A Basic Course in Real Analysis
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Lecture - 14
Tutorial-II

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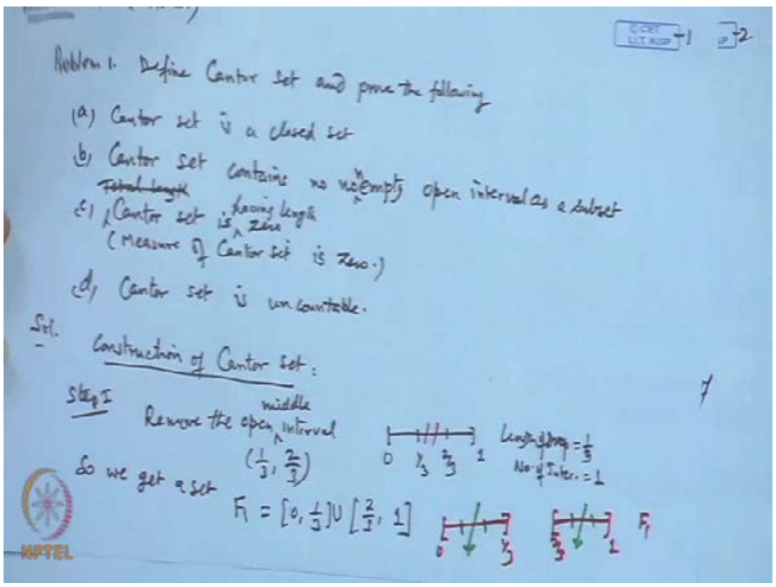

So, we will take a few problems which are itself in interesting problem, itself is a very interesting problem, and also it involves in some other concepts in connection with the sets. So, let us see the first problem. Define the cantor set, cantor set, and so that and prove the following, and prove the following say. Now one or a part, the cantor set is closed set, cantor set is a closed set. Second we wanted to show that cantor set contains, cantor set contains no non empty open interval; cantor set contains or contains no non empty, no non empty, non empty, n o n, non empty open interval as a subset, as a subset.

As third point, which wants to prove that, cantor set is an uncountable set. Cantor set, the length of the cantor set, you can say the, the cantor set has a length, the total length of the cantor set, total length, total length of the cantor set, you can say, cantor set is 0. In fact, the measure of the cantor set is 0. We wanted to show this is that length or we will say the measure is 0. The measure of the cantor set instead of saying total length, we will just say the measure of cantor set is 0. Cantor set is having length 0, length 0.

And d, the measure of the cantor set, I will, because we have not discussed the measure, that is why I wanted to avoid this term measure; and then fourth one is cantor set, cantor set is an uncountable set, is uncountable. So, we wanted to express these following four things about the cantor sets. Let us see the solution. So, first let us define the cantor set. So, what is the cantor sets? The construction of cantor set.

Let us consider a close interval 0 1. Divide this interval into 3 parts. One-third, two-third; and then remove the open interval, and remove the open interval, open interval one, middle one, open interval, one-third open middle interval, one-third two-third; so, close interval 0 1. We are dividing first into three parts and removing this middle portion. What the middle open interval we are removing. So, the set which obtained; so we get a set, let it be denoted by F 2 or say F 1 as the remaining set will be 0, one-third close interval union of two-third, 1. So, in the first step, what we are doing, we are taking a close interval 0 1 and from there, we are dropping the middle portion. This portion we have dropped.

So, the remaining one will be this interval 0 one-third, and then a two-third one. This is our remaining interval; and this we said denoted by F 1. Now what we are doing is now we next again sub divide these intervals, the difference into three parts and let it be divided by this interval again in three parts, this interval also again in three parts and then from here; we drop this one, we drop this portion, this middle open portions.

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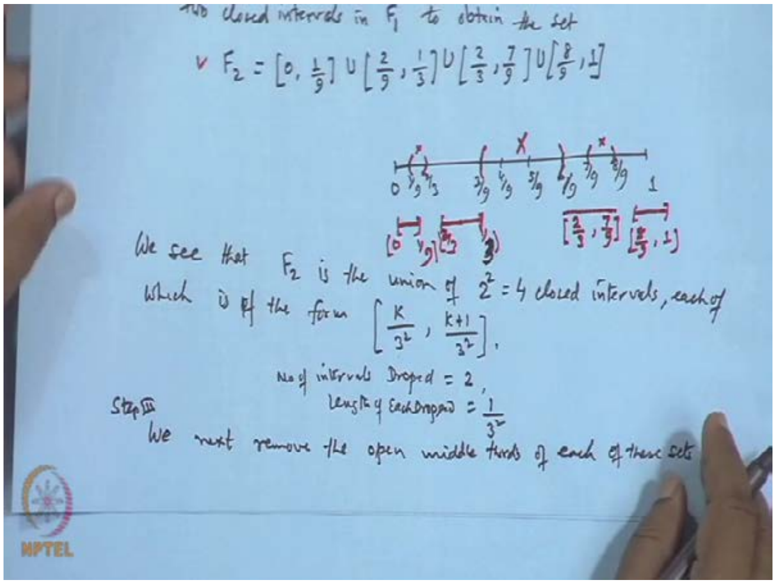

So, what we do we next remove the second step is, we next remove the open middle interval, remove the open middle third, a middle third of each of middle third, we open middle, open middle third of each; each of the two closed, each of the two closed intervals in F 1. Means, we are dividing further these remaining intervals into three parts. And then out of these 3 of intervals we are removing the middle one of an intervals and to get that set to obtained, obtained the set say which we denoted by F 2 as 0, 1 by 9 union 2 by 9, one-third union two-third, 7 by 9 union 8 by 9, 1. That is what. So, basically this is if we have this structure suppose, I have 0 1 then what we if first we have moving this is say 0, 1 by 9, 2 by 9, 2 by 9, 3 by 9 that is one-third, 1 by 9, 2 by 9 and 3 by 9. Let it be then 4 by 9, 5 by 9, 6 by 9 then 7 by 9, 8 by 9. So, these are them.

