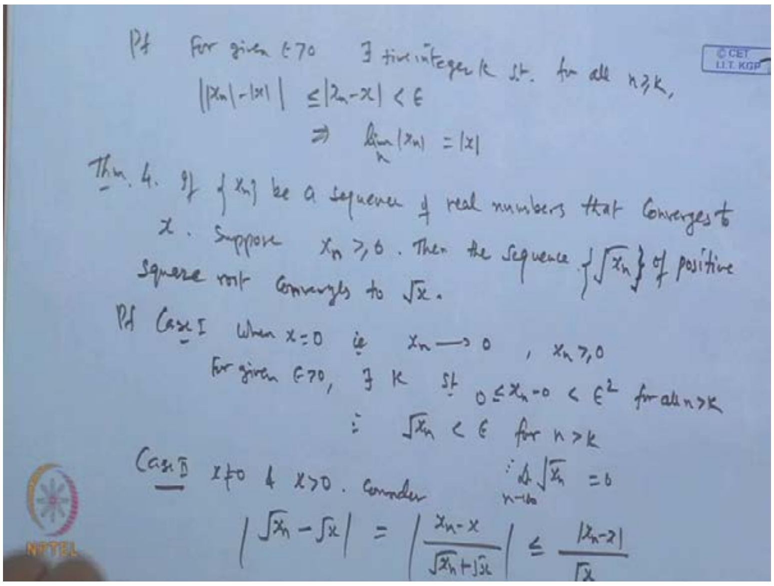

The proof is very simple, this follows from this in equality, x n minus x is greater than equal to mod x n minus mod x; triangular inequality, we know this result. So, if x n converges to x means for a given epsilon greater than 0 there exist a positive integer positive integer k such that for all n greater than equal to k we have this part is less than epsilon. So, this part less than epsilon means this shows the limit of mod x n over n is mod x, so that is what we show. Then, if x n be if x n be a sequence of real numbers a sequence of real numbers that converges that converges to say x, and suppose that x n is greater than equal to 0 x n is greater than equal to 0 then the sequence under root of x n, this sequence of positive square root of positive square root converges to under root x. the proof is, since x n be a sequence of real number that converges to, and we are assuming x n to be greater than 0, so limit point cannot be negative, so we are considering only two case when this x is equal to 0, and x is greater than 0.

So, case one when x is equal to 0. So when x is equal to 0, x n sequence converging to 0 that is x n is tending to 0, where x n are all positive greater than or equal to 0, they are 0, so for a given epsilon. So, by epsilon for given epsilon greater than 0 there exists a positive integer k such that x n which is greater than equal to 0, will less than x n minus 0 is less than say epsilon square for all n greater than for all n greater than k. Therefore, under root of x n will less than epsilon for all n greater than k, so the limit of the sequence x n will. So, limit the of x n when n tends to under root of this will go to 0. Similarly, if we take x to be non 0 then in that case non 0 and greater than 0 because if greater than 0 then we consider under root x n minus under root x, so rationalize it, multiply and divide by under root x n plus, here we get x n minus x over root of x n plus root x.

Now, this is given under root of x n, and both are positive, so this will be this will be less than equal to, so if I take mod of this mod of this then, this is less than equal to mod x n minus x and so this part under root x n plus x is greater than 0. So, this will be greater than equal to root x, so it will be less than equal to root x; this is positive, this is positive, so this will be greater than root x, so one upon root x will be less than. Therefore, the limit of this is same as the limit of this under root x, but this limit is tending to 0, so this implies that limit will go to the. So, this implies limit of under root x n over n is under root x. So, nothing much, then we go for, monotonic sequences we have discussed already, now yes subsequences that part is left.

