A Basic Course in Real Analysis
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Lecture - 15
Concept of limit of a sequence

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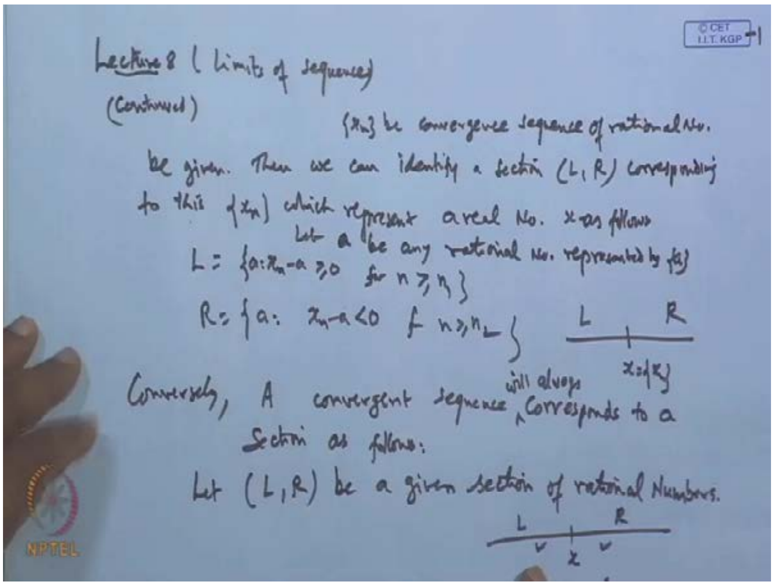

So, in the last lecture, we were discussing about the equivalence of these two theories, that is Cantor's theory and Daniken's theory. Daniken's theory based on the cuts, Cantor's theory is based on sequences of, convergence sequence of rational numbers. And we have seen one, that if a section is given, than convergence sequence is given, than we can identify a section; is it not that? That if x n be convergence, x n be convergence sequence, this is continued; continued from previous lecture. Then limit of sequence will be there later. So, x n be convergent sequence, suppose x n be convergence sequence of rational number is given, rational numbers be given. Then we can identify or we can identify a section (L, R); is it not? Corresponding to this, corresponding to this sequence x n, which represents a number x, which represents a real number x as follows.

We are taking lower class L, contains those classes such that x n minus a, is set of those x n minus a, a is any real let a be any, let a be any rational number represented by the same sequence is upper and the real number x n minus a is represented by x minus a. So, set of those numbers a for which, x n minus a is greater than, that is positive and after sometime is greater than equal to 0, up for n greater than equal to say n y. And upper class R with the set of those sequences we are taking, for which x n minus a is negative from an after, sometimes greater than say n 2. Now these two classes L and R will decompose, this entire sequence x n we can break up and this class x point is correspondent to x n will divide give the section L R; this we have discussed already. So, this is no point of doing, now we will do it today the converse part.

That if suppose the section is given, then corresponding to the section we can generate a sequence of rational number, which will give the same real number as the section L R represent. So, let us see the converse, conversely; we can say, A convergence sequence, convergent sequence corresponds to a section. Correspond will always, will always correspond to a section as follows. So, let us suppose let (L, R) be a section, be a giving section of positive rational number, there are be of rational numbers, rational numbers. Let L R be given section of rational number, now this is here he say some rational numbers say x. So, here this is L, this is R. Now, if you pick up any two rational numbers, then the property of the section a. That we can identify the two rational number one belongs to this class, another belongs to this class; such their difference of these two can be made as a small . So, this is the property, which we are in a.

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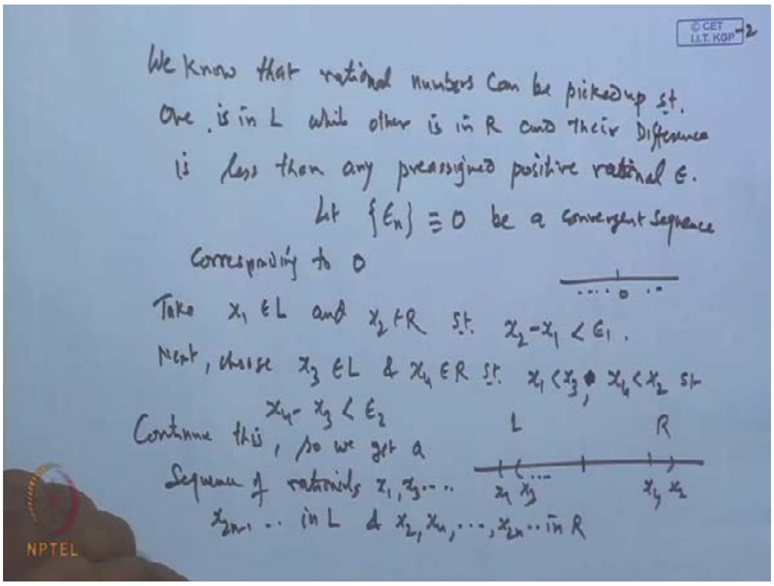

So, since L R is a section and we know, we know that two rational numbers, rational numbers can be picked up, picked up, can be picked up such that one belongs to one is in

L and the other is, while other is in R and their difference, and their difference, and their difference is less than any preassigned number epsilon, is it not? Preassigned positive rational number epsilon, rational epsilon. This we know, that we can identify that proved rational numbers one belongs to L other once to R and their difference can be made less than, less than epsilon. So, for any epsilon one can identify these two numbers. So, this is will be given possible.

So, let us suppose, let a sequence epsilon n be a convergence sequence corresponding to the number 0, this is be a convergent sequence, convergent sequence corresponding to the number 0. So, 0 it means 0 is this point. So, they are the sequences are there, such that mode of epsilon n is can be made as small . So, these sequences correspond to a convergence sequence correspondent to 0. Let us suppose this one.

