A Basic Course in Real Analysis
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Lecture - 16
Some important limits, Ratio tests for sequences of Real numbers

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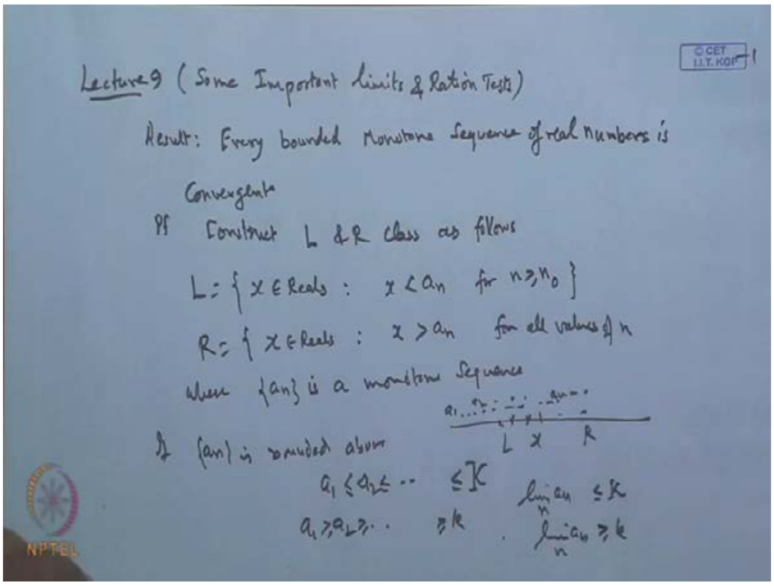

So, today we will discuss of few important limits and the ratio test, but before this let us is with the previous thing, we were discussing about the monotonic sequence. There are two types of monotonic sequence; one is that non-decreasing sequence, other one is the non-increasing sequence. And when these non-increasing or non-decreasing sequences are bounded say increasing sequence or non-decreasing sequence, which are bounded above or the non-increasing sequence which is bounded below. Then such a sequence monotone sequence will always be convergent. And I think the proof is very simple, we can use it with the help of our cuts, it is not cut that is, the result is every bounded monotone sequence, monotone sequence of real numbers, of real numbers is convergent.

So, we will use this result, the proof is just based on this to show the proof of this we will develop the two classes; define or construct L and R classes as follows. L means lower class and R is the upper class. So, the number x is put in the class. So, L is the set of those reals, such that x is less than a n from an after some n for all n greater than equal to say n naught, after certain a stage we are getting; and R is the set of those real numbers reals such that, x is greater than a n for all values of n, we are taking this. So, let a n be a sequence which is a where sequence a n is a monotone sequence, is a monotone sequence, monotone sequence this is bounded above bounded below.

So, when you take this a we are getting this is number a, a ns. So, what a1 a2 a ns, these are the numbers like this. So, set of those real numbers which are less than a n for n greater than n naught; it means these are the points which are less than and so on, and the set of those real number which are greater than a ns will be putted in the upper class lower. So, it will converge from here, it will converge to x lower limit, it will go to the upper a ns will be bounded by x. So, we get the…

For all a n not for all n.

Not for all n, for all a n, it means after certain you say all the terms are x is greater than all a n s. So, those real number are there. So, it means lower class may have a it is some greatest upper least number, but greatest upper bound, but this cannot have, this may not have it the least number like this, So, this where. So, if you take the a n s to be bounded sequence, if sequence a ns bounded above, say monotone sequence which is bounded above; it means a 1 is less than a 2 less than equal to non- decreasing sequence which is bounded by say K for each n. So, limit of a n as n the will be less than equal to K like this; similarly, if it is of this type and each term is greater than equal to K. Then the limit of this a ns as n will be greater than equal to k, is in all the terms will be less; so it will be bounded. So, it means a every bounded monotone sequence of real number is a convergence sequence. So, using this here we will draw few we will discuss some limits which are useful in.

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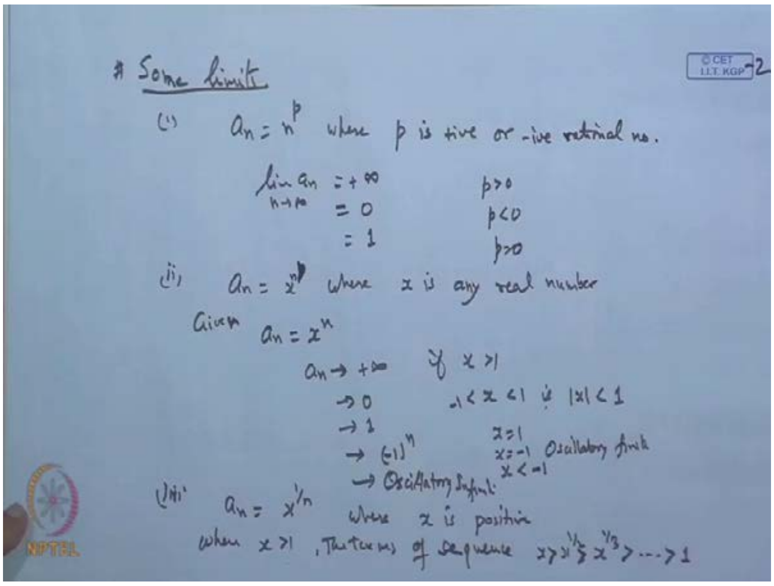