So, what the remaining portions was earlier we have already removed one-third, two-third this portion was already dropped. Is it not? This portion was already dropped. This was dropped. Then in this interval we are dropping this portion. We are dropping from here this portion. So, these portions are dropped. So, remaining portions will be this 0, 1 by 9 ok then this portion is dropped. So, 2 by 3 and then 1 by 9. This portion, then this portion is dropped. So, here 6 by 9 means 2 by 3 then 7 by 9 and this portion is dropped. So, this one is, then this one is 8 by 9, 8 by 9 and 1. So, basically these intervals are there. Close intervals are left. Union of these is we denote by F 2.

Now each of these F 2 we see that, we see that F 2 is the union of, union of 2 to the power 2, that is 4. Close intervals, close intervals each of which, each of which is of the form, is of the form K over 3 square, K plus 1 over 3 square. You know, because this is also K 0. So, you are getting 0 and 1 by 3 square. When you taking case 2 you are getting this one, K is equal to, I think this is 1 by, 1 by 3, here is 1 by 3; so, 1 by 3. So, here getting this one and similarly when you take this K is equal to 2, you are getting k is equal to 6, you are getting this one, k is equal to 8 you are getting this one. Like this. So, all these, of this form and then we length of this, if you look the basically the length of the first one dropped one, which you have dropped, what is the length of the dropped? Interval is one-third and here an only one number of intervals we are dropping, number of intervals which is dropped is 1.

Now here the number of intervals, number of length of these intervals, the number of intervals dropped is how many? 1 and 2 and length of each interval dropped. Dropped interval will be what? A length of each dropped interval will be 1 by say 3 to the power p. is power. So, each one will be of a length say, this one 1 by 3 square so. That is the first interval. 2 intervals each length of 1 by 3 square 9 length; we are dropping for this like this, like this. So, continue this way. So, if we continue, we continue this way.

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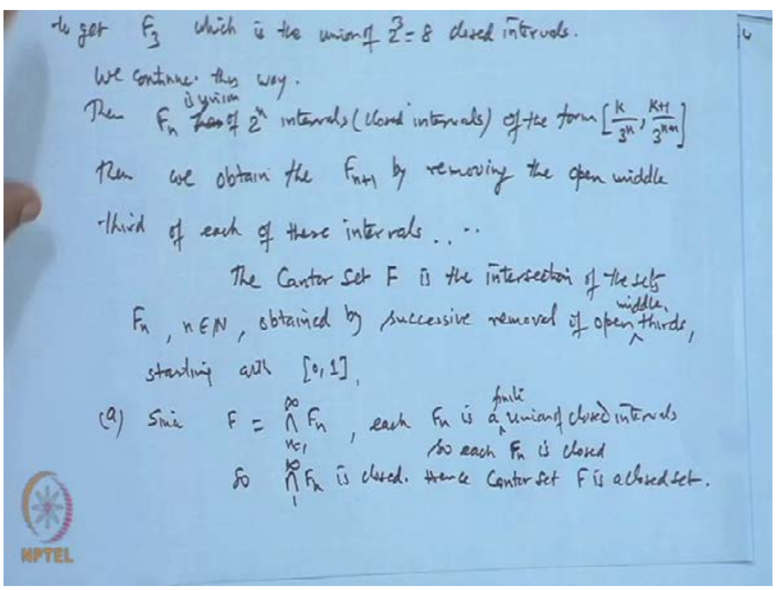

Then what we will next remove, step third, we next remove, we next remove the open middle third, middle third, thirds of each, thirds of each of these sets to get F 3; to get F 3, to get F 3, which is the union of 2 to the power 3 that is 8, closed intervals, close intervals, and like this. We continue this way, continue this way. So, what we get that in the n has step, if we take then, then n, n has step, then F n at the n has is step has, has, we construct it has, has, 2 to the power n, 2 to the power n intervals, close intervals, of course to the closed intervals, intervals of the form, of the form K over 3 to the power n, K plus 1 over 3 to the power n plus 1 like this, then we obtain. Then we obtain the set or then F n is the union of 2 to the power n intervals, intervals of the form this one. Then we obtain the set F 3, the set F n plus 1, by removing the middle again, by removing the open middle, open middle third, open middle third, of each of these intervals, intervals and like this.

Then, what is cantor set? The cantor set, cantor set denoted by say capital F is the intersection, is the intersection of the sets, of the sets F n when n is a integer; a set of a natural number of positive integer, obtained by successive, by successive removal, removal of open third, intervals open thirds, open middle third, open middle thirds, middle thirds, a starting with the close interval 0 1. So, this set F will be the cantor set. So, basically the cantor set will be the collection of all the end points of these, these removed intervals. So, in fact we get since is closes so we know. So, basically this set is the collection of all the points which are the, which are the corner points of this deleted intervals.