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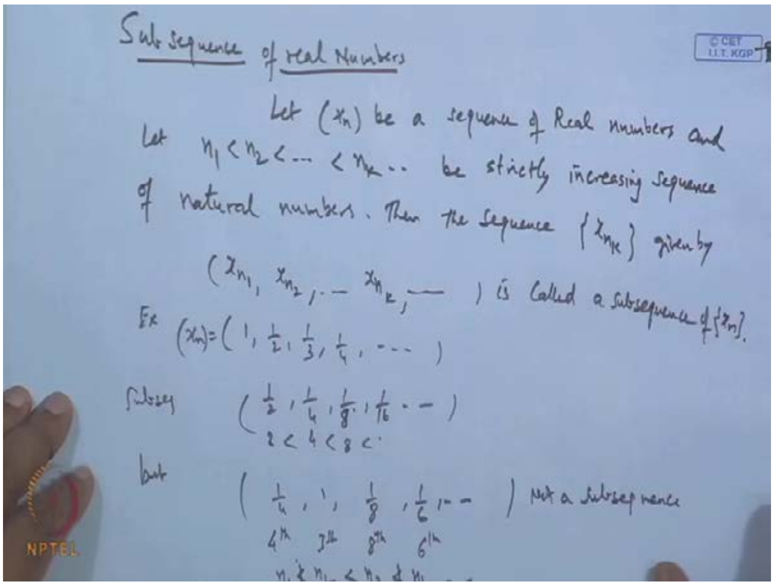

So, let us see concept of the sub sequences. Sub sequence of a real number of real numbers sequences. So, let x n be a sequence of be a sequence of real numbers real numbers and let n one less than n two and so on less than n k be strictly strictly increasing sequence is strictly increasing sequence of natural numbers of natural numbers then the sequence then the sequence x n k then this sequence x n k given by x n one, x n two, x n k and so on given by this is called a sub sequence of x n. sub sequence of x n

For example, if we take the sequence x n is one, one by two, one by three, one by four and so on this is a sequence x n. Now, if we take the sequence half, one by four, one by eight, one by sixteen and so on, then this is a sub sequence because this is the second point of this. So, this is the second term, this is the fourth term, this is the eighth term and like this, so we are getting n one, n two, n one less than n two and less than n three and so on. But if we take a sequence like this, but if we take a sequence like this say one by four, one then let it be one by eight, then one by six and so on, then this is not a sequence not a sub sequence, why why its not a subsequence it is a sequence, but its not a sub sequence why because the order is not n, because here n one this is the fourth term, this is the first term, this is the first term, this is the eighth term this is the sixth term. So, there is no such order like this n one less than n two less than, n one is not less than n two, n two is less than n three, but n three is not less than n four like this, so it does not satisfy the criteria of the sub sequence like this, you check it . So, that will be now this is very interesting result, here the result say.

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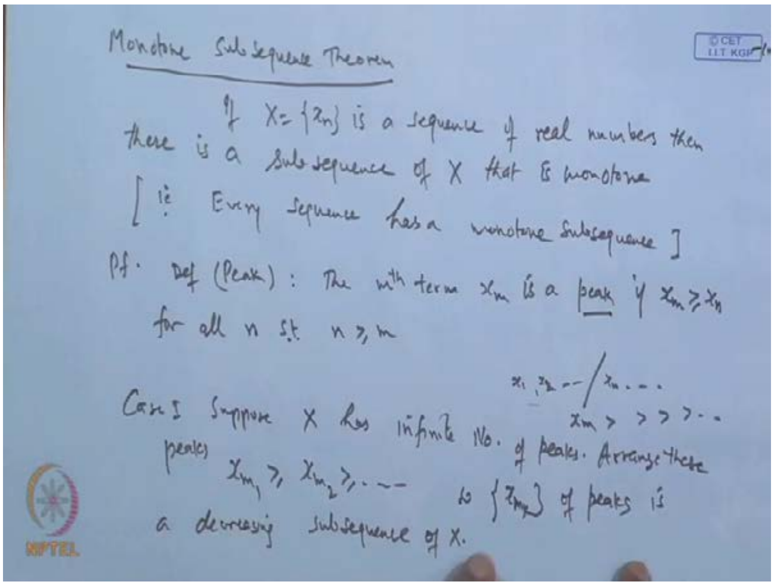

Monotone sub sequence theorem monotone subsequence theorem, the theorem says if x equal to x n is a sequence of real numbers is a sequence of real numbers then there is a subsequence there is a subsequence of x that is monotone.