So, let us see the some limits. The first one is very simple is if suppose a ns is the sequence which is n to the power p, a sequence of n number where p is positive or negative rational numbers, rational numbers then limit of this sequence a n as n tends to infinity will be; if p is positive the limit will come out to be plus infinity. Is it not? That is because as n increases it unbounded sequence, and so it is plus infinity; then p is negative then the limit will come out to be 0, it will tends to 0. So, when p is this less than 0, we are getting l and when p is 0 then the limit will come out to be 1. So, is simple there is nothing to prove just. Then another sequence suppose, a ns x to the power p; where x is any real number, real number, x is any real number, x to the power n, a n. So, let it be n, x to the power n, clear? That is a ns x to the power n given.

Now, this a n will go to plus infinity if x is greater than 1, x is greater than 1 each term of the sequence is greater than 1 and in fact, it keeps on increasing. So, increases to infinity; so then and when it x lying between minus 1 and plus 1, minus 1 and plus 1 it means mode of x is less is strictly less than 1, that is mode of x is strictly less than 1 absolute value of this is less than 1. Then in that case this sequence will keep on decreasing and decreases to 0 and when x is equal to 1, the limit is tending to1. What happen to that when x is, strictly less than minus 1, when x is, strictly less than minus 1 or when x is equal to 1 minus 1; when x is equal to minus 1 it is basically of the form minus 1 to the power n. So, this is an oscillatory, oscillatory finite series, oscillatory finite because the limit will it will go to plus 1 minus 1 like this. So, mode of this thing is bounded, but if an x is strictly less than minus 2; then it is, it is a oscillatory infinite in a... Is it not? It will go oscillatory in finite, plus infinity- minus infinity it will go; so we get this. And third limits, if a ns be suppose x to the power 1 by n where x is positive, x is positive; now when x is positive and n is any integer, n is any integer then what happens there when x is greater than 1 each term of the sequence is greater than 1. So, when x is greater than 1 the terms of the sequence, the terms of the sequence, terms of the sequence, sequences is that is 1, x, x to the power half, x to the power one- third and so on. These are satisfying this x is greater than this, x to the power half is greater than this, is greater than this and all the terms are greater than 1; it means it is a monotonic decreasing sequence bounded below. Is it not?

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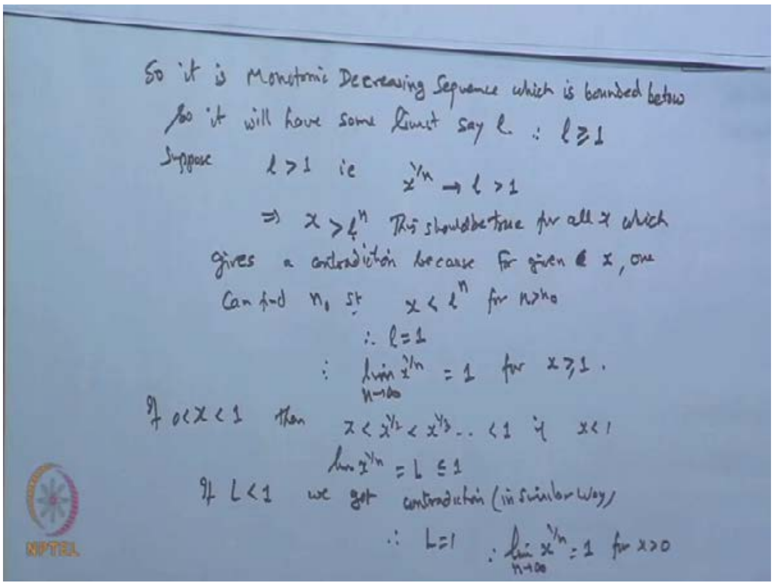

So, it is a, So, it is a monotonic decreasing is strictly decreasing, decreasing sequence which is bounded below, bounded below, monotonic decreasing sequence is bounded below. So, it must have a some limit because every bound is a sequence limit. So, it will have, it will have the limit, it will have some limit say l, and this limit therefore, l will be either greater than or equal to 1, l will be either greater than or equal to 1. Suppose l is, l is, suppose l is strictly greater than 1 suppose we say. Then what happens this that is x to the power1 by n, limit of this x to the power 1 by n this limit as n tends to infinity tends to l which is greater than 1; So, that is only possible when x to the power1 by n should be strictly greater than 1 for all n. Now this shows that, x must be x must be greater than l to the power n. Is it not? Limited, x to the and this is to for all x, but when l is greater than, 1,l is greater than 1 then this is not true for all x this should be, this should be true for all x which gives a contradiction, which gives a contradiction. Why contradiction? Because for given x, for given x, one can find n naught, such that x will be strictly less than l to the power n for n greater than n. Because this keeps on increasing right hand side is keeps on increasing, but the left hand side is fixed once you choose the x left hand side is fixed and what we are saying that, if limit of this is suppose l which is strictly greater than 1 it means of large values of n all when x, what is the behavior of x; x will be exceeding l to the power n. Is it not? It will be greater than l 2 then it will goes on decreasing and decreases a something like, but l is give leads to a contradiction once you fix up the x and since l is greater than 1; so when n is sufficiently large this will go keep on increasing. So, a number n naught can be obtained. So, that x can be made as can be made less than the term l to the power after certain a stage, and this contradiction is because our wrong assumption that l is a greater than 1. Therefore, l must be equal to 1. Is this clear?