Now we wanted to estimate these results. The first result says cantor set is a closed set, which we want to show. So, now, first is since cantor set F is the intersection of F n, n is one to infinity countable intersection of F n and since each F n is closed, is closed interval, is a union of closed intervals. Is it not? And closed intervals finite union of, is a finite union of closed intervals. So, each F n is closed, is closed and when you take, the countable intersection of the close set, then it is closed. So, the countable intersection of F n 1 to infinity is closed. Hence, the cantor set F is a closed set. So, this is the first we wanted to prove.

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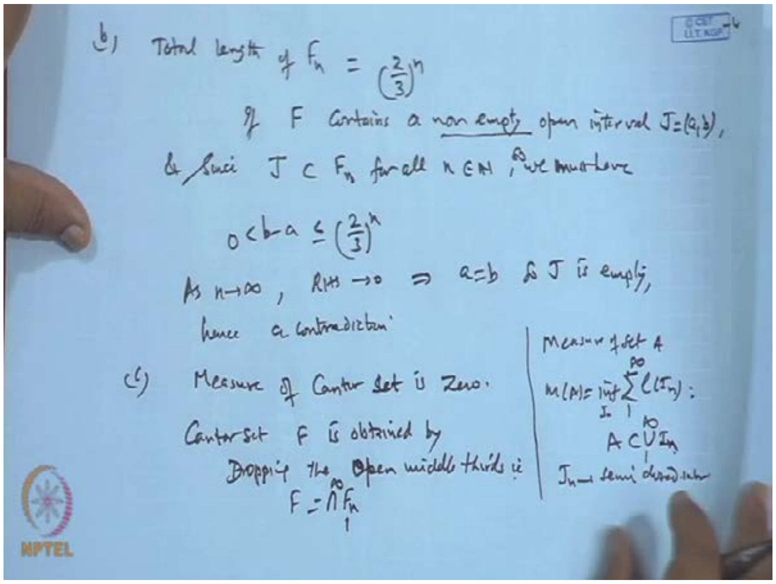

Second one we wanted to show that, cantor set contains non non-empty intervals as a subset and then length of this cantor set or measure of the set is 0. So, suppose what is our append b part. The total length of F n, the total length of F n is basically what? Is nothing but the 2 by 3 n, because if we look that one the first one interval is of length when you choose the F 1, the F 1 this length is one-third plus one-third. So, it is equal to one-third plus one-third that is 2 by 3. When n is 1 it is 2 by 3, when n is 2 the total length is coming to be this, n is 2. So, what is this is, this is 1 by 9, 1 by 9, 1 by 9, 1 by 9. The four intervals of length 1 by 9. So, it is 2 square by 3 square, that is when n is 2. It is 2 square by 3 square. So, 4 by 9. So, similarly when F n come it length will be two by three n power of n, so in general.

Now, if suppose, if suppose F contains, if F contains a, suppose F contains a non-empty, non-empty open intervals say J, of say a b, then F is the intersection of all these opens intervals so; obviously, F1 covers F2 cover F n and so on. So, basically we say this interval a b will be so since and since J this is this opinion twice contain in F n for all n this is true, is it not? For all, J has contained in this.

So, the corresponding; so, we must have, must have the length of this is b minus a which is positive, must be less than equal to the length of this F n, but as n tends to infinity, this right hand side of this goes to 0, the right hand side goes to 0. So, this implies that a is equal to b. It means we do not get any intersection. So, J is empty, J is an empty set. Empty hence a contradiction, hence a contradiction because what we assume that F contains a non-empty open interval. So, which is not true contradiction. Hence the cantor set will not contain any of non-empty open interval, interval as a subset. Now, third part is the measure of this cantor set, measure of cantor set is 0. In fact, I do not want to interview the measure, what is the measure suppose I take the interval a b; then when we say a b is interval, open interval close interval the length of the interval is b minus a. So, that is called the measure of the interval.

Similarly when we say other set, then the set is there then it is difficult to find out the interval, because if it says set solve, set order. Then a line interval a b, then if you take b minus a, then the length of the set or the measure length is not exactly same. Because it is more what it will contain some more points it is not available in the set. So, in order to get the measure of the set we introduced the concept of the measure in terms of the length, and in fact, when you say measure of the set.

Then this is the infimum of measure of a set A means, it is denoted by A, it is the infimum value of a sigma, the length of the interval I n 1 to infinity, such that the countable union of I n cover say. Where I n is a intervals, open intervals or semi closed interval which covers I n. I n s are semi closed all open intervals, intervals containing this. So, when you take the length of the interval is possible, find the sum and take the infimum all over the such I ns; we get this infimum exists, then we set the measure. So, not sure in the rough sense which say, measure of the interval is nothing, but the length of the interval. Since the set a F which is a cantor set contains the points, basically the points end points of this removed intervals. So, what we want is so, the length of the total set is 0. A measure of this set is 0.

So, let us see the, the total length of the removed interval. The cantor set F is basically is obtained, is obtained by dropping the middle, the open middle thirds. When you divide the whole close interval in to the 3 parts and opening and then middle third you are dropping. So, is obtained by dropping this. So, remaining one is nothing, but the cantor set. When n is sufficiently large is obtained by this n, for n is sufficiently large, middle third in successive, middle third of this is. So, that is intersection of F n, that is the intersection of F n will be intersection of F n 1 2 infinity that will be the cantor set is it not that is F. Then F ns are the remaining one and then take the interest.