Now take any x 1, take a point x 1 in belongs to L and x 2 in R, such that the difference of this, difference of this x 2 minus x 1 is less than epsilon; because any element of L is less than element, any element of R; so, x 2 minus x 1 is positive, but this difference should be less than epsilon 1. Now choose another one now. Now, next choose x 3, choose x 3 in L and x 4 in R, such that x 1 is less than x 3, less than x 4, x 3 and x 4 is less than x 2.

So, what we are doing is, this is our number say, I am taking x 1 here, x 2 here; the difference of this is less than epsilon 1. Than I am picking up the number x 3 in L, L and a number R, x 4 in R. So, obviously x 3 is less than R, such that x 4 minus, such that x 4 minus x 3 is less than say epsilon 2. Continue this. So, we continue this, then what happen, we are getting a sequence. So, we are getting continue this.

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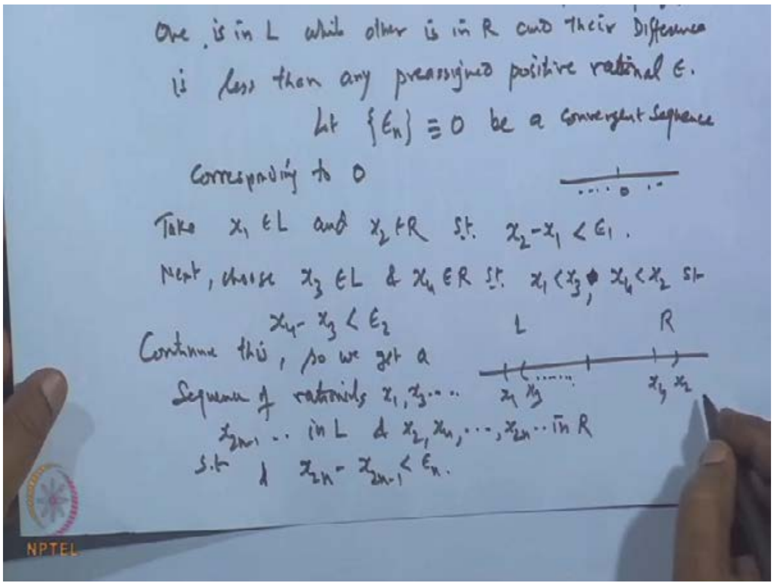

So, we get a sequence of rational numbers x 1, x 3 and so on; x 2 and minus 1 and so on; in L and x 2, x 4, x 2 n in R; such that, such that the difference of these two is it not? Such that difference of x 2 n minus x 2 n minus 1 is less than epsilon n. We are choosing in this process, now this way we have. So, corresponding to a section, we have now generated a sequence. Now this sequence expand odd, odd suffix index is in L, all evince index is in . Now we claim that these two sequences, x 2 n minus 1 and x 2 n; they will be a convergent sequence, why?

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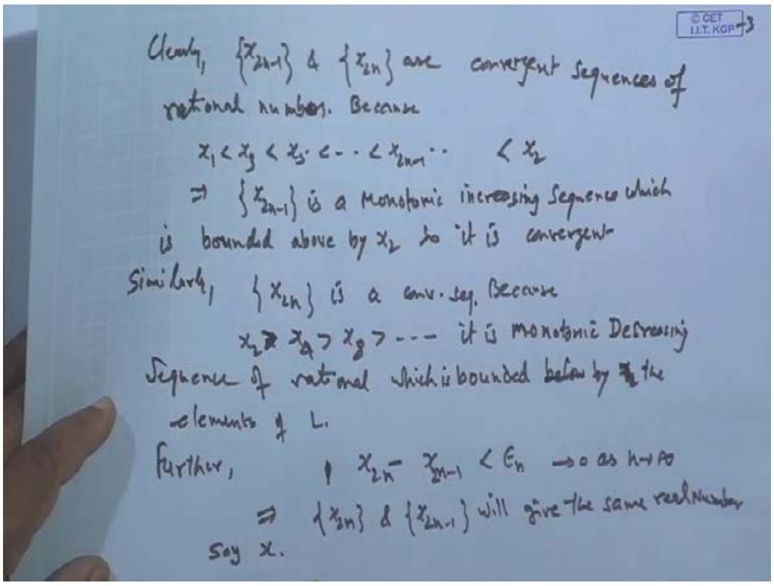

Clearly the sequence x 2 n minus 1 and x 2 n are convergent, are convergent sequences of rational numbers, of rational numbers. Why? Why it is so? Because, the reason is the x 1, x 2 because, x 1 is less than x 3 is less than x 5 less than and so on. x 2 and minus 1 and all are less than x 2. x 2 lying in R 2, x 2 lies in R 2. So, whatever the sequence you are choosing it is n R 2. The every element will be in less than the element of x 2. Now, so this sequence, this imply the sequence x 2 n minus 1 is a monotonic, monotonic increasing sequence, is it not? Increasing sequence which is bounded above, which is bounded above by x 2; and we know every monotonic increasing sequence which is bounded above is convergent. So, it is convergent.

Similarly, the sequence x 2 n is a convergent sequence, because x 2 n is also what? x 2 is because, because, x 2 is less than, is greater than x 4, is greater than x 8, is greater than and so on. It is a decreasing sequence and bounded below, it is monotonic decreasing sequence of rationales, which is bounded below by x 2, is it not? It is a monotonic decreasing sequence which is bounded above sorry, which is bounded, yes which is bounded below or may be bounded above. Is a decreasing sequence which is bounded below by which number, by x, by which number this is? x 1, x 2, x 3 sub which is bounded below by what? Every element of this L, which is bounded below, by the elements of L, is it not? So, this L therefore it is convergent and converges. So, these two sequences are convergence sequence.