So, what we conclude is the limit of this therefore, limit of x to the power1 by n when n is sufficiently large is 1, when x is greater than 1, for x greater than 1 and for x equal to 1 already true for x equal to 1 also it is true; so let is. Now if x is strictly less than 1. Is it not? Then x we are choosing positive remember this is x is a positive. So, we cannot choose the minus or something x is positive greater than 0. So, if 0 less than x less than this then then what happens to this l limit is there, is it not? So, we can get a contradiction here again because x is less than 1, x is less than 1 and we are choosing the limit l because if x is less than 1 then what is the behavior of this sequence? If x is less than 1; the terms of the sequence this sequence will be what it keeps on increasing, this keeps on increasing. Is it not?

So, then x is less than x to the power half, less than x to the power one-third and so on; if x is less than 1, x is less than 1. So, what happen is increasing function of this and then what happens when x is equal to 1, but it is always be less than 1, and less than 1. So, when you take the limit of this, limit of this x raise to the 1 by n is suppose L which is less than or equal to 1, but again contradiction; if l is strictly less than 1 we get a contradiction in a similar way, in similar way, therefore, l must be 1. So, what we conclude is that limit of this therefore, limit of x to the power1 by n when n is tends to1 is always1 for x greater than 0; x negative is not defined. So, that is why we are not choosing because x negative the roots are not reals, so we cannot get. So, this is what.

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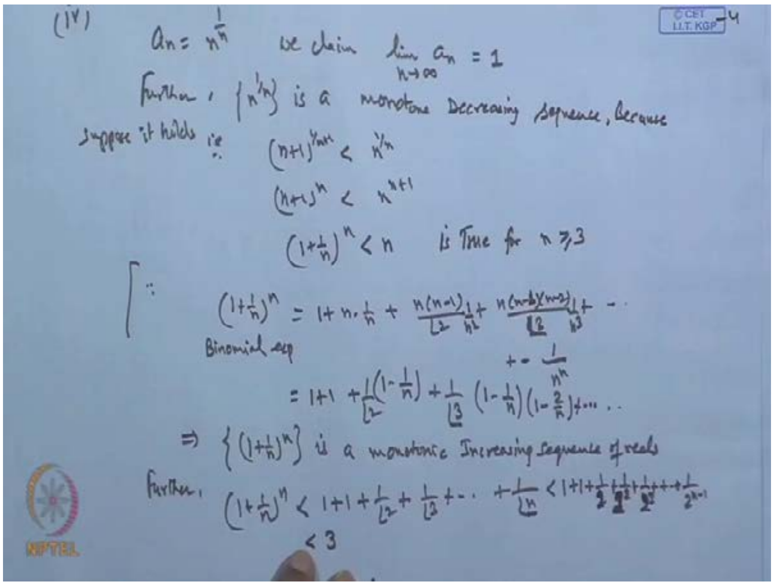

Now if we take this is our fourth problem limits, suppose we take a ns to be n raise to the power1 by n, now we claim that limit of this sequence is, we claim that limit of this a n as n tends to infinity is 1. As n is sufficiently large, this basically tends to like this; so how to justify limit to be 1. So, first what we will show a that this sequence is a monotone sequence, if we prove it is a monotone decreasing or increasing, if increasing bounded above if decreasing then it is bounded below then limit will exist. So, first is be claim this further, the sequence n to the power1 by n is and monotone decreasing sequence, is a monotone, tone decreasing sequence, monotone decreasing sequence. Why? This is our claim because suppose it is true then n plus 1 to the power1 by n, because suppose it is true, suppose it hold then it means that is this is should be less than n to the power 1 by n; as n increases it keep decreases.

So, we claim that this is a decreasing sequence, monotone decreasing, n plus 1, n plus 1 exactly; so we are getting this. So, what we get is n plus 1 to the power n is less than n to the power n plus 1; now this is equal to when you divide by n then we get 1 plus 1 by n to the power n is less than n, is less than n. Now this is true, this is true, for n greater than or equal to 3, as n is sufficiently large for n is equal to 1 and 2 it may it will not be true, but when n is greater than or equal to 3 it holds; the reason is because 1 plus 1 by n to the power n apply the binomial, binomial expansion. So,1 plus n x, plus n n minus 1 over factorial 2 and plus n n minus 1 n minus 2 over factorial 3, like this, so suppose I apply the factorial then what will be this say last term up to n term then we get1 by n to the power n. Clear? Just I 1 plus n x plus n n minus 1 and so on, just binomial; now this will be equal to what?1 plus 1 plus 1 by no1 by n square will also come n n minus 1 of factorial 2 into x square, so 1 by x square this is 1 by n cube and so on. So, n gets cancel we get 1 minus 1 by n by factorial 2, then 1 by factorial 3, 1 minus n minus n n minus 1,1 minus 1 by n,1 minus 2 by n, and like this up to so on.