So, this is a very interesting result that every sequence has a monotone subsequence every sequence that is the result is very that every sequence every sequence has an monotone subsequence sub sequence. So, the proof, let us suppose we have, first we define the term say x one, x two, x n, lets define this term which we call it as a peak, peak the mth term x m is a peak the m' th term m' th the term x m in the sequence x n is a peak is a peak if x n x m is greater than equal to x n for all n for all n such that n is greater than equal to m. Means, like this if suppose we are having a sequence x one, x two, x n and so on, and if we identify some term x m here such that all after this all the terms of the sequence are of decreasing nature and above it is bounded by x m, this term x one, x two, x n may behave (( )); may be x one is greater than x two, x two may be less than x three and so on, we do not care, but if we find out some m such that x m is greater than after a certain stage, all the terms of the sequence, then we say the peak mx m is the peak of the sequence x n .

So, what we do here is that every sequence using this concept we will prove that every sequence will have a convergent will have a monotone subsequence. So, let us see the two cases when the sequence has an infinite number of peaks, and the case when the sequence has only a finite number of peaks. So, suppose sequence x has infinite number of peaks, infinite number of peaks let them arrange these peaks, arrange these peaks arrange these peaks suppose the peak x m one where then x m one then x m 1 is greater than equal to x m two greater than and so on, because m one, m two, m three these are the terms, m one is less than m two less than m three, but the peak x m one is the peak after this all the terms of the sequence are less than n then x m two is [vocalized noise] another term such that after this all the term are less than this and like this, so arrange this thing. So, this sequence of will have a decreasing. So, the sequence of the peaks this sequence of peaks peaks is a decreasing subsequence is a decreasing subsequence decreasing subsequence of x.

So, in case of infinite we identify the points, first peak which is such that after this all the terms of the sequence are less than this identify x one, then after certain stage again you will find out some term for which the condition that x m is greater than x n satisfy for all n greater than m and like this, so arrange these three you will get a monotonic decreasing sequence for that.

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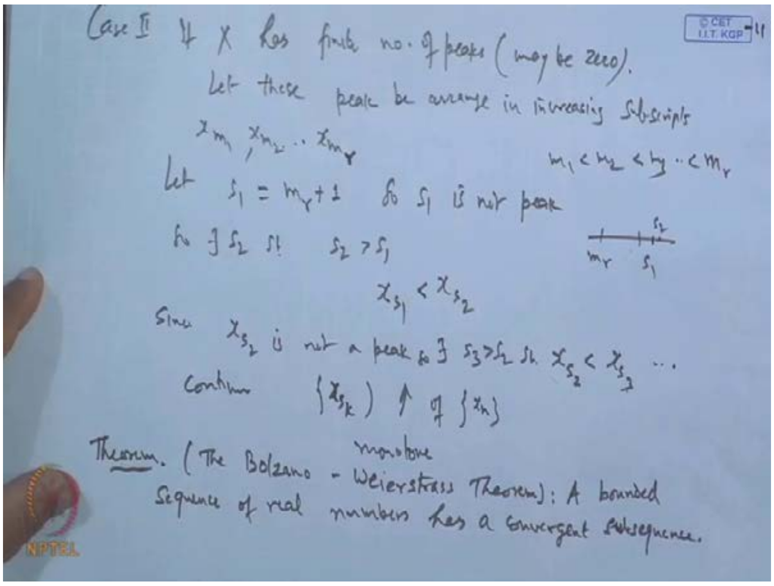

Now, if x has a finite number of if x has a a finite number finite number of peaks may be 0 also may be 0 no peak, this is also possible, so if the finite number of 0; let these peaks be in increasing, these will be listed by increasing order. So, let these peaks be arranged in increasing order, let us suppose in increasing subscripts that is if m one is less than m two, then x m one, x m two these are the peaks x m say and arrange in the increasing subscript that is m one is less than m two less than m three and this is m r, arrange in this.

