A Basic Course in Real Analysis
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Lecture - 17
Cauchy theorems on limit of sequences with examples

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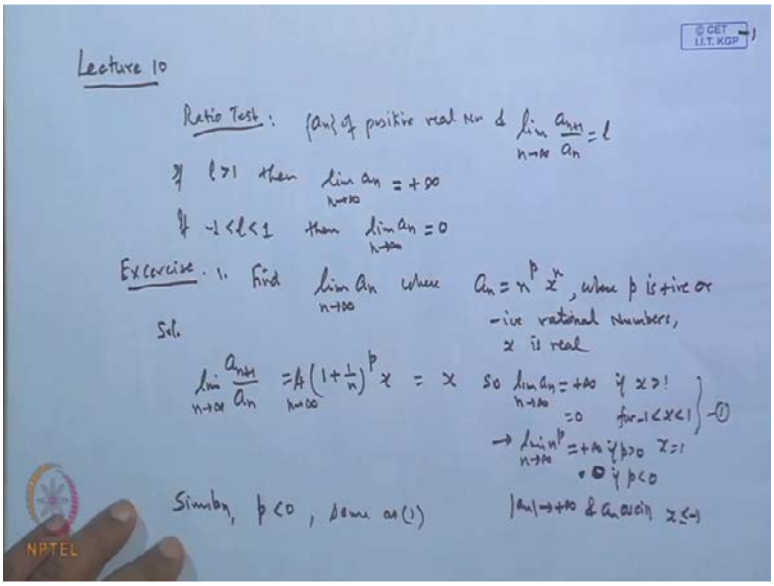

So, today we will discuss the few problems based on the ratio test; using the ratio test we can solve those problems. And also we will discuss the few of the results, which are also needed for further study. So, let us see, we have gone through the ratio test. The ratio test says that, if a n be a sequence of positive numbers, if a n's are sequence of the positive numbers, of positive real numbers, and if limit of this a n plus 1 over a n, as n tends to infinity, is say l. And when if l is greater than 1, then limit of the sequence a n as n tends to infinity will be diverging, and diverges to infinity. And the second part of it shows that, if limit of this l lies between minus 1 and plus 1, then the limit of a n as n tends to infinity is 0. And particular case is, when a n s are, when l is less than 1, then in that case, we are negative, then we will see that, all negative terms, then we can apply, take the minus sign and get the results (( )).

So, let us see the exercise based on this. Suppose, it is given that, we have the sequence, one is, find the, find limit of a n, as n tends to infinity, where a n s are, is of the form n to the power p, x to the power n, where p is positive or negative rational number, where p is p is a rational number, positive, or negative rational number, rational numbers, and x is a real quantity, x is real, x is real.

Now, if we look this, apply the formula. So, what is a n plus 1 over a n, if we look this, it comes out to be, 1 plus 1 by n raised to the power p x. Now, when n tends to infinity, p is fixed. So, this term will go to 1 and basically, the limit of this, when n tends to infinity, is nothing but x. So, when x is greater than 1, so according to the ratio test, if the limit of n plus 1 or n is l, where l is greater than 1, the limit of n will be plus infinity; when l lies between minus 1 to plus 1, the limit will be 0. So, limit of this a n over n is infinity, if x is greater than 1 and 0, for x lying between minus 1 and plus 1. And when x is equal to 1, then it reduces to the form n to the power p.

And the behavior of n to the power p is already there; if p is positive, it will go to plus infinity; when p is negative, it will go to 0. So, when x is 0, it is equivalent to the limit n to the power p, as n tends to infinity, which will be plus infinity, if p is positive and minus infinity, and 0, and 0, if p is negative, and for p is equal to 1 also, it will be 0. So, this is… And when x is less than equal to 1, if x is less than or equal to minus 1, then what happens, this sequence mod x n less than equal to 1, the sequence is minus 1 to the power n, minus 1 to the power n and then it will keep on oscillatory, and oscillating; so the mod of a n will go to plus infinity and o n s, a n s are, is oscillatory infinite, oscillatory infinite.

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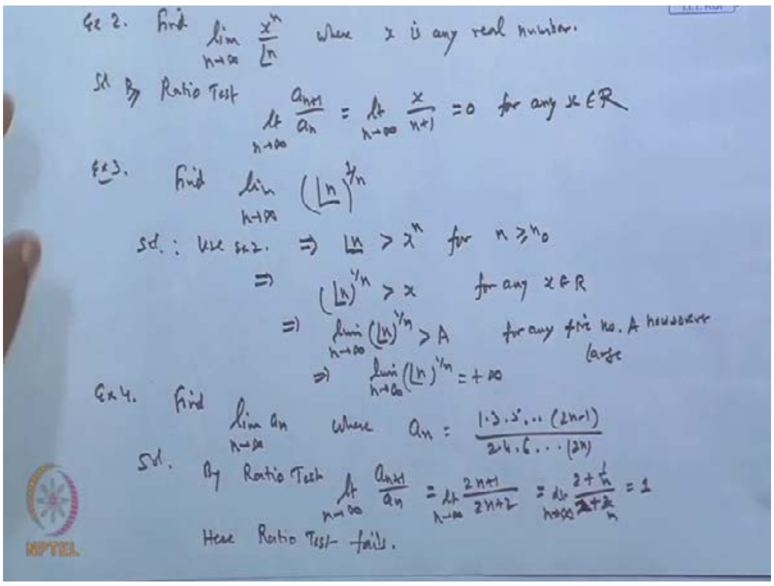