Now, once they have convergent and also they satisfy , further mode of x 2 n minus x 2 n minus 1; this is less than epsilon n, which tends to 0, when 2 n minus 1, 2 n minus 1. So, this is less than which goes to 0, as n tends to infinity; is it not? Because epsilon is a sequence convergent to 0. So, this sequence, it means these two sequences are identical sequence. So, this implies, that sequence x 2 n and sequence x 2 n minus 1 will give the same real number, same real number, say x, is it ok? Now we are, so we have with the help of this section, we have generated a sequence. Now we claim that, this section if you take any element in the lower class; then if L is does not have a upper bound list, say upper bound; then element of the lower class will be less than the element of upper, upper class and so on.

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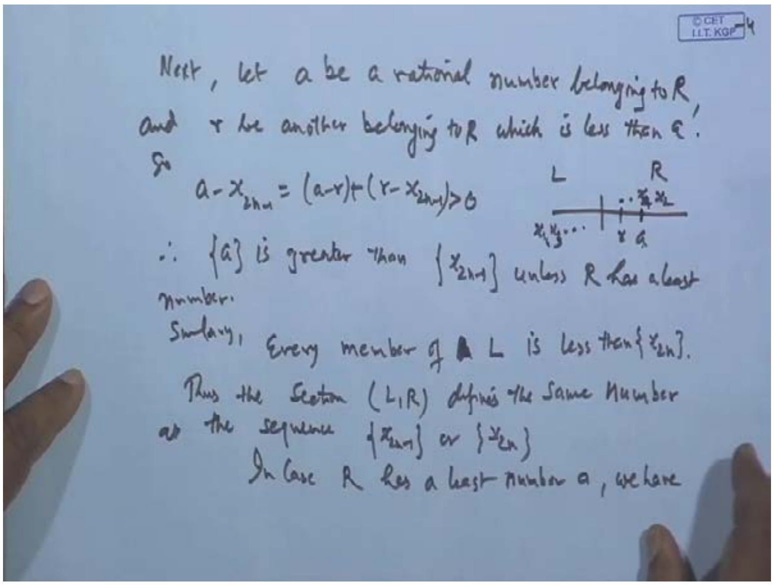

So, next we saw, next let, next let a be a rational number, a be a rational number belonging to R, belonging to R upper class and r be another number and r be another number belonging to R; which is less than a, less than a. So, this is the point here we are taking R, this is L, here we are taking a point a, and this is our r. Now these are the sequences x 1, x 3 and so on odd sequences, here we are getting x 2, x 4 and so on. These are the sequences . So, if you take the difference of this from the lower term, what you get; a minus, a minus x 2 n minus 1, odd terms. This is equal to a minus r, plus r minus x 2 n minus 1. Now a is here, a is greater than r. So, this term is positive, now r is a rational number, belong into the every element of this will be greater than the element of this; so, it is positive. It means, you choose any number in the upper class; it is always be greater than the elements of the L, unless R has a some least number.

So, therefore, this number a is always greater than, greater than this any element of this sequence, is it not? Unless R is, unless R has a least number; similarly we can proof the other, similarly we can saw that any element of this. So, similarly, similarly every member of R, member of R, every member of L sorry; this every member of L is less than the terms of this sequences x 2 n. But these 2 sequences are equivalent, they give the same number.

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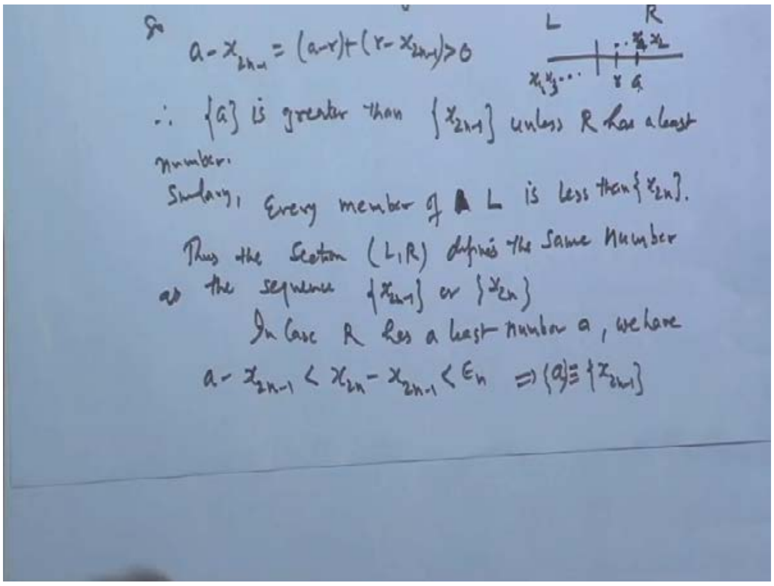

So, thus the section (L, R) defines the same number, defines the same number as the sequence either x 2 n minus 1 or x 2 n; because, both are give the same number, is it not? But if R has any least number, then it will belongs to one of the class, either all n in that case also. If, we in case R has a least number a, number a, then, then we have a minus x 2 n minus 1 is less than x 2 n minus, 2 n minus, 2 n minus 1; which is less than say epsilon n. So, what this shows, that thing goes to 0. It means, a is identical to the sequence x 2 n minus 1, this number identical to this sequence; similarly a is also an identical to x n. So, this shows that both are equivalent. And ok? So, this case so, this proves the eqalence of the two terminology. So, this is complete the concept of this numbers, real numbers with the help of Cantor's and Daniken's. Now let us come to the .