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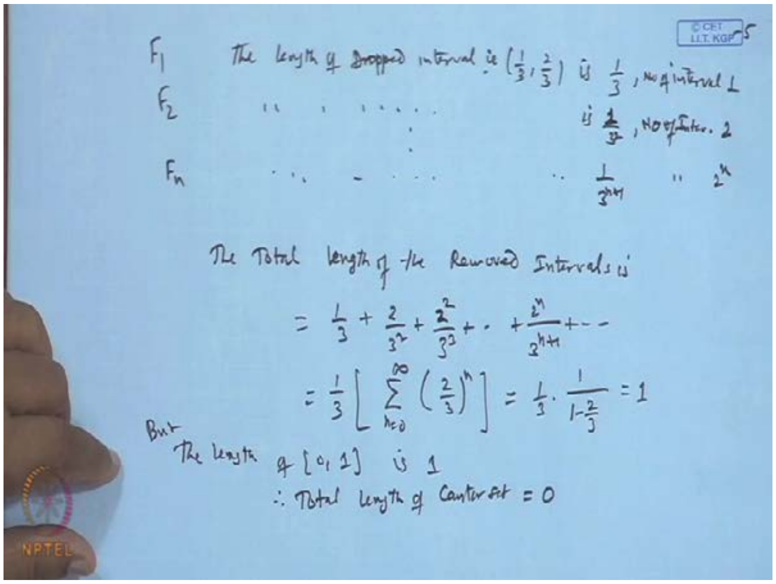

So, when it drop the middle one then what happens to this in the first case F 1, the length of the dropped interval that is one-third, two-third, is one-third. The in case of F 2 the length of the dropped interval is two-third; and number of the intervals and numbers of the interval is 1. Here, the numbers of the intervals is 2. Then continue this. So, in case of F n, when you taking the F n then basically it is the length of the dropped interval will be 1 by 3 to the power n plus 1, 3 to the power n. This is number is 1. Sorry This 1 and then each 1 is 1, but the length will be F 2, 3 square sorry 3 square number is 2. So, here when it is in that second case the number of the interval will be 2 to the power n like this. So, it is 1.

So, the total length, the total length of the removed interval, intervals is nothing, but what one-third plus 2 over 3 square plus 2 square by 3 cube and so on plus 2 to the power n by 3 to power n plus so on. Basically this is equal to one-third sigma n is 0 to infinity 2 by 3 power n. Now if I yet this is a geometric series, the geometric series the first term is 1. So, it is one-third, 1 over a minus r and the length is coming to 1. So, it means when the close interval is there from the close interval, the length of the close interval from where we are started is 1 and the dropped interval is also length 1 and the remaining one is the cantor set. Therefore, the total length of the cantor set is 0, the total length of the cantor set is 0, that a measure of the set is 0. Clear now?

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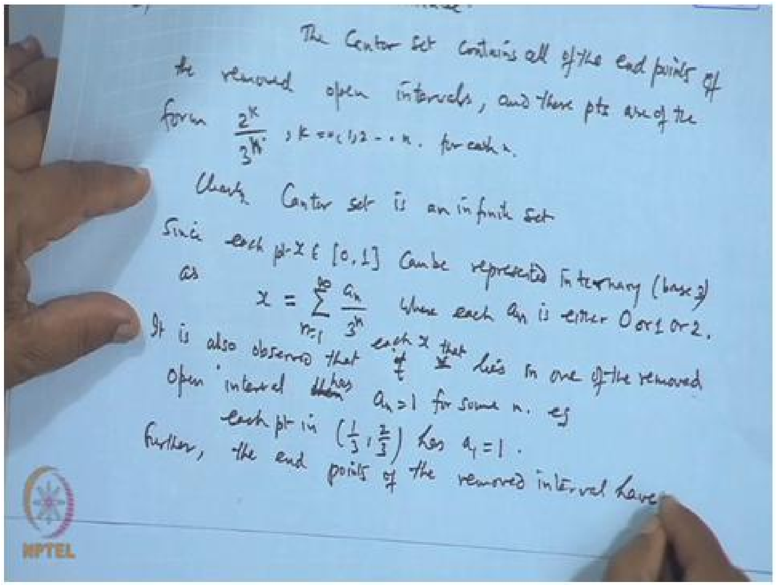

The next the last one which we want to show the d part, the cantor set is uncountable, is uncountable, As we seen the cantor set the, cantor set contains all, all of the end points, all of the end points of the removed open interval, open intervals and these points and these points, these points are of the form, are of the form 2 to the power k over 3 to the power k n 3 to the power n where k is 0, 1, 3, up to say n for each n, for each n. The collection of these points is the cantor set; we wanted to show this cantor set is uncountable. So, these are the infinite first thing is the cantor set is the infinite set because, there are so many points are available; So, it is an infinite set. Clearly cantor set is an infinite set, infinite set.

Now let us take if we take any point x belongs to the interval 0 to 1, then we can be since each point x belongs to n can be represented, represented in ternary form, in ternary that is with the base 3, this is the base 3 and its expansion will be as expansion will be x equal to sigma n is 1, 2, infinity a n over 3 to the power n where each a n, each a n is either 0 or 1 or 2. Any point in the close interval 0, 1, a real number; we can express it in the decimal expansion in that expansion in ternary expansion with base 3 is for. So, we can write this form is a1 by 3, a2 by 3 square, a3 and we are a1, a2, an, may be 0, 1 or 2 this will be done.

Now if x lies in one of the a it is observe, it is also observed that if x lies in, that if x lies in one of the removed, one of the removed open intervals, open interval in one of the removal is a then, then at least a then a n is will be 1 for some n. for example, suppose I take the interval the each point suppose I take for example, each point in the interval one-third, two-third this is the dropped interval has first term a1 is 1; similarly the end points of the removed interval similarly. So, this is to for each x it is observed that you can write it that each x, each x that lies, that lies in the one of the open interval has, has a n is to be 1 for some n, this is true.