Now you see this each term is positive, each term is positive. So, the sequence, so the sequence 1 plus 1 by n to the power n is a monotonic, monotonic increasing sequence of reals because when n increases the term increases here. So, this is positive this is positive; you are getting some positive terms here again, so it is a monotonic increasing sequence. Further the this sequence all the for all n this at least greater than 2 because these are positive things, but this sequence further 1 plus 1 by n to the power n is strictly less than 1 plus 1 plus 1 by factorial 2, 1 by factorial3 plus 1 by factorial n, is it not? That will be this is last term n to the, clear?

That will be factorial n; so this one. Now this will be further less than or this is further less than 1 plus 1 plus 1 by 2, 1 by 3, 1 by 4 and so on. Sorry, 1 by 2 square, 1 by 2 cube and so on; like this, so it is 1 by 2 to the power n minus 1, 1 by 2 to the power n minus 1. Because factorial 2 is greater is equal to 2 basically, so 1 so first term is at the factorial 3 is greater than 2 square. So,1 by factorial 3 is less than 2 square, 1 by factorial 4 is less than 1 by 2 cube, 1 by factorial and less than 2 to the power n minus 1, now this is a geometrical series. So, what is the sum of this geometrical series is, 3 less than a over 1 minus, so it is less than 3 is it 2 cube1 minus. So, this it means this is a monotonic increasing sequence of real number which is dominated by 3 bounded by 3; so it must be a convergent sequence, and limit of this cannot exceed by 3 and will always be strictly greater than 2 at the most it may be 3.

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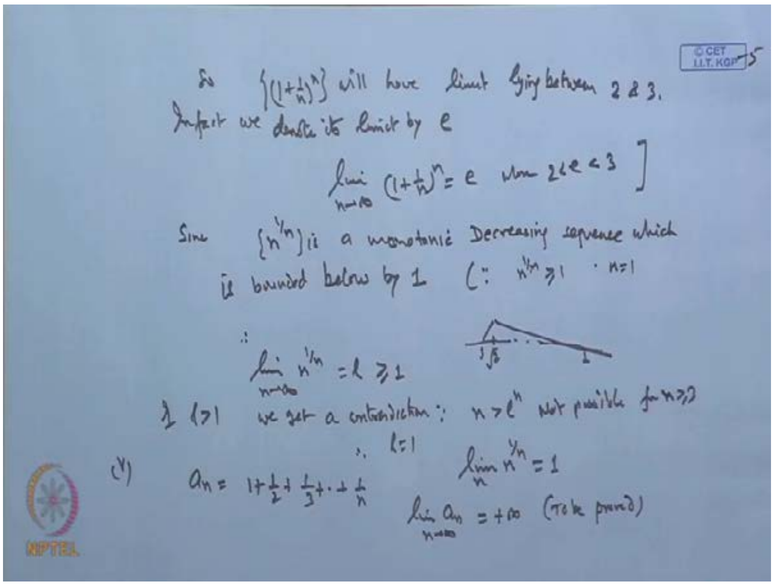

So, limit of this a n, so limit of this, so sequence1 plus 1 by n to the power n will have, will have limit, will have limit lying between 2 and 3. And in fact, we denote it is limit by e; and it was shown later we shown the limit of this as n tends to infinity1 plus 1 by n to the power n e where e lies between 2 and3, and this is irrational number, 2 .7 something like that. So, this, so this will be the limit.

Now here we are discussing about this part, we are it is given the n to the power 1 we are interested in finding the limit of this, Now what we claim this sequence is a monotonic decreasing sequence which is already proved. When n is sufficiently large greater this is a monotonic decreasing sequence monotonic, but each term of this sequence is greater than 1 or equal to 1 because n is 1. So, it is a monotonic decreasing sequence, is it not? And each term n is greater than equal to 1. So, now, so since n to the power1 by n is a monotonic decreasing, monotonic increasing, sorry it is a monotonic, this sequence is monotonic in decreasing. Is it not? Decreasing sequence, decreasing sequence which is bounded below by 1, is it not? Because n to the power1 by n is greater than equal to 1 at least for when n is equal to 1 you get1, n is1 you get 1. 2 it is greater than 1, is it not? So, it is monotonic decreasing sequence.

After 3 it is there, is it not? n to the power1 by n because this is 2 for n is greater than equal to 3. So, after 3 this is decreasing sequence, but all the terms will definitely greater than 1, did you get or not? (( )) You cannot say when n is equal to 1 and 2 you cannot say that n is equal to 1 rather than n equal to 2 it is greater than 1,1 n equal to 3 onward it decreases for n is equal to 1 and 2 it is increases. Clear? But when n is 3 onward it decreases; so it is just like this it guess like this and then decreases. So, this is 1, this is 1 and here is something 2 to the power under root and then it keeps on decreasing, and decreases to basically again 1; like this. So, this is a therefore, the limit of this sequence a ns when you say limit of a n exist and we get the limit to be 1 limit of the n to the power1 by n, then n tends to a is suppose l; we get limit.