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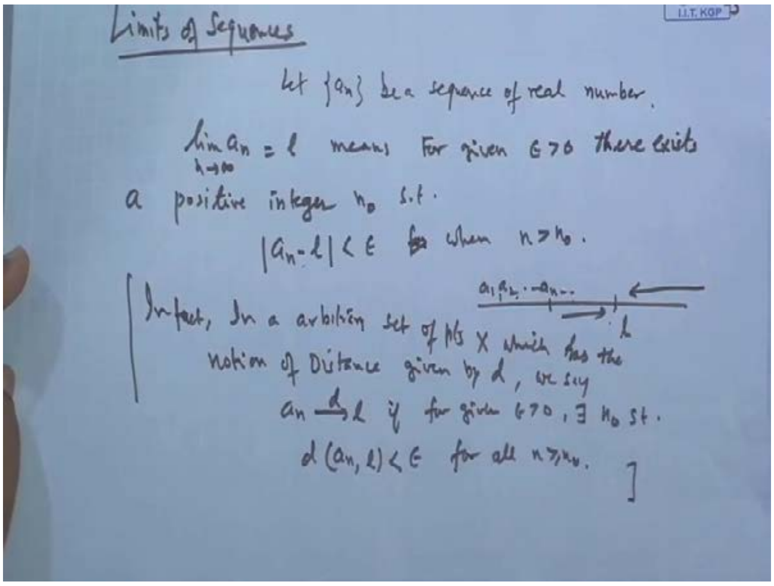

Now we will discuss the Limits of sequences, limit of sequences; suppose a n be a sequence, we say the sequence a n is the limit. Let a n be a sequence of real number, I am just saying real number, I am not now taking any rational or something, real numbers.

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Let a n be a sequence of real numbers, suppose we say limit of the a n has an tens infinity is say l. Means for given epsilon greater than 0, for given epsilon greater than 0. There exist, there exist a positive integer n naught. Such that the difference between a n minus l, this difference can be made less than epsilon for all n, when n is greater than n naught. The meaning of this is very clear, suppose we have this number L and there are the sequences a 1, a 2, a n and so on. We say this sequence converges to L; it means, that if we find out the distance from each L, each term distance of each term from L, then this distance keep on reducing and reduce it to 0. Then we say the sequence and converges to L. Basically this mode is the distance, this mode means difference between the two values, now a n and l both are real numbers, they can be represented by means of a point on the real on the axis.

So, once you have the point you can identify the distance and this is the distance a n minus l. Why absolute value, because there may be a sequence which may converse from this side, it may go from left hand side. So, L may be less than this, L may be greater than this, but in absolute value the distance must tends to 0 as n tends to. So, we say the sequence is set to be convergent, when the difference between a n minus l, or the distance of a n from l keep on reducing and reduce to 0. So, this is the way we can. Now, this has been generalized to an arbitrary metric space. Because this is the case, when we are dealing with the real numbers only or complex numbers, then when you have the real or complex number, the distance notion is simply the absolute value, is it not? The absolute difference between the two points is the distance of the two real number or distance between the two complex number. But suppose x is an arbitrary set of points. Then the notion of the distance will be define in such a way so, that of the usual notation must be take. That is this. So, that we will take, in fact, in fact in a arbitrary space, in a arbitrary set, in a arbitrary set of points X; which has, which has the notion of the distance by d, notion of distance given by d, given by d; we say a sequence a n converges to l under d, if for given epsilon greater than 0, their exist n n naught. Such that the distance between (a n, l) is less than epsilon, for all n greater than equal to n naught.

This is a general way, but we are not dealing with a general, that is why I restricted only , but what is the distance function? Distance means, that a distance d, a function d is a mapping what is the metric or distance function. This is non-negative, it always greater than equal to 0 and 0 when a, the two points are coincident, then we get the reverse, if a n and l position is reverse, we get the same value and then the tend learn inequalities. So, these conditions are satisfy; then we, so, we are not that is why we will drop this one and we will simply take up the mouth side just to say. Because here, in fact, I wanted to introduce that metric distance, but because it is early, that is why; it is not .

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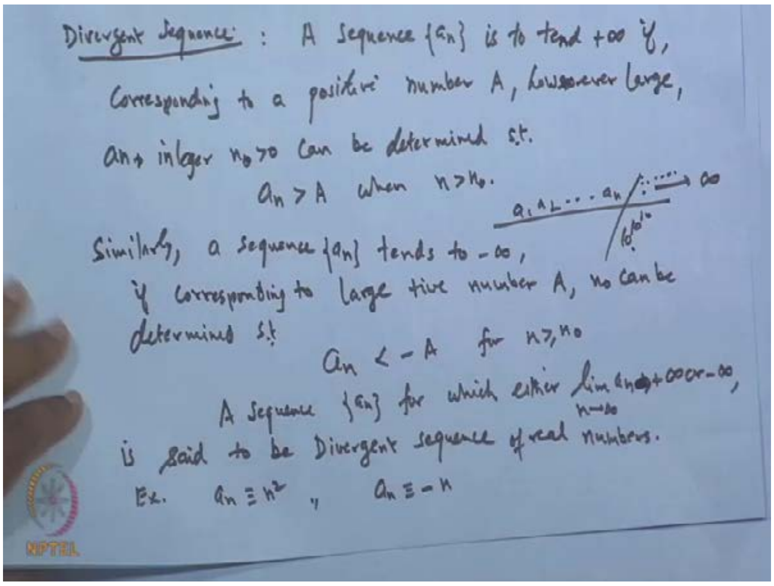

So, we will take this. Now, divergent sequences, divergent sequence, we define the divergent sequence a n set to be divergent; A sequence a n is set to, is set to tend, is set to tend plus infinity, is set to plus infinity; if corresponding to a positive number, if corresponding to a positive, corresponding to a positive number A. How so ever a large, how save a large however large, how save a large however large, a number a n integer n not, integer n not greater than 0 can be find, can be determine such that a n are greater than A, for whenever n is greater than n naught; we say the sequence a n tends to plus infinity. So, these are the sequence a 1, a 2, a n and so on and this tends to plus infinity. It means the limit of this sequence a n is not finite, it is infinite. So, it is a divergent sequence; a sequence is set be divergent, when the limit of the sequence does not exist; either it will be a plus infinity or minus infinity, then it is set to be a diverging sequence.