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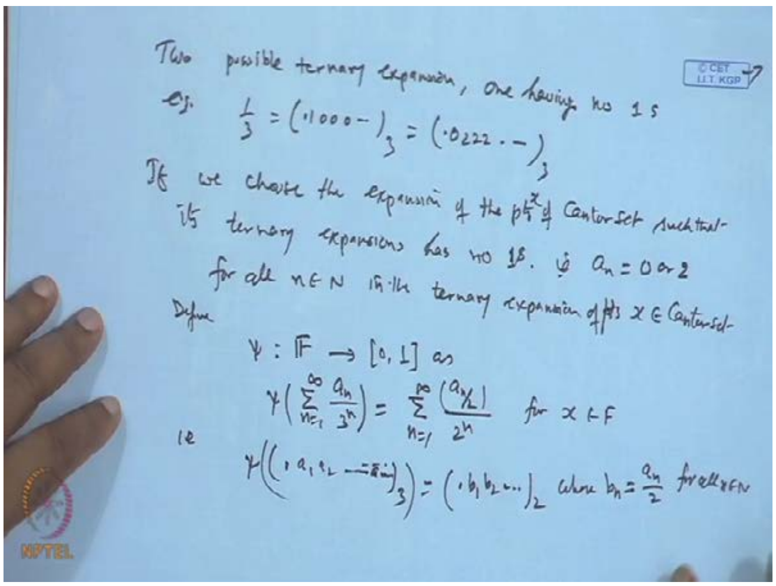

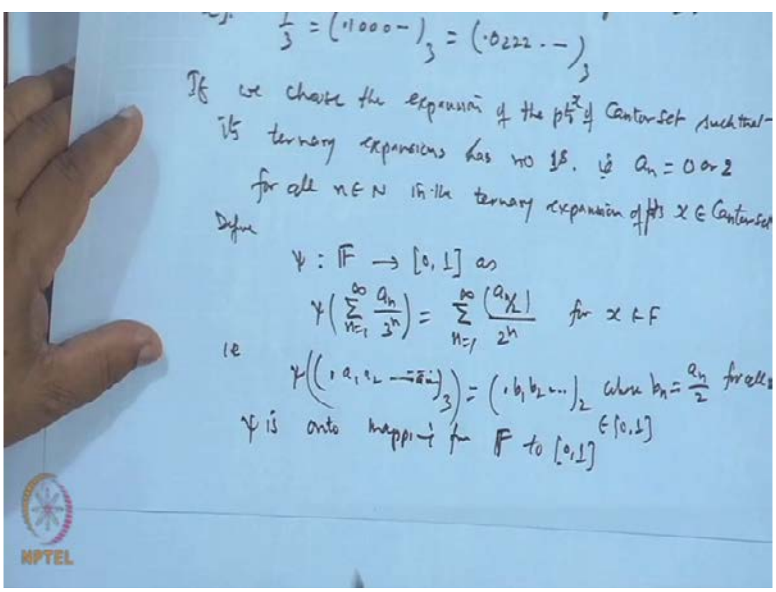

Now in the interval say end points further the end points of the, end points of the removed interval say one-third, two-third have two possible ternary expansion, have two possibly in ternary expansion, possibly in ternary expansion, expansion one having, one having no 1s, no 1s and set other may be some 1s or 2. For example, for example if I take this one-third, the expansion of the one-third in ternary will be 0.1000 third is. If I take the same one then we can also put it in this form 0.0222 if is a approximately you will get 1.

So, one of these expansions will involve 1 s and others may not have a 1s also. So, at least 1 will have no ns will not be 1. So, these were. So, what, will advantage now we choose that expansion, if we choose and these are the n points all in the cantor, cantor sets now we choose the expansion, we choose the expansion of the points of cantor set x, expansion of the points say x of cantor set in such a way such that it is ternary expansion, ternary expansion, expansions has, has no 1s, no 1s, that is we wanted to have it no 1s. That is the ternary expansion will involve a n which may be either 0 or 2 for all n belongs to N in, in the ternary expansion of each point of points x belongs to cantor set; this we are assuming.

Now define a mapping psi from the cantor set F to say close interval 0,1 as follows, if I take a point x whose ternary expansion will be 1 to infinity a n 3 to the power n, this is the ternary expansion and we are doing is we are taking the image of this under psi as the point whose binary representation is this an by 2 divided by 2 to the power n; that is for each x belongs to F. That is what we are doing is that is the image of this psi a 1, a 2, a n, and so on, in ternary expansion sorry and so on, this is ternary expansion of this point in ternary expansion is nothing, but the expansion of this same point in binary mode b 1 b 2 b n etcetera in binary 2; where b n is will be an by 2 for all n belongs to N, for all n belongs to N. So, this is about this each point I am picking up from here writing down is ternary expansion and then image of this we are taking the image as a binary expansion of this end points, binary expansion for this . So, what we are doing this binary expansion will also be a point in 0,1. So, each point which is in F will have a point in 0, 1; which is in 0,1. So, this psi is a onto mapping from F to, F to 0,1.