Similarly, when p is negative, similarly, when p is negative, then in that case also, the same thing happens. It is, again, this will be, x greater than 1; it is infinity; x lying between minus 1 is 0 and for other, it… So, the same results as 1; same as 1; that is one, so this will… Second exercise, let us see. So, exercise 2, suppose, find the limit of, limit of x to the power n, factorial n, as tends to infinity, where x is any real number, any real number. Now, again, apply the ratio test. So, by ratio test, what happens to this? a n plus 1 over a n, as n tends to infinity is, comes out to be what? Limit of this, as n tends to infinity, x over n plus 1. So, whatever the x may be, this limit will go to 0, for any x, real. It means, this limit will always be 0, for all x.

Third, find the limit of this, limit of factorial n to the power 1 by n, as n tends to infinity; factorial n 1 by n. Now, to show this limit, let us use the previous one. We know, the limit of x to the power factorial n is tending to 0, is 0 (( )). So, use the, use exercise 2. So, from exercise 2, it implies that, factorial n goes much faster than comparative to x to the power n; that is, this is always be greater than n, for n sufficiently large, say, n naught; because then only it, because it is tending to 0; it means that denominator is going much faster to infinity, comparative to x to the power n.

Therefore, it will, factorial n will be greater than x, after a certain stage; say, n is greater than n naught. Therefore, x factorial n to the power 1 by n, will be greater than x and this is true for any x, for any x belongs to R; it means, the limit of this thing, limit of this factorial n to the power 1 by n, as n tends to infinity, will exceed to any number a, will exceed to any number a, for any, any positive number a, howsoever large, howsoever large; and this shows, this is only possible if the limit of this is infinity, limit of this will be infinity. So, it will be followed by this, ok.

Now, this can exercise 4. Find the limit of, find limit of a n, when n tends to infinity, where a n is 1, 3, 5, then 2 n minus 1, over 2, 4, 6, up to 2n, suppose this is our n. Now, these are all positive terms. So, apply the ratio test. By ratio test, a n plus 1 over a n, this comes out to be what? 2 n plus 1 over 2 n plus 2, n plus 1 is, n is, n plus 1, 2 n plus 1, and 2 n plus 2. So, the limit of this, as n tends to infinity, is the limit of this, as n tends to infinity; divide by n. So, when you divide by n, it comes out to be 2 plus 1 by n over 2 plus 2 by n; and limit as n tends to infinity; and, that will come out to be 1. It means the limit of this a n, as n tends to 1 is convergent and it goes to, limit is 2 n plus 1 by 2 n. This limit, what is this, this is 1, sorry, is 1.

Now, what the ratio test says, the ratio test, if you look the ratio test, the ratio test says, when a n is a sequence of positive numbers such that limit a n plus 1 over a n is l, when l is greater than 1, limit will be plus infinity; when l lying between minus 1 to 1, limit is 0; but it does not say anything about l is 1. When l is 1, the series, the sequence a n may converge, may not converge; because it depends on the type of the sequence; we cannot say the limit is 0, limit is infinity or limit is finite. We have to compute that. So, here, the ratio test fails. Here, ratio test fails. We are unable to get it. So, what to do? In that case, we have to apply our previous knowledge; that is, we know, if a sequence, which is a monotonically decreasing, or monotonically increasing sequence, and if it is bounded, monotonic increasing bounded above, monotonic decreasing bounded below, then such a sequence, will definitely have a limit.

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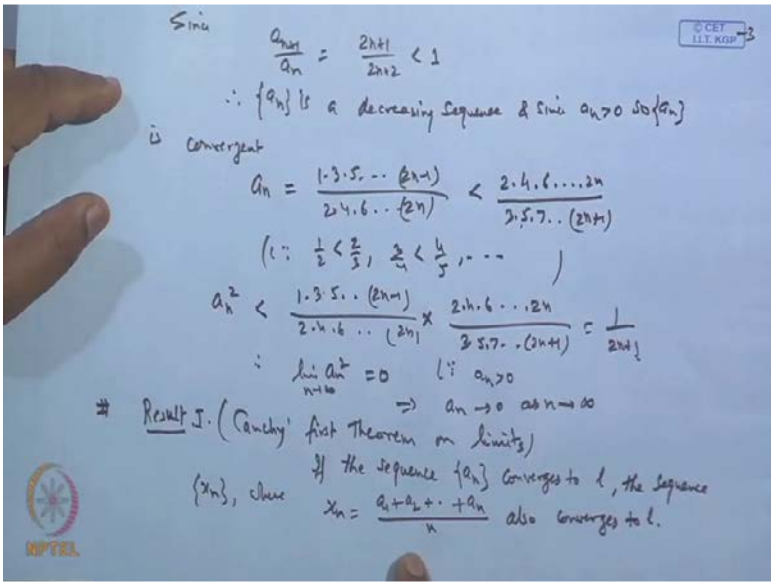