Now, let us picked up this term let pick up the term let s one is the term which is after this peak m r in this first index beyond this, then this s one cannot be a peak because this is the peak and this is the last peak, so we cannot take any number which is greater than this can be a peak, so s one is not a peak, means x is a finite then s one (( )). Once s one is not a peak it means there will be some point here is m r, so s one is this point something, now s one is not a peak it means there may be a some number s two which is greater than s one then only s two we can identify, so there exists an s two such that s two is greater than s one because s one is not a peak. So, once it is greater than then the corresponding sequence x s one; obviously, this peak is less than x s two, this term will be less than this. Again, since x two is not a peak is not a peak. So, we can there exists. So, there exists another term s three greater than s two such that the term of the sequence x s 2 is less than x s 3, because this is not a peak, a peak term will be there which is greater than this and continue this. So, once you continue then we get a sequence x s k an increasing sequence of x which is monotone. So, this shows that every sequence will have a monotone subsequence; it has a subsequence of x that is monotone.

Following these two we have an important result which is known as the Bolzano Weierstrass theorem. The Bolzano Weierstrass theorem say that what this is a bounded sequence a bounded sequence a bounded sequence of real numbers has a convergent subsequence. So, that is very important result because it shows that every bounded sequence of real it may not be a convergence like minus 1 to the power is not a convergent, but it has a bounded convergent subsequence, at least one bounded convergent subsequence may be there.

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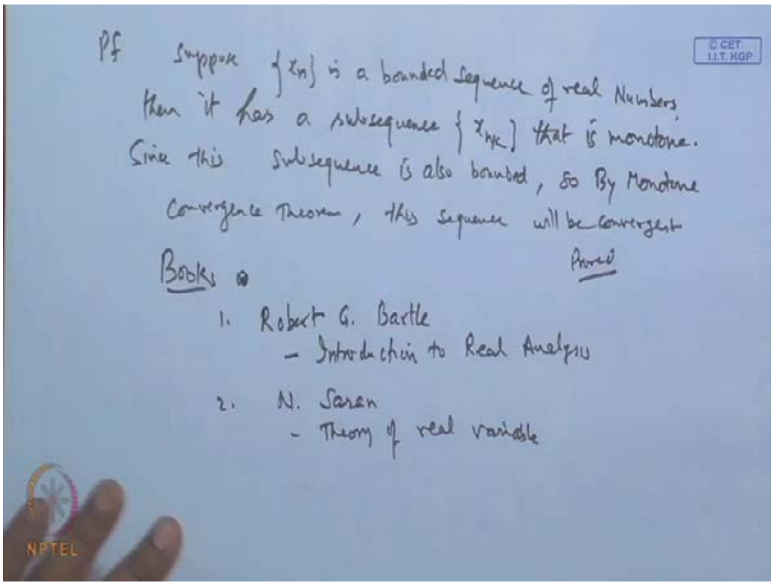

The proof is based on the previous result. Suppose, x n is a bounded sequence is a bounded sequence of real numbers of bounded sequence of real numbers then it has subsequence x n that is monotone. Now, since it is a bounded sequence of real number, so it will have then it has suppose bounded sequence of real number then it has a subsequence because every sequence has subsequence had a subsequence x n k that is monotone by the previous result that is monotone by the previous result. Now, since subsequence is also bounded since this subsequence this subsequence is also bounded,it follows from the monoton[e]. So, so every now, one this is a subsequence which is monotone and it is bounded, so by monotone convergence theorem monotone convergence theorem says that if a sequence which is monotone or and bounded up and below must be convergent, so by monotone convergence theorem this sequence will be convergent. So, this proves that a bounded sequence of the number has a convergent subsequence that proves the result.

Now, this completes your first module that is cantor's Dedekind's and theory of sequences. Now here the books which I have followed, books which are used reference the first book which I have taken as an introductory by Robert G Bartle, introductory introduction to real analysis, second book which I followed is by n saran that is theory of real variables, I think that is the real variables or something, I will give the name exactly theory of real variables or theory of real functions like that. So, mainly these two books I followed for this section. Thank you very much, thanks.