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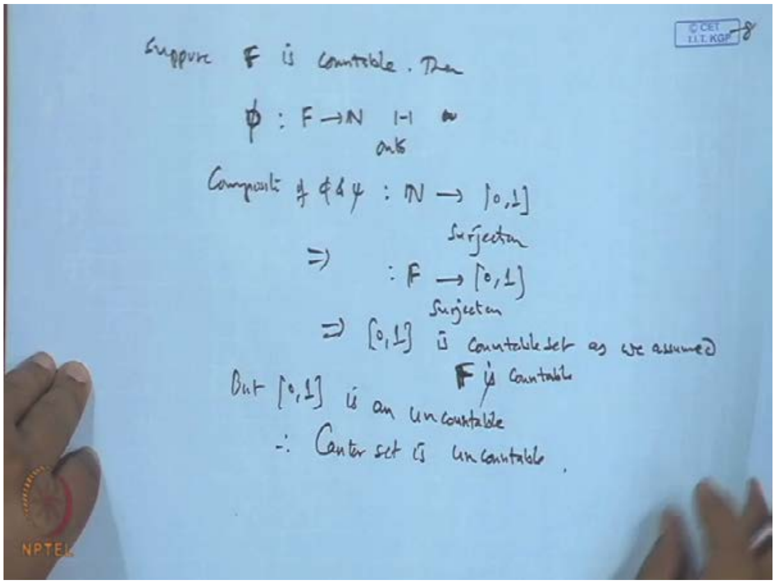

Now let us suppose the F is countable, suppose F is countable, this cantor set F is countable. So, if it is countable then there exist a mapping say phi, phi from F to n or n to F; 1 to 1 correspondence with the which is 1 1 correspondence 1 2, 1 1, 1 2, 1 1 mapping correspondence for this. So, they will exist, F psi which is 1 1 from this to this surjection mapping. F is countable the surjection of 1 1, 1 2, onto.

So, if we combine this thing it means when we combine this then what you are getting is, that the composite mapping composition of phi and psi this will a mapping which brings the n to 0,1, n to 0,1 as surjection, this is a surjection mapping 1 2 mapping. So, if we assume F to be countable then basically this implies that this is F and this are having the surjection; there is a mapping from F to1 which is surjection 1 1, 1 2 a 1 2 mapping. So, once it is surjection then since we have assume; F to be countable. So, this implies that 0 1 is a countable set as we assumed, assumed F is countable, F is countable. This is our assumption, but 0 1 is an uncountable set, but 0 1 set of all real numbers in between the 0 1 that continual is an uncountable which we have already sawn. So, our assumption is wrong; therefore, this implies that cantor set is uncountable and that is proved that. So, this is what we.

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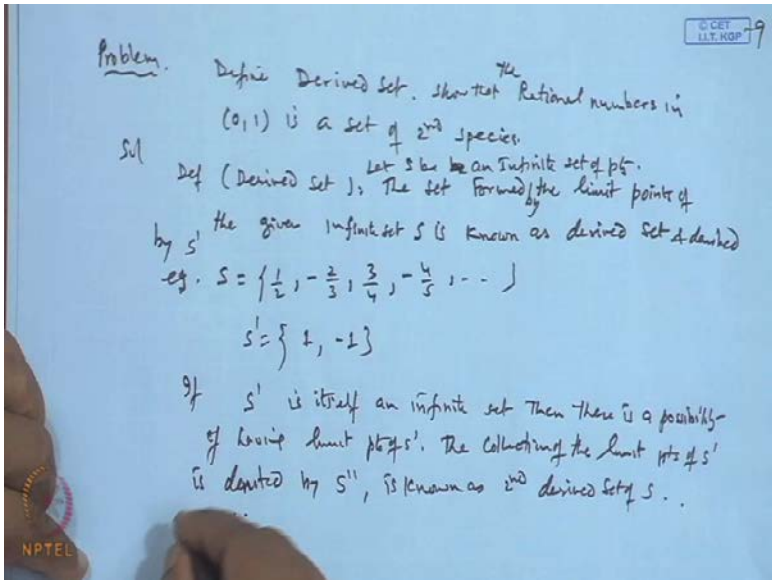

Now another examples, which want to show; define the drive sets, problem- define drive set, define drive set, and show that the rational numbers, the rational numbers, numbers, in the interval 0,1 is a set of second is species, is a set of second species. So, let us see first what is the drives set, what is the species, species we mean; drive set if the set S is infinite set, then what we do is, we can find out the limits of this; we can find out the limit of this say infinite set may have a many limit point or may not have a limit point also depending on this set; so if the set is infinite set, there is a possibility of having the limit points. So, we define the drive set as follows.

The set found, let the set found by the limiting points, by the limit points of the set s of the given infinite set, infinite set is known as the derived set, because finite set does not have limit point, that we have already seen. So, that is why we are taking derived set, derived set. So, suppose I take the set s which having say point half, minus two-third, 3 by 4, minus 4 by 5, and so on. Now the derived set and is denoted by, denoted by s dash by s dash. So, let s be the I will write like let s be an infinite set, be an infinite set of points, points then the set form, form by the limit points, by the limit points of the given infinite set s is known as the derived set and denoted by s dash. So, for example, if we take this one, then what are the derived set 1 by 2, 3by 4 etcetera the limit point will be 1 and minus 2 by 3 minus limit point will be minus 1. So, this is the derived set having this point.

Now in case if derived set s is itself an infinite set then there is a possibility of, there is a possibility of having limit points of s dash and then the collection of these limit points, the collection of the limit points of s dash is denoted by, is denoted by s double dash and is called and is known as the second, the second derivative, second derived set of s. Continue this like this. So, suppose if we continue this, so, if we continue this suppose we have after proceeding n we have the s n comes out to be finite. So, proceeding this one continue, if the if s dash this is the first derived set, s double dash is the second derived set, s n is the n eth derived set of s these are the derived sets of s.