So, let us find out, whether this sequence is a monotonically sequence or not. Now, here, if we look to a n plus 1 over n, this comes out to be, 2 n plus 1 over 2 n plus 2. It means that, this is, total thing, this total is less than 1. Since, since the ratio a n plus 1 over a n is coming to be 2 n plus 1 over 2 n plus 2, denominator is higher than the numerator. So, ratio is strictly less than 1. So, it means, the, any term is larger than a n plus 1. So, when n 1, 2, 3, etcetera, it keeps on decreasing. Therefore, the sequence a n is a monotonically decreasing sequence, is a decreasing sequence; and, since a n s are all positive, and since a n s are positive, so it decreases and at the most, it will go to positive number. So, and…So, sequence a n is convergent; this is one thing which is clear from this concept. We are interested in finding the limit now.

So, it is, it is, that convergence part is clear, that, the sequence has to converge; but what will be the limit? That we will see. So, let us see the a n s again. What is a n? a n was 1, 3, 5, up to 2 n minus 1, over 2, 4, 6, up to 2 n. Now, can you not say, it is less than, if I write 2, 3, is 4, 5, 6 up to 2 n; and, this part, I am just increasing 3, 5, 7, 2 n plus 1. So, this term, because 1 is less than 2, or 1 by 2 is less than 2 by 3, you can say like that. So, 1 by or 2 is less than... So, 1 by 2 greater than 3, 1 by, 1 by 3 is greater than 1 by 2 and like this. So, this will be 1 by 2 is less than 2 by 3; 3 by 4 is less than 4 by 5; 5 by 6 less than 7, like this.

So, we can say, this is less than, because half is less than 2 by 3; 3 by 4 is strictly less than the 4 by 5 and continue. So, we can say this. So, a n s is less than this. a n s is this. So, a n square will be less than, less than the product of this two. 1, 3, 5, 2 n minus 1, 2, 4, 6, 2 n into product of this, 2, 4, 6, 2 n, divide by 3, 5, 7, 2 n plus 1 and that comes out to be 1 by, over 2 n plus 1. So, n square is coming to be 1 by 2 n. Therefore, the limit of a n square is tending to 0, because a n s are positive. So, limit cannot go negative; it can go at the most 0 and a n square is less than this, as n is sufficient large, the limit is tending to 0. So, it means the, a n limit must go to 0 as n tends to infinity. So, that will be the answer for this. So, this is what. Now, there are others also, but before going few, let us take one result, which is given by the Cauchy.

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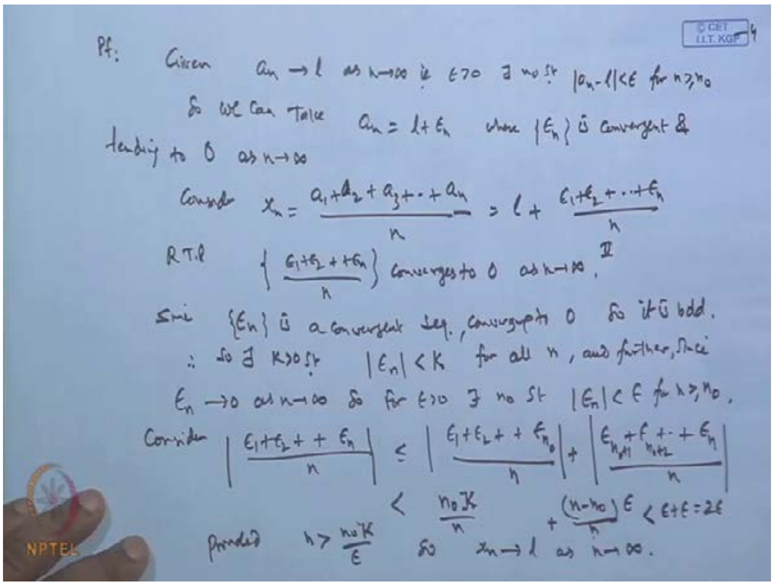

And, so before going this, let us see the result which is known as the Cauchy's first theorem on limit, on limits. So, what this theorem says, if, if the sequence a n, if the sequence a n converges to l, then the sequence, the sequence x n, where x n is a 1 plus a 2 plus a n by n, also converges to l, also converges to l. Let this be result 1; can be put it in the exercise also, but because this is an extended result; so we can say, put it as a result 1. So, what he says is, if suppose, a n sequence are convergent, then their mean arithmetic mean, basically, it is a arithmetic mean, mean of this, will also converge to l; that is the first result.

So, let us see the proof of this result, or solution of this. What is given is, a n converges to l, as n tends to infinity. It means, the difference between a n and l, that is, for a given epsilon greater than 0, there exists an n naught such that, the mod of a n minus l is less than epsilon, for n sufficiently large. So, what we can do, a, we can say that… So, we can assume or we can take, a n s as l plus epsilon n, where epsilon n is convergent; this you can, is convergent and tending to 0,0, as n tends to infinity. So, when n tends to infinity, the sequence epsilon goes to 0. It means a n minus l can be made as small as we please after a certain stage. So, let us suppose…

Now, consider x n. So, consider x n. x n is a 1 plus a 2 plus a 3, a n divided by n. So, substitute a 1 in terms of epsilon 1. So, we what we get is, if I substitute a 1 is l plus epsilon 1, a 2 is l plus epsilon 2, as…So, this n times l divide by n. So, l plus epsilon 1, epsilon 2, epsilon n divided by n; we get this. Now, we wanted the sequence x n goes to l. It means, if I prove that, this second term, this second term goes to 0, as n tends to infinity, then it will...