So, when you say it is a sequence a n goes to plus infinity means that, whatever the number you choose, you can always find an integer n naught. Such that the value of the coordinate of this sequence will exceed by that number; suppose, I would say a is equal to 10 to the power 10 to the power 10. Then this number is their say 10 to the power 10 to the power 10 like this; then one can identify number n not here, that all the terms of sequence after this will greater than this number. So, we say it is tending to plus infinity. Similarly, we say a sequence, a sequence a n similarly you can tends to minus infinity, if corresponding to, if corresponding to a large positive number A, large positive number A and not can be determined, and not can be determined, determined such that, a n are less than minus a for all n greater than equal to n naught. Then it is tending to minus infinity, a sequence which is either so, A sequence a n for which either the limit of a n, as n tends to infinity is plus infinity or minus infinity. Limit of this tends to not tends to plus infinity or minus infinity or minus infinity, is it not? Plus infinity or minus infinity; is said to be, is said to be a diverging sequence, sequence of real numbers, of real numbers. For examples are, suppose I take a n, the sequence say n square this will diverse similarly, other sequences also you can say it will diverse to plus infinity. If, I take a n equivalent to say minus n, it will diverse to minus infinity like this; and so on so.

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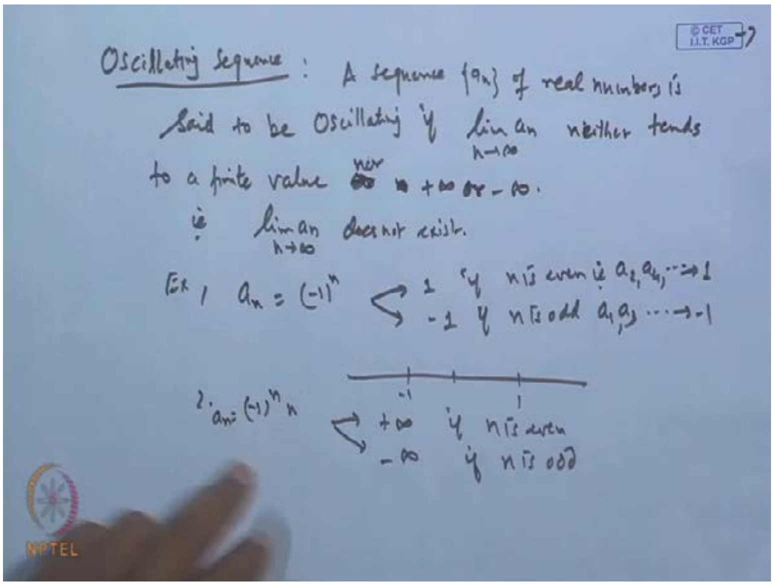

Then oscillating sequence, oscillating, oscillating sequence: A sequence a n of real numbers is set to be oscillating, is set to be oscillating. If, if the limit of a n as n tends to infinity does not exist and limit of this, neither does not exist or neither, neither tends to a finite value, neither tends to a finite value or, or nor, or plus infinity or minus, nor plus infinity or minus infinity, plus infinity or minus infinity. The meaning is that a n is such, where the limit does not exist, limit does not exist means, that is limit of the sequence a n, when n tends to infinity does not exist.

When we say the limit exist it means, whatever the path you choose because, the definition of the limit; when the limit is there, this is the definition of limit, that if the limit exist means this is less than. So, whatever the path you choose, limit of a n minus l can be made . The difference between a n a l should be made a smaller in this edge; one can desire, desire, that should not be fluctuation, but if such a sequence are there, where this difference cannot be made a smaller, some time it is small; some time it is become very large; then in that case limit does not exist or along different sub sequences, it has different values, then the limit does not exist.

For example, if suppose I take a n to be minus 1 to the power n, then along the positive path it will go to 1; If n is even, that is when the sequence are choosing like this a 2, a 4 etcetera. The limit will go to 1, but if the sequence is chosen to be odd, then n is odd; that is a 1, a 3; this limit will go to minus 1. So, the sequence when n tends to infinity, does not tends to a one value. Because, it fluctuate like this; here this minus 1, here is plus 1. So, what happened is a n minus, you cannot find any suppose, I take 1 is the value, then a n minus 1 cannot be made epsilon because, as soon as a n odd becomes than it will be minus 1, minus 1, minus 2. So, it becomes very large similarly so, it does not go to that does not have a finite value limit; similarly, if we take the sequence minus 1 to the power n, n say this one, then what happen? When n is sufficiently large, when even number it will go to plus infinity, it goes to plus infinity. If n is even, n goes to minus infinity; if n is odd so, it does not have the limit.