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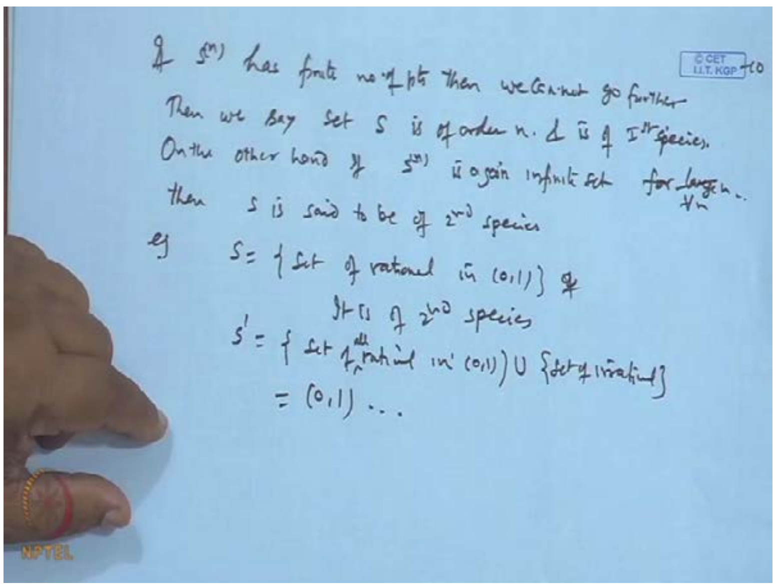

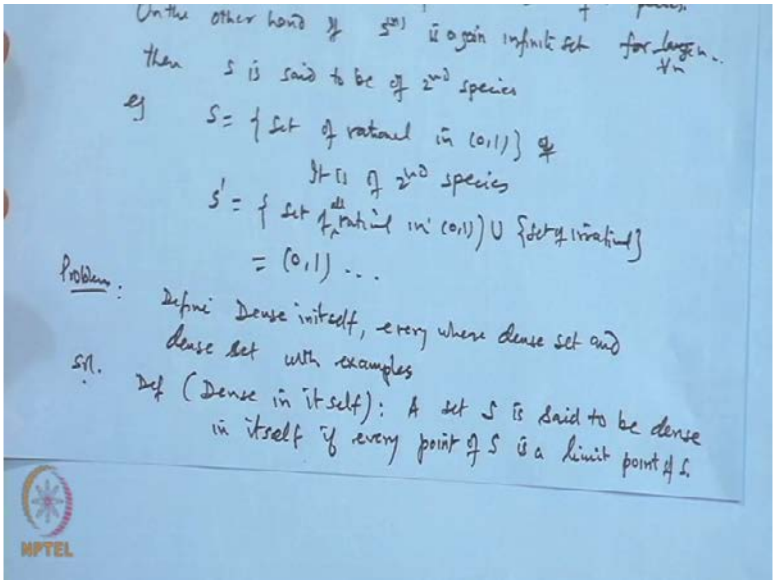

And if s n is having finite number of terms and if s n, the n eth derived set has finite numbers of points only; then we cannot go further, we cannot go further that is we are unable to get the limit point of s n and then we stop it here then we say, we say that set s, the set s if it is the s is there set s contains in next sem and it is a derived set s is of order is of order n and is of first category, and is of first category that is first species, species.

On the other hand, on the other hand if, if s n is again infinite set for large n for large n, large n greater than then s is set to be large n means for every n, for every n you can say then s is said to be, s is said to be of second species. So, this is the way. So, for example, the set of rational number, if is the set of rational numbers in the interval 0, 1 it is of second species, it is of second species. Why? Because the set of limit point of this is the set of rational in the all rationales in the interval 0,1; then set of all irrational will also be the limit of this and basically, this is nothing but the interval 0,1 is self. So, continue this we get every times the derived sets is comes out to be with sets set, that is this possible to get the limit point. So, it is of second category; So, that is what.

Now them we next problems; So, that problem which are define there So, the irrational number of second case species. Then define the dense, define the dense in itself, itself everywhere dense set, dense set and dense set with examples. So, let us see the solution for it, what is our definition for the dense in self. So, let us see the solution; we define the dense in itself, itself. A set s is said to be, is said to be dense in itself, in itself if every point of s is a limit point of s, is a limit point of s. If every point of s is a limit point of s, then we say is this set is dense.

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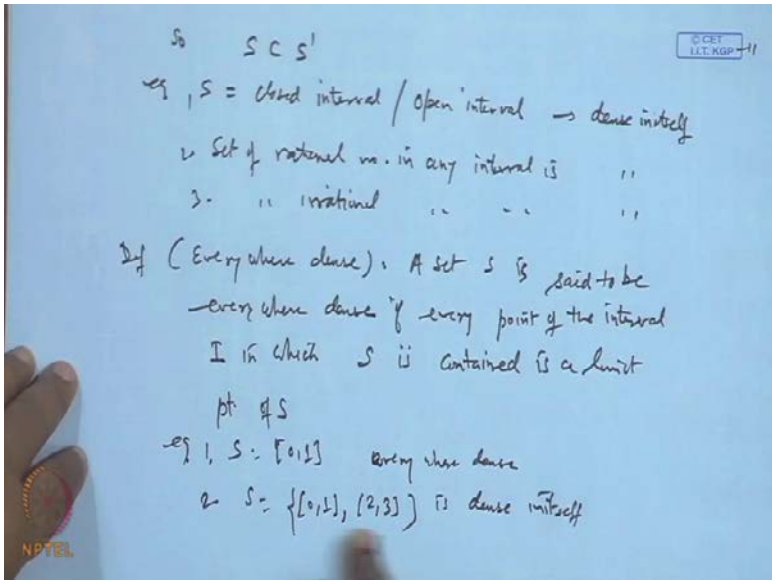