So, required to prove is, the sequence epsilon 1, epsilon 2, plus epsilon n divide by n, this sequence, basically converges to 0, as n tends to infinity; this here. Once we prove, then x n will go to l. Now, since epsilon n is a convergent sequence, converging to 0, converging to 0, so it is a bounded sequence. So, it is bounded sequence because every convergent sequence is bounded. So, there exists a k, such that, mod of epsilon n is less than or equal to k, for all n; for all n, there exists a positive k. And further, epsilon n tends to 0; so we get… And further, since epsilon n tends to 0, as n tends to infinity, so for a given epsilon greater than 0, there exists an n naught, such that, mod of epsilon can be made less than epsilon, for n greater than equal to, say, n naught. Now, consider, now, this one. Consider, mod of epsilon 1, epsilon 2, epsilon n, divided by n.

Now, this will be less than equal to mod epsilon 1, epsilon 2, plus epsilon n naught, divided by n, plus epsilon n naught plus 1, epsilon n naught plus 2, up to epsilon n, divided by n. I have chosen this. Now, when n is greater than n naught, the mod of epsilon is less than…It means, each of this term is less than epsilon. So, total becomes n minus n naught. So, this will be less than n minus n naught times, n minus n naught times and then this is what, this is equal to, less than epsilon divided by n, plus. Now, this term, each term is bounded by k. So, this is bounded by k. It means, this is less than n naught times k, divided by n, is it not? Now, n naught k, if I choose, n naught is fixed, k is fixed. So, if I take epsilon such that, that, n naught k by n is less than, say, epsilon, then what happens?

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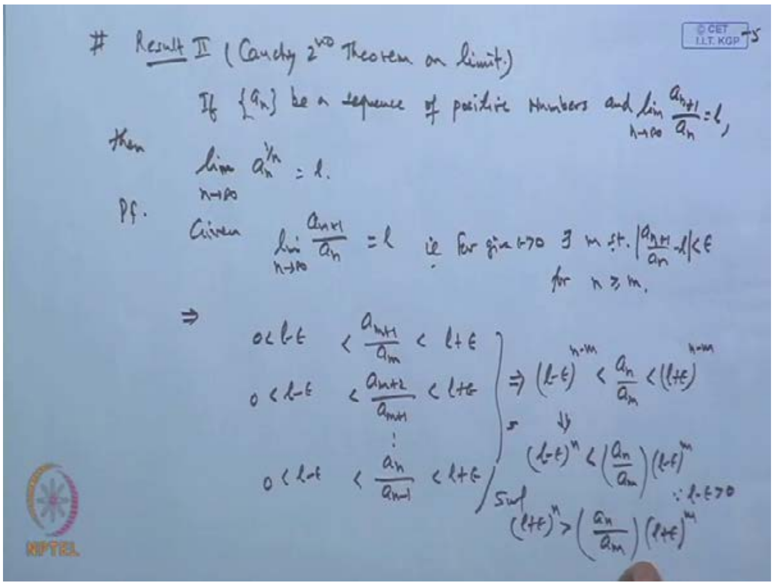

So, is, this is less than epsilon plus epsilon, provided n is greater than, n is greater than n naught k by epsilon. If I take n to be greater than this, then this part is less than epsilon, this is less than 1. So, this is already less than epsilon. So, total is less than 2 epsilon. So, this part, when n is sufficiently large, goes to 0; then once it goes to 0, then this sequence x n will go to l. So, x n will go to l, as n tends to infinity, is it okay? That is what. This is the first Cauchy theorem on the limits.

The second Cauchy theorem, that is, the result 2, which is also, we call it as a Cauchy's, Cauchy's second theorem on limit, on limits. What this theorem says is, if, if sequence a n be a sequence of positive number, positive numbers, sequence of positive numbers, and limit, and limit of a n, a n plus 1 divided by a n, as n tends to infinity is l, and limit of this is, say l, then limit of a n to the power 1 by n, as n tends to infinity, will also be l. So, that (( )) means, if the ratio of this sequence is l, then nth root of this n will also have the limit l. So, let us see the proof of this. What is given is that, this ratio is l; given, a n plus 1 over n, limit of this, as n tends to infinity is l; it means, that is, for given epsilon greater than 0, there exists, say, m, such that, mod of a n minus a n plus 1 over a n minus l is less than epsilon, for all n greater than equal to m. So, what you, it means, this implies that, a m plus 1 by a m, this term lies between l minus epsilon and l plus epsilon.

Now, if we continue this, a m plus 2 by a m plus 1 lies between l plus epsilon, l minus epsilon, and like this, upt o, say, any term which is a n over a n minus 1 lies between l minus epsilon, l plus epsilon; just continue. Now, find the product; because these are all positive quantity, remember; because a n s is a sequence of positive, these are all positive, greater than 0, greater than 0, greater than 0. So, this is also positive. So, once they are all positive, we can multiply, without getting the change in the inequality.