Now this limit is greater than or equal to 1; is it ok? Now we claim l cannot be greater than 1, now if l is greater than 1 then again we lead a contradiction then we get a contradiction. Why? Because n is greater than l to the power n l is greater than 1 So, not possible, not possible; for n greater than equal to 3, 3 cannot be greater than some number power 3, so that will be more. Suppose l is equal to 2 then what happen, 3 cannot be greater than 2 to the power 3; so it is contradiction. Therefore, l must be 1. So, limit of this n to the power1 by n over n that is also in testing limit, then another limit is a ns are suppose we have a sequence; which in form of a series, 1 plus 1 by 2, 1 by 3, 1 by n, first n terms of this series some series therefore, some terms is denoted by n. Now limit of this a ns, when you take the limit of this a n we say it will not exist limit of, limit of a n s when n tends to will be plus infinity this is to be proved, limit of this will be true.

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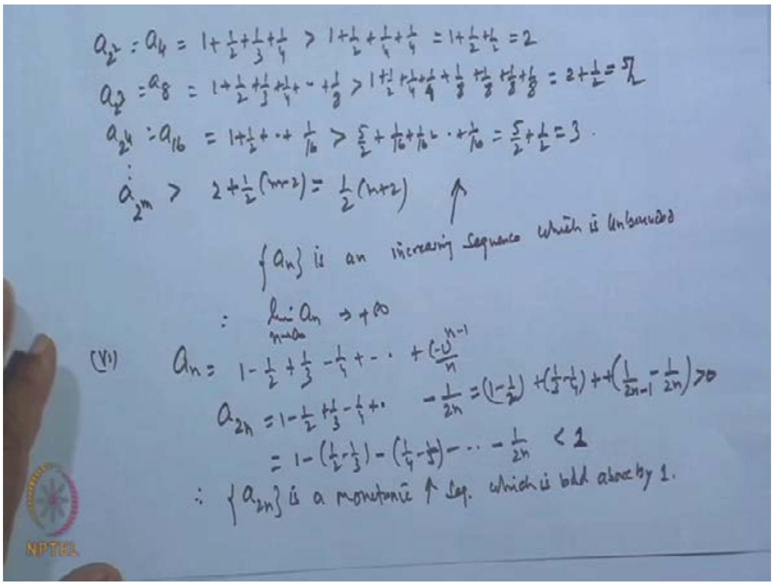

Now, how to show this part again we will justify it by our (( )). What is our a 4, a 4 is 1 plus half, one-third, one-fourth. Now this will be greater than 1 plus half plus one-fourth plus one-fourth. That is equal to 1 plus half plus half that is equal to 2. One-third is a greater than 1 by 4. Is it not? So, this is. Then a 8, a 8 will be 1 by 8. Now, this will be greater than 1. This part 2 plus first (( )) 1 plus half 1 by 4, 1 by 4 and then 1 by 5 onward you write 1 by 8, 1 by 8. So, what happen is, this is get equal to what? 2 plus 4 by 8, 4 by 8 means half that is equal to 5 by 2 is increasing. It means this is a square. This is a 2 cube. So, if you take a 16, that is a 2 to the power 4.

Then we get this is equal to 1 plus half plus 1 by 16, it is greater than 5 by 2 plus 1 by 16, 1 by 16 up to 1 by 16 means up to here 8 terms then this is 8 terms. So, it is equal to 5 by 2 plus half that is 3. So, like this and continue this process then what we get it, that there is, if I take a to the power 2 m, then we can say that this number is greater than 2 plus half m minus 2, that is equal to half m plus 2. This satisfies this condition. You just see m is here 2. So, this can be written as 2 plus 2 minus 2 0. So, this is satisfied; m is equal to 3. So, 3 minus 2 half, this is satisfied; m equal to 4. So, this will be 4 minus 2 by 2, that is 3 and continue. So, what we say is, that this sequence keeps on increasing, increasing. So, the sequence a ns is an increasing sequence, increasing sequence which is unbounded. So, the limit of this a n, limit of this sequence a n as n tends to infinity is tends to plus infinity. This is known as the harmonic series. Basically it is known as the harmonic series.

Now next is, suppose we have a sequence a n is 1 minus half plus one-third minus 1 by 4 and so on, plus minus 1 to the power n minus 1 by n. Alternate positive negative terms. We claim the series this sequence converges. Alternate positive negative terms, this will come how. a 2 n even number terms this is 1 minus half plus one-third minus one-fourth and so on and last term is n is even, n is even so n minus 1 will comes odd so minus 1 by 2. Now, this we can write it like this 1 minus half plus one-third minus one-fourth and then plus 1 by 2 n minus 1, 1 minus n combined this because these are even numbers. So, we can make the pair. Now each 1 is positive, each 1 is positive, 1 minus. So, the sequence of even terms, some of the even terms comes out to be greater than 0. Is it ok? And then this can also be written like this, 1 minus half minus one-third minus 1 by 4 minus 1 by 5 and so on and last term will be1 by 2 n.