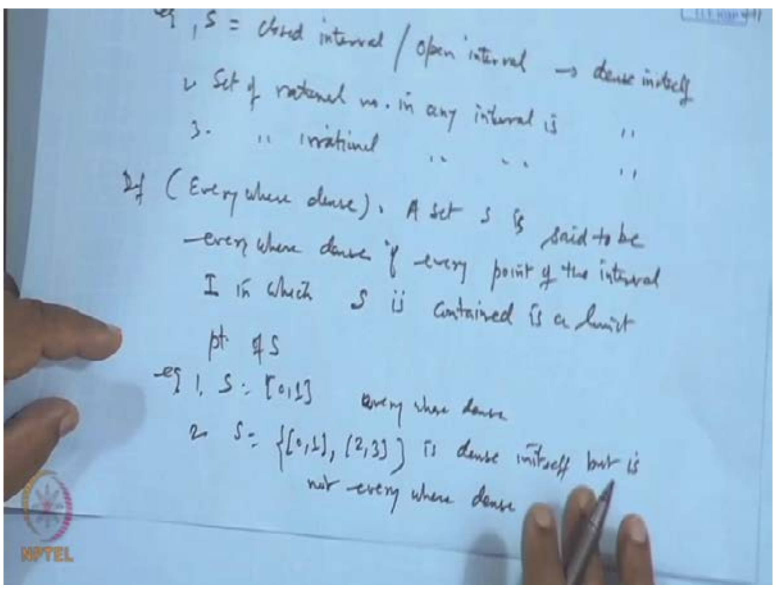

So, in such a case, so, if s is a dense set itself then it is subset then. So, s will always be subset of s dash, because every point of s is the limit point; So it is the s dash will contain all the point of s, s dash the. For example if you take the set s as a closed set, set s is the closed set closed intervals or may be open intervals, closed intervals or may be open intervals, these are all dense in itself, dense in itself dense in; then set of rational number, irrational number and the set of irrational number, rational number, set of rational numbers in any interval is dense in itself; set of irrational numbers in any interval is dense in itself and like this. Now we define the everywhere dense it, a set s is said to be, is said to be everywhere dense if every point of the interval I, interval I in which s is contained s is contained is a limiting point, limiting point of s means what a set is said to be dense everywhere dense if the interval in which s is lies every point of that interval which s is lies must be limit point of s.

For example, for example, it will take this closed set then for example, if we take this closed sets or closed interval s is the close interval 0,1 then this is everywhere dense because, the interval which it lies will be that set closed interval, but if we take this one dense, but this is everywhere dense, everywhere dense because the intervals. But if we take the set S as the union of this interval 0,1 and then 2,3 if this set if I take collection all point in between this then this is dense in itself, in itself because dense in itself means every point of this is a limit point, but is not everywhere dense .Why? It is not everywhere dense set, because a interval 0, 1, 0, 3 does not contain the point in between 1and 2. So, it is not a dense in.

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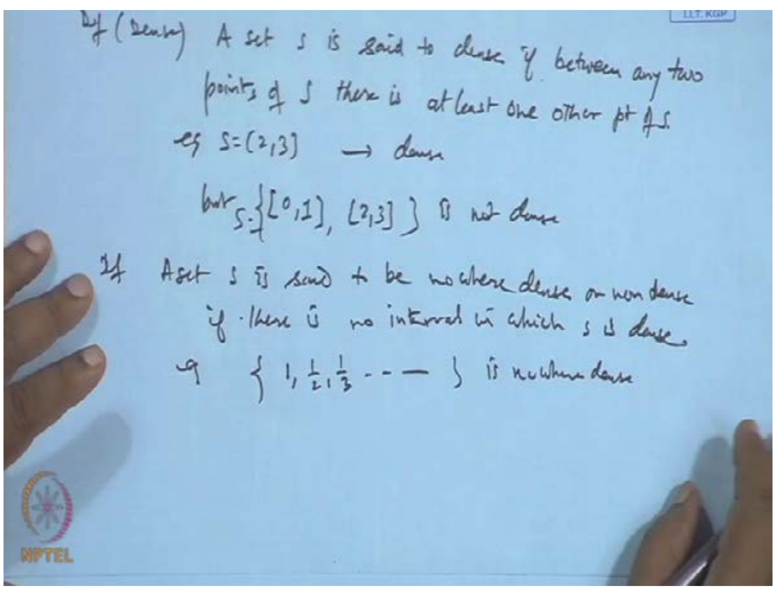

Then a set is said to be dense set and last dense which a set s is said to be dense said to be dense, if between, if between any two point, between any two points of s, s there is, there is at least one, at least one other point of s, other point of s, then it is a dense. For example, if we take the for example, if we take the interval 2, 3 this is our s, then this is dense set because between any two point there is a point of the interval, but if we take this set say 0,1, 2,3 this collection s is not dense, is not dense because, between any 2 point we do not get one between 1and2 the point which is available here; So, that is what is dense. A set s nowhere dense set a set s is said to be nowhere dense or non dense if, if there is no interval, no interval in which s is dense, s is dense; for example, for example, will be take the set of rational numbers 1, 1 by 2, 1 by 3 this collection is nowhere dense because, we cannot get an interval in between this whichever.

Thank you very much.

Thanks.