So, when you multiply these things, what you get? These are total n minus m terms. So, here, we get l minus m. So, this implies that, l minus epsilon to the power n minus m is less than, if you multiply this… So, the cross, this scopically getting cancelled, so a n over a m is left only. So, a n over a m, which is less than l plus epsilon to the power n minus m, where n is greater than m. Now, divide by l minus epsilon to the power minus n. So, if I divide, because this is positive quantity… So, again, when you divide, so this implies that, which implies that, l minus epsilon to the power n is less, l minus a is less than, is less than… If I divide by this, then what you are getting, a n over a m, a n over a m into l minus epsilon to the power m; because this will, because this is positive, because l minus epsilon is positive. So, we can do like that. Now, take the power 1 by n. So, what we get from here is… And similarly, when you take l plus epsilon to the power n, here also, we get, this is greater than a n over a m, a n over a m into l plus epsilon to the power m.

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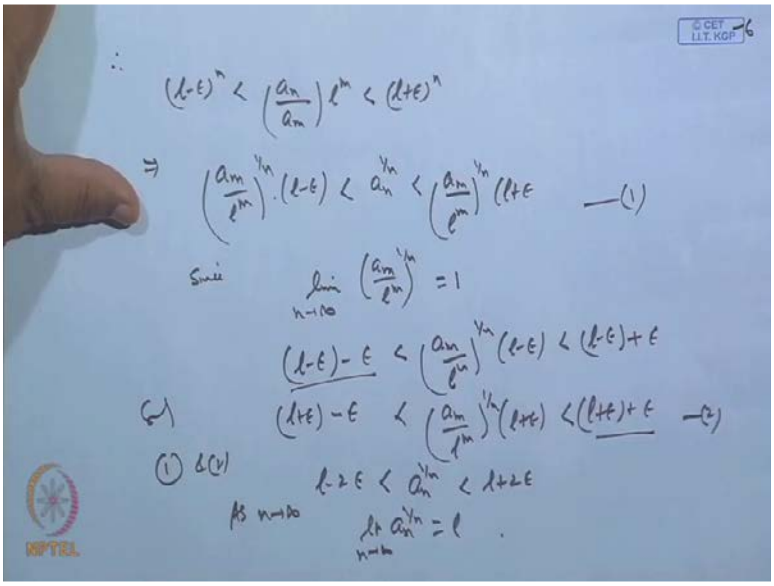

Now, this part, l plus m is greater than 1; sorry, this is a n by a m… So, now, from here, again, so what we get basically, so we get… Therefore, we get l minus epsilon to the power n is less than a n over a m, a n over a m, l m is less than, is less than l plus epsilon to the power n; just like, is it not? So, that is not…Now, take this term, here. So, we get l minus epsilon a n a m over l m, raised to the power 1 by n into l minus epsilon, into l minus epsilon, is less than a n to the power 1 by n, which is less than a n a m over l to the power a m, raised to the power 1 by n l plus epsilon. Let it be 1, clear? Now, since limit of this a m over l to the power m by 1 by n, as n tends to infinity, is 1. Why, because this is fixed; a m, sorry, a m fixed point; l m is also fixed. So, this is some constant power 1 by n.

So, when n is sufficiently large, the limit will go to 1. It means, this term will lie between 1 minus epsilon and 1 plus epsilon. So, we can say that, this limit, entire thing, l minus epsilon, this part can be…So, we can say, this limit when so what we get is… Therefore, this entire thing lies between, yes. So, therefore, we get l minus epsilon minus epsilon is less than, less than a m by l m, a m by l m raised to the power 1 by n, l minus epsilon, l minus epsilon, which is less than l minus epsilon plus epsilon.

Why? This a minus 1 by l, because it is, this entire thing, this entire thing lies between l minus epsilon and then minus 1. So, this will be, this one tending to 1, basically. So, this limit will go to l minus epsilon only, basically, l minus epsilon. So, this total thing will lie l minus epsilon minus 1 and l minus 1 epsilon. Similarly, this term will also lie between there. Similarly, we can say, l plus epsilon, l plus epsilon minus epsilon, less than a m over l m raised to the power 1 by n l plus epsilon, which is less than l plus epsilon plus epsilon. So, what we conclude is that, if I look this entire thing a n to the power 1 by n, a n to the power, the lower bound will be l minus 2 epsilon; this will be the lower bound; and for a n by 1 by n, upper bound will be this. So, this 1 and 2, if you combine, then we get from here is, l minus 2 epsilon is less than a n to the power 1 by n, (( )) l plus 2 epsilon.