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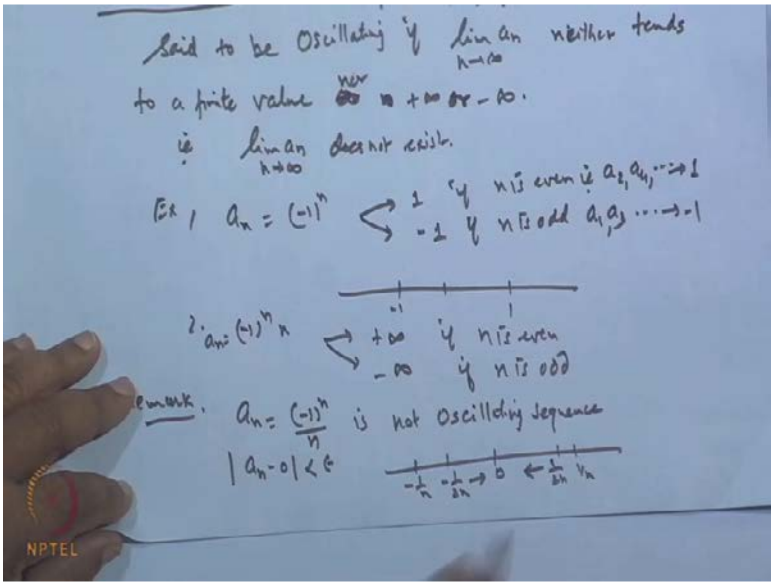

Similar, however the sequence a n which is minus 1 to the power n by n is not an oscillating series; is not, is not a sequence, is not an oscillating sequence; why? Oscillating sequence are those sequence which limit does not exist, I the limit is not tending to a finite value or plus infinity or minus infinity. Now this sequence tends to value 0, though it is alternately positive, negative, but what happens if, this is the value 0 you are getting minus 1 by n, plus 1 by n. Then as n increases you are taking minus 2 by n, plus 1 by 2 n, like this. So, this goes to here, this goes to here. So, after certain stage the difference between a n minus 0 can be made as a small as be . So, that is why this sequence is convergence sequence, converges to 0. So, it is not , but convergent, this convergent, then bounded sequence.

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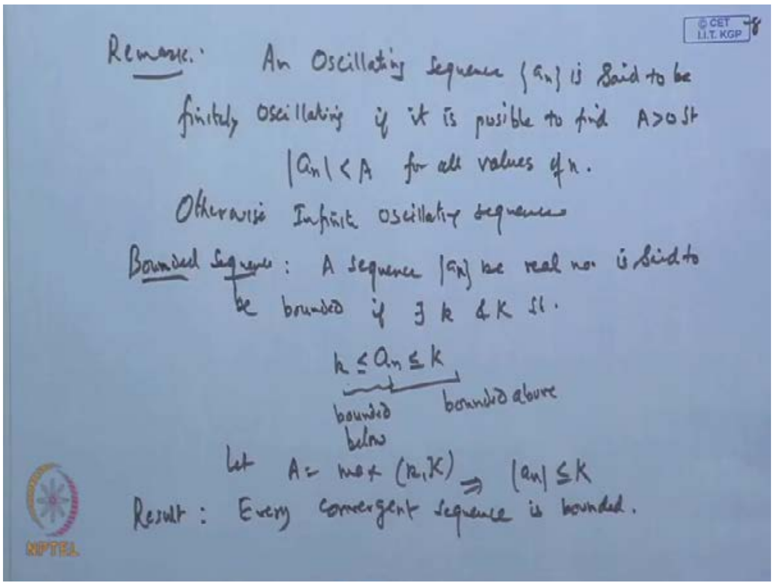

So, we say here also we can when the limit tends to a does not tends to finite value, but does not go to plus infinity, minus infinity, but it is alternates and finite means, that is when we are unable to get, that is a sequence is set to be a finite no remark, a sequence an oscillating sequence; sequence a n is set to be, is set to be finitely oscillating. If, if there exist number A, if it is possible to find a number A greater than 0. Such that all the terms of the sequence remain less than A, all the terms of the sequence remain less than A; for all values of n. We are able to get it just like this series; this sequence is a finitely oscillating because, a number 1 can be , but this not a, because it does not even a you cannot find otherwise, otherwise infinite oscillating sequence. So, we carkatise this into . Bounded sequence we have already discussed so, no kind of bounded sequence and then every convergence sequence are bounded. A sequence a n is set to be bounded; a sequence a n of real numbers is set to be bounded, if they are exist k or K; such that a n is greater than equal to k, greater than equal to is less than equal to K. Suppose, we have two bound, it is not necessary that we have the same bound so, a sequence a n is to be set is to be bounded below. If there is a k such that all the terms of the sequence are greater than equal to k; then this is, this will give the bounded below, bounded below. Where this thing will give bounded above, bounded above and if you combined both the and let A be the maximum of k and K; in fact this is the K only, then in that case mode of a n is less than equal to k; then we say this sequence is bounded, bounded. So, lower bounded, upper bounded and bounded; all like this.

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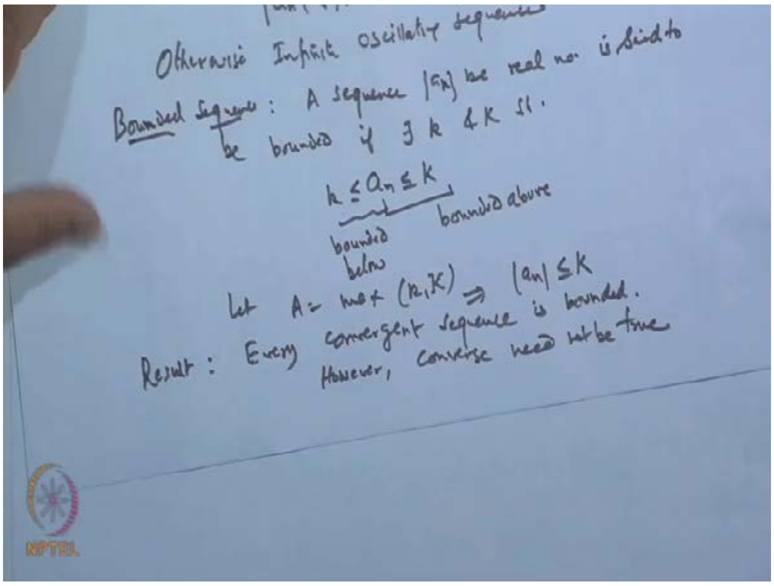

So, we get this one and every convergent sequence is a bounded sequence that we have seen in there. So, result is every convergent sequence is a bounded sequence, is bounded sequence and is bounded. I think this proof we have done is whatever the converse; can you say every bounded sequence is convergent? The answer is no, but converse is not true.