Now, these terms are positive, this is positive. So, you are subtracting a positive quantity from 1. It will be strictly less than 1. Is it not? It means the sequence a 2 n when n is even, a n when n is even is a sequence of positive terms and upper bounded is; and not only this, it is an increasing sequence, monotonic increasing, because when n increases, suppose 2 n plus 2. One mode term will come here, which becomes positive.

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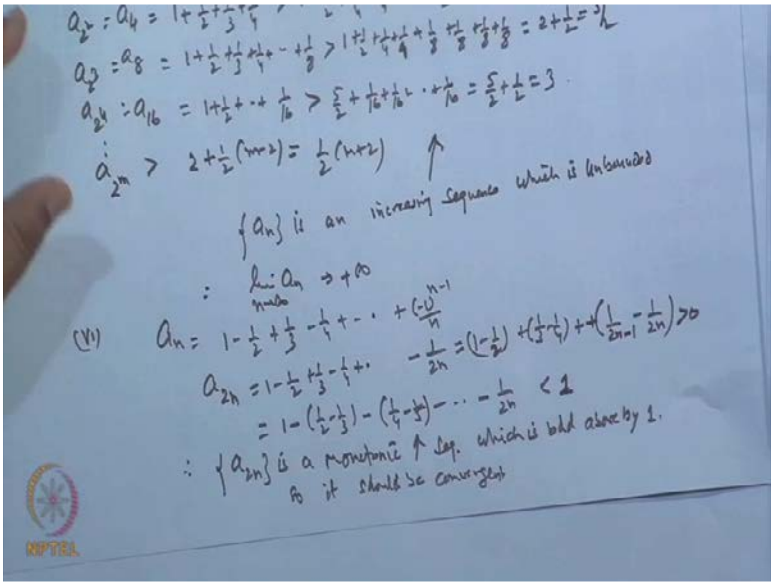

So, this shows the sequence a 2 n is a monotonic, monotonic increasing sequence, increasing sequence which is bounded above by 1, which is bounded above by 1. Is it not? Now, if I take the odd sequence. So, it must be convergent. So, it will should be convergent and once it is convergent the limit it must sent to a limit. So, we get a limit.

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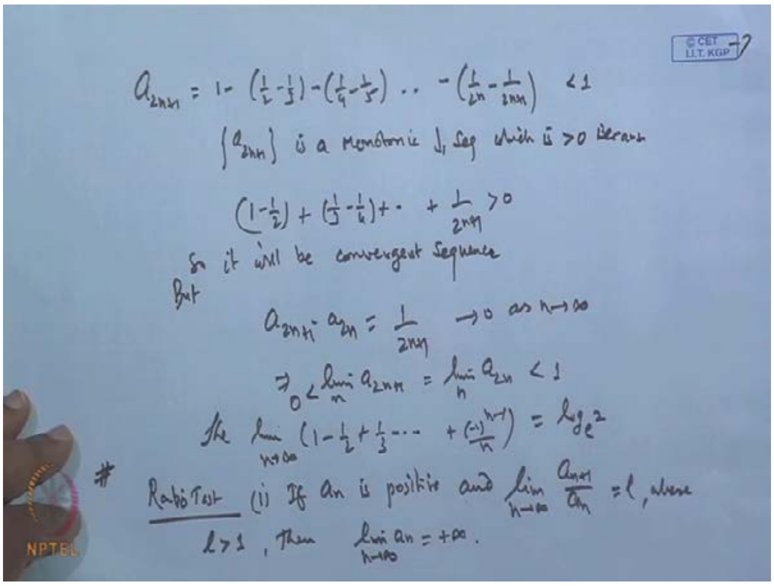

Now take the odd terms. If you go for the odd terms a 2 n plus 1, then what we get from here, 1 minus half minus one-third minus one-fourth minus 1 by 5 and so on; and then we get minus 2 n minus something like 2 n plus 1, odd terms. Now again these are positive quantity. So, it will be less than 1. As n increases it keeps on decreasing, no? Keeps on decreasing. So, it is a decreases as increases. So, sequence a 2 n plus 1 is a monotonic, monotonic decreasing sequence because as increases means you are subtracting something is decreasing sequence and then what about this? Each term is positive because a ns, a 2 n these are all the terms which are greater than this less than 1 and this is positive quantity which you are subtracting, is it not? So, why it is a positive? Monotonic decreasing sequence which is greater than 0, why because monotonic decreasing is when n increases that term.

Because if I take 1 minus half, one-third minus one-fourth, like this, then what happen to this, last term will be what, last term will be 1 by 2 n plus 1 because when n is odd, this term will be positive. So, last term will be this, this is positive. So, it means this sequence is greater than 0. So, a sequence as a monotonic decreasing sequence, whose lower bound is greater than greater than 0, so limitedly. So, it will be convergent sequence, it will be convergent sequence. Is it not? It will be convergent sequence. So, the sequence with the even number of terms is a convergent sequence; sequence with their odd number of terms also convergent. Then whether this will give the same limit or not because one may got some other limit, other one get other limit.