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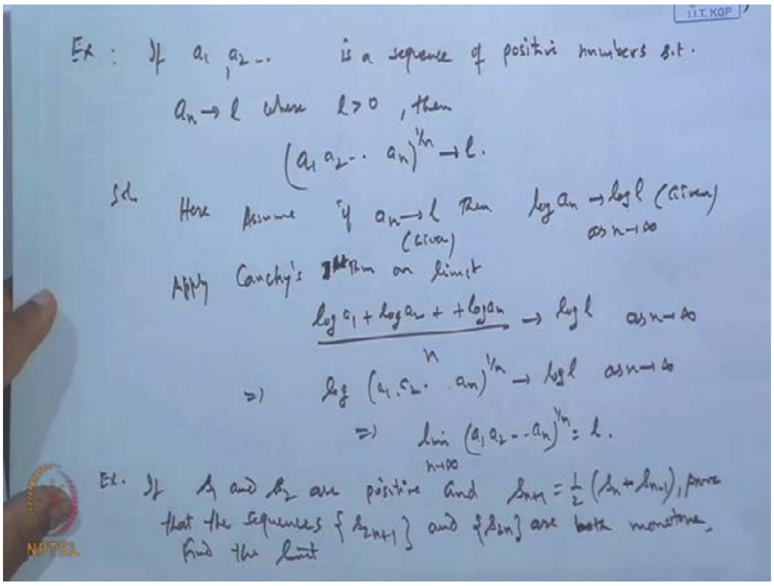

So, as n tends to infinity, the limit of this a n to the power 1 by n, limit of this, is nothing but l, and that is proves the result. Now, here, when it is, l is, when l is greater than, sufficiently large, this limit will go to infinity, also, infinity, in fact, ok. So, that is… Now, as a corollary to this results, as an exercise, we can say, if a 1, a 2, a 1, a 2, a n, etcetera, is a sequence of positive number, positive number, number such that, such that, a n goes to l, a n goes to l, where l is greater than 0, then a 1, a 2, a n raised to the power 1 by n, goes to l.

So, let us see the solution of this. a 1, a 2, a n is a sequence of positive number such that, limit of a n is l, where l is positive; then the nth root of this product, that is, geometric, basically, geometric mean of this, will also tends to l. The solution based on the previous Cauchy's theorem, first theorem on the limit, that is, if a n goes to l, then the mean value of a 1 plus a 2 plus a n, will also go to l. So, here, we assume that, if a n converges to l, then log of n will go to log l; this is our assumption. Here, assume that, if a n goes to l, then log of a n will go to log l.

In fact, this we will show it, that, log is a continuous function, when n is positive, of course, well defined. Then continuous function has a property that, it transfers the convergence sequence to the convergence sequence. So, log, because of, it is a continuous function, so it will transfer the convergence sequence to convergence. But here, since we have not gone to the continuous function so far. So, let us assume here that, when a n goes to l, then logarithm of this will also go to l. Now, so it is given. So, this is now given; because a n is given. So, we can assume that, log n goes to l. Apply the Cauchy's first theorem on limit. So, what we get, their mean, that is log of geometric mean of this, second, sorry, this is the second theorem, second theorem. So, apply the mean of this, first theorem, we will get the first theorem. So, log of a 1, plus log of a 2, plus log of a n, a n divided by n, will also go to log l by that theorem.

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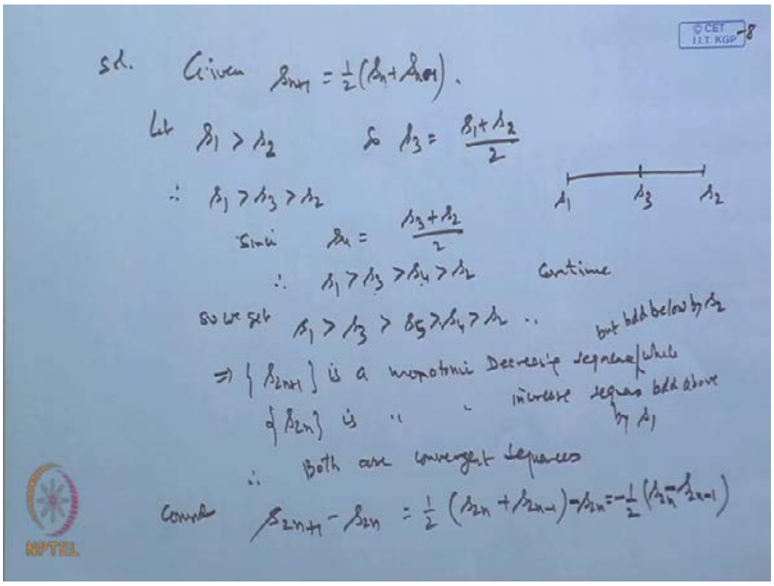

But this product is nothing but the log of a 1, a 2, a n and power 1 by n will go to log l by property of log. So, this shows that, as n tends to infinity, as n tends to infinity...So, this implies, limit of this a 1, a 2, a n, raised to the power 1 by n, as n tends to infinity will be l and that proves the result for…Now, let us see the other exercise. If s 1 and s 2 are positive, are positive, are positive and s n plus 1, s 1, s 2 are positive and s n plus 1 is half of s n plus s n minus 1, half of s n minus 1, then prove that, prove that the sequence, prove that the sequences s 2 n plus 1 and sequence s 2 n are both monotone, are both monotone; the one increases and other decreases, are both monotone and find the limit of this; find the limit, limit.