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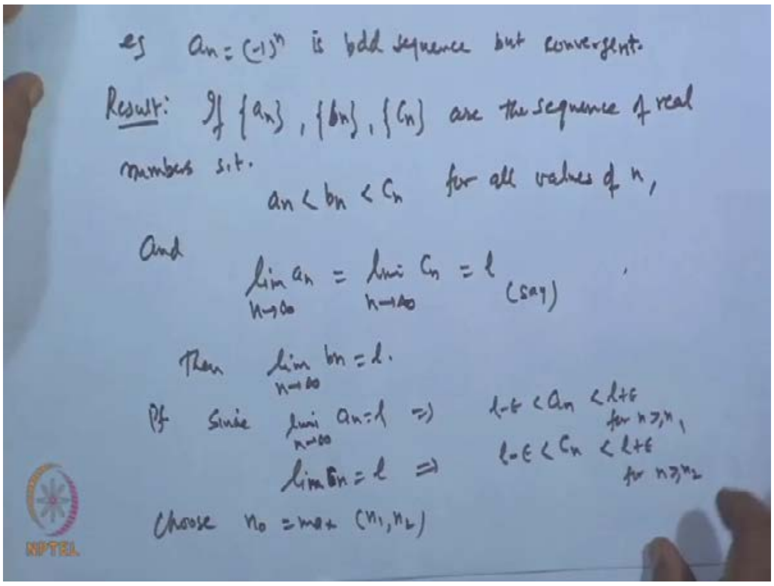

However, converse need not be true because, if we take for example, if you take the yes, if u take the sequence a n minus 1 to the power n; this is a bounded sequence, but not convergent. So, we can get this clear; now fundamental theorems of this limit are the same as they so, we are not touching those thing. Let us see the results, which is very important result. Their one result which is known as the Sandwich Theorem, Sandwich Theorem; what this say is, if sequence a n, b n and sequence c n. Let a n, b n and c n be the are the sequences; such that of real numbers, such that are the sequences of real numbers, such that a n are less than b n less than c n. Suppose, we have this sequence and this is true for all values of n, values of n. That is three sequence are given and they satisfy this inequalities, for all n and limit of a n as n tends to infinity is the same, as the limit of c n as n tends to infinity and suppose it is l; then what this result says, then the limit of this sequence b n will also be l, this is known as the Sandwich Theorem.

That if you want to find the limit of the sequence b n; if we are able to identify that lower and the upper bounds for each n, that is a n sequences; they are the corresponding terms are satisfying this condition and if these left hand sequence and right hand sequence converges to the same limit; then the middle sequence will also converse to the . The proof is very simple, proof is not that because, proof is why it is so because, a n is given to be l. So, since limit of this a n is l so, it implies that a n must lie between l minus epsilon and l plus epsilon for n greater than equal to n 1; similarly, limit of this b n is l.

So, this implies that l minus say same epsilon we can choose or different also ; c n sorry, limit of c n.

So, l minus epsilon less than c l, less than l plus epsilon; after integer n greater than equal to n 2. Now, if I picked up the n greater than n 1 and n 2, choose n naught is the maximum of n 1 and n 2. So, for all n greater than n naught this condition is satisfy for all n greater than n 2, this condition be satisfy.

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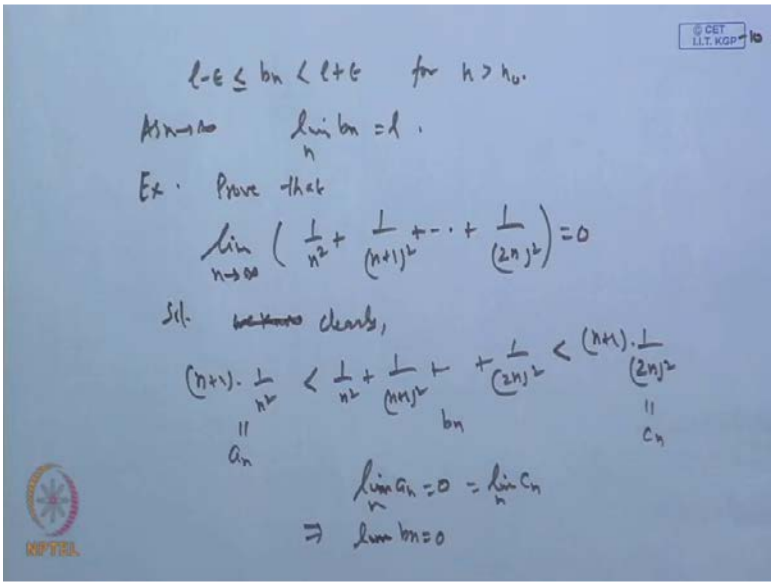

So, we get from here a n and b n lying. So, b n is the number lying between a n and c n. So, we can say l minus epsilon is less than b n, is less than b n; at the most equal to less than l plus epsilon, for all n greater than n naught, is it know. Clear?

So, as n tends to infinity, limit of the b n will go to l ; now, this will, this is used to find the limit of the complicated express for, what is the use? Suppose, I take this problem prove that, limit of this as n tends to infinity; 1 by n square, 1 by n plus 1 whole square and so on. 1 by 2 n square is 0, limit of this is 0. We wanted to saw this one. So, we know, we know that 1 by n square is 1 plus, the lowest term is 1 by 2 n up and largest term is this. So, this calculation shows that 1 by n square, 1 oval n plus 1 whole square and 1 by 2 n square.