But, but mode a n 2 n plus 1, minus a n what is this? a 2 n plus 1 minus a 2 n. If I find out this difference, then this difference comes out to what? a 2 n will be this term, a 2 n plus 1 will be up to here. So, only this term will be left out 1 by 2 n plus 1 and now, this goes to 0 as n is sufficiently large. It means both the sequence are basically identical sequence. So, this implies the limit of a 2 n plus 1 over n, will be the same as the limit of a 2 n over n limit of this n. It will exist limit will exist and lying between 0 and 1. This limit, the limit of this sequence, limit of this 1 minus half plus one-third minus so and so for, plus minus 1 to the power n minus 1 to the power n as n tends to infinity is denoted by log of 2 to the base. This will denote. In fact, this will be proved to be log 2. So, this may exist. Then that is all.

Now, let us come from some test is there, which is important also ratio test. So, sometimes, lets one more example it is taken then not less you tests it is ok. Ratio test. Now, so far what we have seen is a given sequence is given then by hook and cook by some other means. We before getting the limit we wanted to make sure the limit exist or limit does not exist. Then what we do is we try to justify or try to find out whether the sequence is monotonic sequence or monotonic increasing, bounded or monotonic decreasing bounded below and then we say limit will exist but if the sequence is monotonic increasing unbounded then will be plus infinity and like the minus infinity.

So, basically we are trying to get the limit by using our tricks, but the term the sequence is may be very complicated terms. So, once it is very complicated term, this type of tricks may not help you much. It will be time consuming. So, without going through this process can you just identify whether, the given sequence is a convergent sequence or divergent sequence and that, for that we require certain test. So, those tests one of the test is the ratio test, there others also, but we are stick with the, we stick only with the ratio test.

So, first ration test is, if a n, if the sequence a n, if a n is positive means term of the sequence is positive and the limit of this ratio and the limit of a n plus 1 by a n as n tends to infinity is l, where l is greater than 1. Then the sequence a n will diverge and the limit will be plus infinity. So, you need not to go through anything, just you will find the ration of this and take the limit a n is sufficiently large. If the limits come out to be l which is to be a greater than 1, then it will diverge otherwise it will converge.

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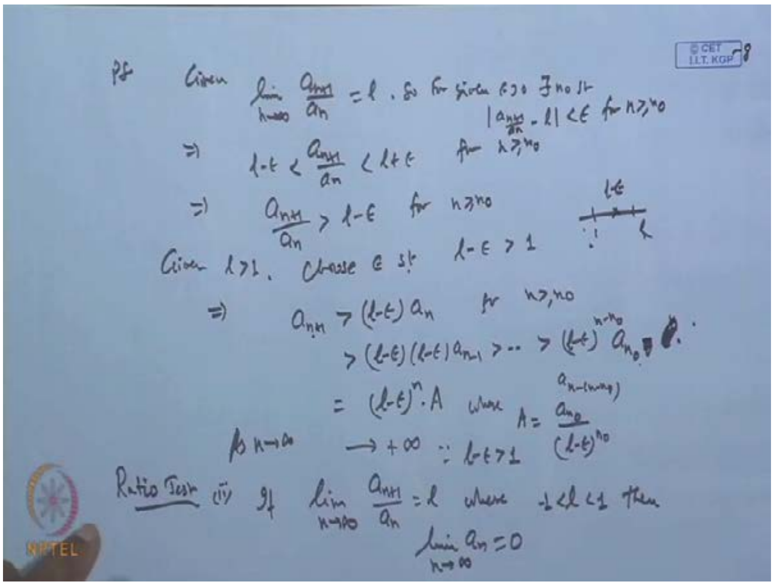

So, let us see the proof of this. What is given is? Given limit of this sequence a n plus 1 by a n over a n is l. So, by definition the a n plus 1 by a n will lie between l minus epsilon l plus epsilon for n greater than equal to n naught. Is it not? When limit a for a given epsilon so for given epsilon greater than 0 there exist n naught, such that mode of this thing, mode of a n plus 1 by a n minus l is less than epsilon for n greater than n naught. Is it not? So, they will exist some n naught depending on epsilon. It means the term this thing lies between l minus epsilon l plus epsilon.

So, this shows that a n plus 1 by a n is greater than l minus epsilon for n greater than equal to n naught. Now, given l is greater than 1, this given, given l is greater than 1. Can you not choose epsilon such that l minus epsilon is also greater than 1? There is a gear. This is 1; this is l. So, l is strictly greater than 1. I choose epsilon such a ways so that the l minus epsilon is also a greater than 1. So, from here we get that a n plus 1 is greater than l minus epsilon into a n, which is greater than 1. Now, this is true for all n greater than equal to n naught.

So, suppose I start with this n, it is greater than l minus epsilon into l minus epsilon a n minus 1. Is it not? Because a n substitute n is equal to a n minus 1, then again greater than like this. So, what happens this will be l minus epsilon, up to say term n minus this thing, is it not? Total term is n naught. So, n minus n naught if I just said, then we get n minus n naught into a n naught. Keep on just substituting the value means a n minus 1 replaced by a n minus 2, a n minus 2 replaced by a n, up to a n, minus a n minus naught up to a n, a n minus n minus n naught. So, n minus n not comes here and this comes out to the n naught.