So, let us see the solution for it. What is given is that, given s n plus 1 is half of s n, s n plus 1, s n minus 1, sorry, s n minus 1; this is given. So, let us assume, s 1 and s 2 are 2 numbers, where s 1 is greater than s 2. Let us assume this. So, s 3, from here, is s 1 plus s 2, no, s 3. So, n is equal to 2. So, s 2 plus s 1 by 2, this will be the s 3. It means, between s 1 and s 2, here is s 1; this is s 2; here, we are getting s 3, which is the average of this; the mean of s 1 and s 2. So, we get, s 1 greater than, s 3 greater than s 2, similarly. Now, s 4, s 4 is nothing but what?

When n is 3, we get s 3, s 2 by 2. So, s 4 is lying between what? Because this s 3 and s 2, in between s 3 and s 2. So, we get, there is, s 1 greater than, s 3 greater than, s 4 greater than s 2, and like this, continue this. So, what we get? So, we get a sequence, who, which have this s 1 is greater than s 3, greater than s 3, greater than s 4, greater than s 2, greater than s 5, greater than s 4, greater than s 2; like this, say 5 terms. What will happen, 5 is 4 plus 3, s 4 plus s 3 by 2. So, s 4, s 3, in between s 3 and s 4, it will lie, and like this. So, what we get, that, when a suffix are odd, s 1, s 3, s 5, it is decreasing; when suffix are even, it is increasing; s 2, s 4, s 6 and so on.

So, basically, this implies, the sequence of the odd numbers s 2 n plus 1 is a monotonic decreasing sequence, decreasing sequence, while the sequence 2 n is a monotonic increasing sequence; s 2 n is a monotonic; s 2 n, that is the same, s 2 n. So, this is n, s 3, s 5; it is a, yes, decreasing, and s 2 n is monotonic increasing sequence, increasing sequence. So, we get, this is a monotonic decreasing and monotonic increasing sequence. So, converge. Now, when it is a monotonic decreasing sequence, but it is bounded below by s 2, but bounded below by what, bounded below by s 2, because s 1, s 2 are fixed.

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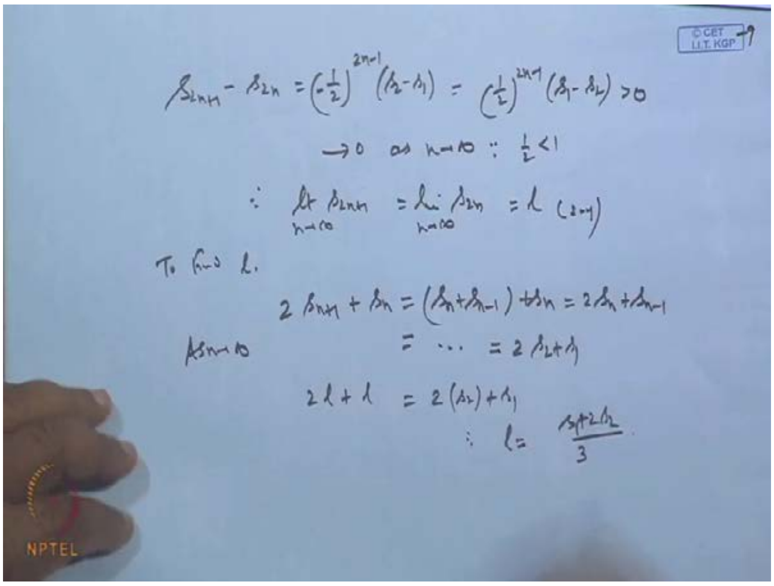

So, it is convergent; s 2 n is again, because s 2 n is increasing, but it is bounded above by s 1; bounded above by s 1. So, this shows that, both the sequences are convergent. So, both are convergent sequence; convergent sequences. Then whether they will converge to the same limit or not, let us see. So, if we consider this difference, s 2 n plus 1 minus s 2 n, consider this difference. So, what we are getting is, half s 2 n plus s 2 n minus 1; because s 2 n, apply the formula, minus s 2 n. So, that comes out to be minus half s 2 n minus s 2 n minus 1. It means, when you take s 2 n plus 1 minus s 2 n, it is half of the s 2 n minus s 2. So, continue this process, if you continue this, then what you are getting? You are getting s 2 n plus 1 minus s 2 n, basically comes out to be what, minus half raised to the power, that is, minus half raised to the power 2 n minus 1, 2 n minus 1, s 2 minus s 1, s 2 n minus 1. Now, this will be equal to s 1, because s 1 is greater than s 2. So, we can take half, 2 to the power n minus 1, s 1 minus s 2. Now, this is positive, but this is fixed; here, this will go to 0.