This will be less than total terms are what n plus 1, stating with n to n plus 1. So, total term is n plus 1 into 1 by 2 n square and greater than n plus 1 into 1 by n square or clearly , we get this, is it not? Now, as n tends to infinity, this is about a ns, this is about c ns. So, as n tends to infinity because, the denominator is having larger degree than denominator. So, this will go to 0, this will go 0. So, a n and b n, c n limit of this a n is 0, is the same as the limit of c n; therefore, limit of b n must go to 0. So, this implies limit of this b n must this is about b ns. So, this shows the very interesting things is it not? Like this.

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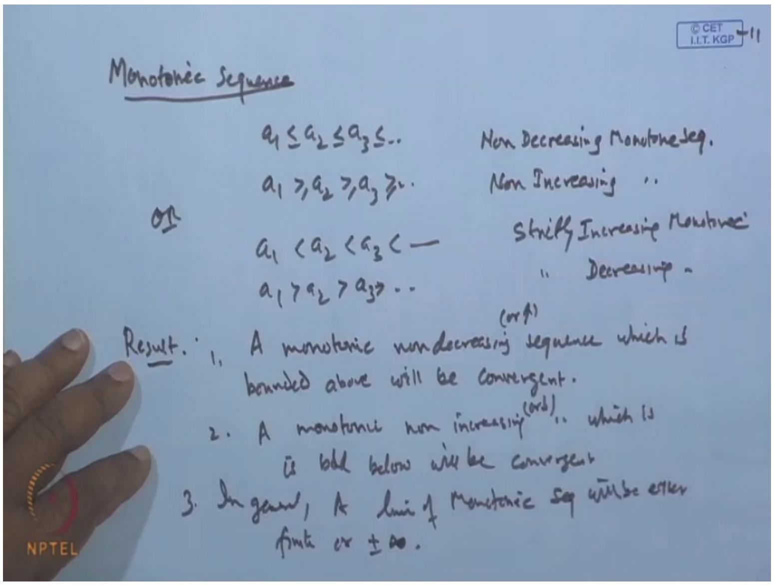

Now, Monotonic sequence there are also similar type of this, monotone, monotonic sequence; sequences we have seen that, there are two types of sequences, which are either in non-decreasing means a 1 is less than equal to a 2, less than equal to a 3 etcetera. These are call the non-decreasing sequence, monotone sequences, non-decreasing; it keep on may be constant also or a 1 greater than equal to a 3; this is called non-increasing sequence, monotone sequence or if is have this one, then it is called the strictly increasing monotone sequences, monotonic and if we have this one, then we say strictly decreasing.

So, we get this one; now, if these monotone sequence are there; if it is bounded above say non-decreasing sequence, which is bounded above. Then it will have the limit, if it is a monitoring decreases bounded below. Then it will also have a limit. So, these two results we have already discussed it; is it not? So, just I will just result, the monotonic decreasing sequence, a monotonic increasing sequence, to either to limit , a monotonic increase, non-decreasing sequence, non-decreasing sequence; which is bounded above, bounded above will be convergent, will be convergent. Similarly a monotonic non-increasing sequence which is bounded below, which is bounded below will be convergent in general; a monotonic sequence the limit of the monotone sequence will be either finite or plus minus infinity in general, but these are the convergent . So, either monotonic in non-decreasing or strictly increasing or strictly increasing or strictly increasing or strictly decreasing; that is all this also now, this will be used also to find the limits.

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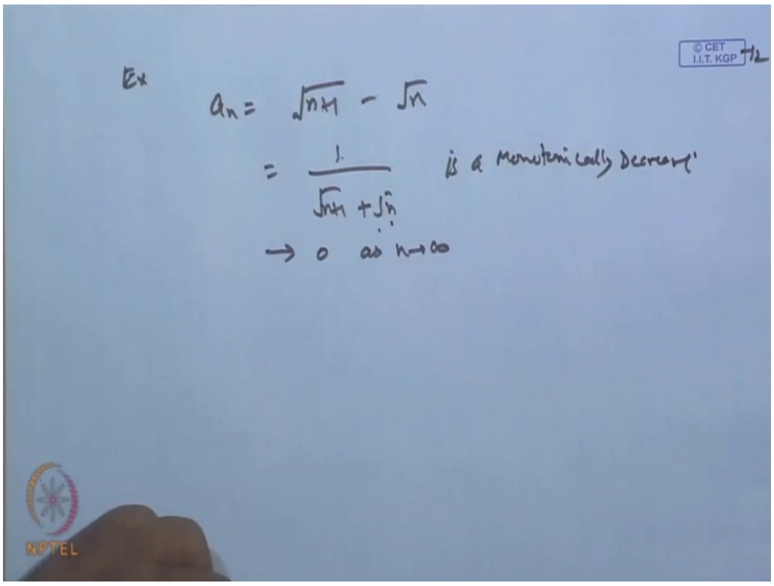

Suppose for example, if we take this a n sequence, as under root n plus 1 minus under root n and I ask what is the limit of this? So, if you got to the limit as n tends the infinity; what you are getting, infinity minus infinity; which in, indeterminant. We cannot get it, but if we slightly, if we manipulated; we get divide and multiply by this. So, when we multiply by this number, then we get a square minus a square the becomes 1 and this is equivalent to this sequence is it not? Now, this sequence is monotonically, is a monotonically what? Increasing or decreasing? Decreasing and tends to 0 as n tends to. So, limit is 0 that is what; is it not? So, we will look the some limits also; which are very interesting particularly that two three elements, which we get as an exercise next time talking.

Thank you very much.