Now this is greater than 1. Now this can be written as what? Can you say this is equal to l to the power n into a constant term say a, l minus epsilon, sorry, I will just write. This is equal to l minus epsilon to the power n into a constant terms A, where what is a? A is equal to what? a n naught, a n naught divided by l minus epsilon n naught. Because l is fixed; epsilon also fixed. So, l minus epsilon is fixed; n not is fixed. So, this is a fixed value. So, a n naught is fixed. So, A is fixed. Now once A is fixed and this I am taking l minus n. So, as n tends to infinity what happens? This keeps on increasing. So, this limit will go to plus infinity because this is greater than because l minus epsilon is strictly greater than 1. So, this limit goes to increasing 1. So, if l is greater than 1 the this sequence will converge, diverge.

Now second test said, second part of this, the second part says that if this is less than 1, if this limit, if limit of this, as n tends to infinity a n plus 1 by a n. This limit is l where, n lies between minus 1 and plus 1, minus 1 and plus 1. Then the limit of this sequence a n as n tends to infinity will be 0, limit n will be 0. This (( )) Again the proof will be on parallel lines, but what we will do it here is, instead of taking this side now we will take this side, I will take this side and because l lying between minus 1 and plus 1 minus so will use the mod l here.

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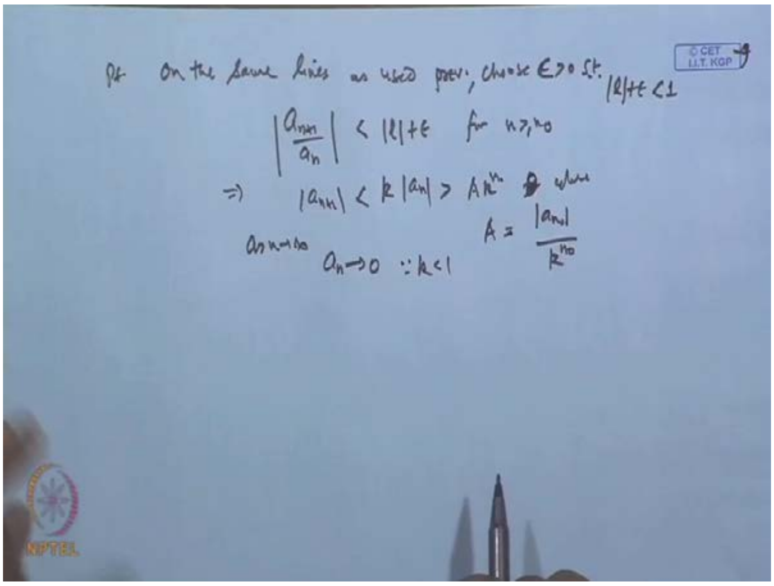

So, followed on the similar paternal. The proof is on the previous line on the same lines as used previously. We can say that a n plus 1 by a n this thing is mode of this is less than mode l plus epsilon for n greater than equal to n naught. Where, where, choose epsilon, epsilon greater than 0 such that mode l plus epsilon strictly less than 1. Clear? And then from here what will be this shows that mode a n plus 1 is less than some number k, this is number k, is it not? k and then mode of a n. Now this will be keep on increasing like this and finally, what we get this is to less than A times k to the power n for n large. Is it not? n naught where, where A is equal to what, A is nothing but mode a n naught divide by k n naught.

Just like, you know previous. So, I am not doing. Now k is less than 1. This k is less than 1. So, as n tends to infinity this limit a n goes to 0 because k is less than 1. So, this shows limit will be 1 0. So, a ns limit will be 0. Now what happens to that is if suppose I take, because here in the ratio test, I have taken this condition. If, a ns are positive then this (( )) l. Ratio of this is l greater than 1 sequence will diverge, when l lying between minus 1 to plus 1 the sequence will lie. In fact, when n is positive, this l, this ratio of this part, a n over n plus 1, we will always get the positive terms.

Now, if l the terms are negative, then if they are having the same signs, then I can remove the sign and get the result, but if the terms are alternative positive and negatives, then in that case we have to see the other results, because this result is not true for a ns to be the negative. Ratio test is always applicable when the sequence is having the same sign. So, I can remove the sign and apply, because the behavior of the sequence, convergence or divergence of behavior of the sequence, does not depend on the sign of the sequence. Because if all the terms are the positive sign or if all the terms are negative sign then both will sequence will behave properly same way. Because if the first sequence convert divergent, other will divergent, only thing is 1 will go to plus infinity, this will go to minus infinity. If first sequence converges limit converges, converges to a, another sequence will converge to the limit minus a, but the mix type of the terms will not help you.

This ratio test may not be a huge applicable, will not help both sequence when the terms are alternatively positive negative and so on. So, all the terms will be (( )). So, that is the one (( )). So, I think this, we will just stop it and then we will see how the this result can be used to compute the complicated limits, the complicated limits how does this follows.

Thank you very much.

Thanks.