So, it tends to 0, as n tends to infinity, because half is less than 1 and therefore, both the sequence, limit of this s 2 n plus 1, as n tends to infinity is the same as limit of s 2 n as n. So, both converges to the same limits, say l. Now, to find out the limit of this, so we get to find l. We consider 2 s n plus 1, s n, twice s n plus 1 plus s n. This will be equal to what? s n plus s n minus 1 plus s n, because s n plus 1 is s n plus s n minus 1 by 2; so two of this. So, this will be equal to what, 2 s n plus s n minus 1. Now, as n tends to infinity, as n tends to infinity, this s n plus 1... Now, again, when you take 2 s n plus 1 is 2 s n is 2 s n; similarly, if I continue this process, what we get? This is also equal to…In the final term, we get 2 s 2 plus s 1, because this is… So, this is equal to basically this. So, as n tends to infinity, this will be our 2 l. This is l, equal to 2 s 2 plus s 1. Therefore, l becomes s 1 plus 2 s 2 by 3, and that is the answer for it. So, is this clear? So, we get… Now, I will write few problems that will be used as tutorials. So, one can try those problems. So, tutorials and then you can find the solution, which is very easy.

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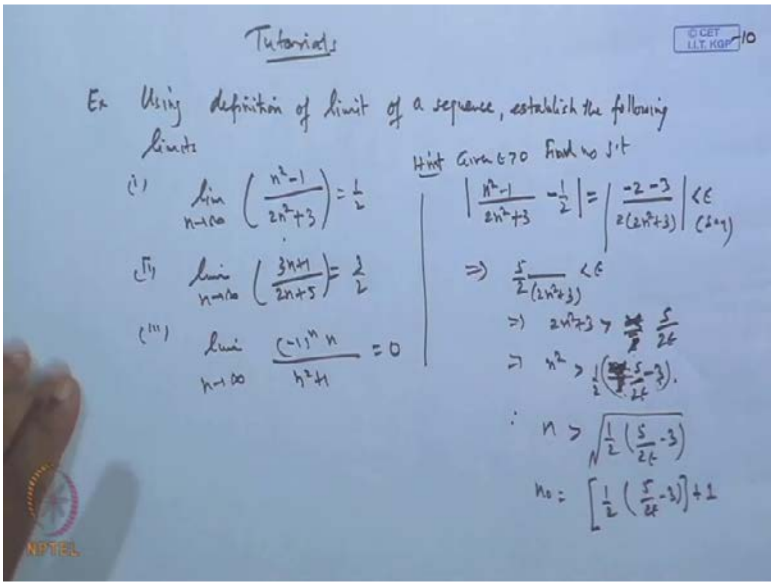

Say, first example, we have to do this part, using the epsilon delta definition, using the definition, using definition of limit of a sequence, limit of a sequence, sequence, to establish the following, the using the, establish the following limits, following limits. Limit of this, as n tends to infinity, is n square minus 1 over 2 n square plus 3 equal to half, n square plus 1. Second, b is, second is, limit, as n tends to infinity, 3 n plus 1 over 2 n plus 5 is 3 by 2; like this. Limit, third, limit of this, minus 1 to the power n into n divided by n square plus 1 is 0, as n tends to infinity. Then limit of this that is all. Now, if we look that, without epsilon definition, we can just find the limit very quickly. So, n is sufficiently large. So, when we substitute n infinity, infinity by infinity is in determinant.

So, what we do is, we divide by n square and get the result; 1 minus 1 by n square 2 plus 3 n square. So, when n is sufficiently large, it becomes 1 by 2; but how to get that result? Say, hint, suppose, I wanted to for (( )); so for any epsilon greater than 0, for given epsilon greater than 0, find n naught, such that, mod of this part, n square minus 1, 2 n square plus 3, minus half should be made less than epsilon. So, from here, you started; consider, this minus this. Now, this can be written as, when you take the LCM, 2, 2 n square plus 3 and what you get, 2 n square minus 1. So, minus 2 n square is cancelling. So, minus 2, and then minus 3; so this will be.

Now, here we are getting minus 5 by 2. So, what we are getting is, this we wanted to be less than epsilon; say, this we wanted to be less than epsilon. So, what is it means, that is, this will give 5 by 2, 2 n square plus 3 is less than epsilon; that is, 2 n square plus 3 is greater than 2 epsilon over 5. So, what we get, n square is greater than 2 epsilon by 5 minus 3 divided by 2, divided by 2; 2 epsilon over 5 minus 3 divided by 2. So, this will be the n square; take the positive value of this. So, n is sufficiently, take the positive value. So, n square must be this number. So, n must be greater than this, under root of this part. So, if you take n square to be this number, we get the results. So, that is what, mod is there, 2 n square minus 2 and 2 n square minus 3. So, that becomes that, is it okay? So, we can choose because this is, this may, we get, epsilon by 5 and then 2 by 5. So, minus 3, here is some problem.

So, let it be, it is, but what we can do is, we can choose the positive part of it, okay, greater than this; that is, this has to be positive epsilon to be chosen, so that this becomes positive; 2 n square is greater than, it is wrong number 5 by 2, I am sorry; that is what is a mistake. Here, mistake is, when you write it, this becomes 5 by 2 epsilon; yes, that is the mistake; 5 by 2 epsilon; because otherwise, it is a negative. So, it will be problem.

Now, epsilon is very small quantity. So, we can choose epsilon so small, so that, this becomes positive; therefore, n can be greater than half 5 by 2 epsilon minus 3 under root; that is, and take the integral value of this, so n naught becomes integral part of this, plus some number 1. So, when you choose n to be greater than this number, you will get already this part. So, do it with that; I think, thank you very much, but we will do some small problem later.

Thank you.

Thanks.