**A Basic Course in Real Analysis**
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**Lecture - 13**
**Weierstrass Theorem, Heine Borel Theorem, Connected set**

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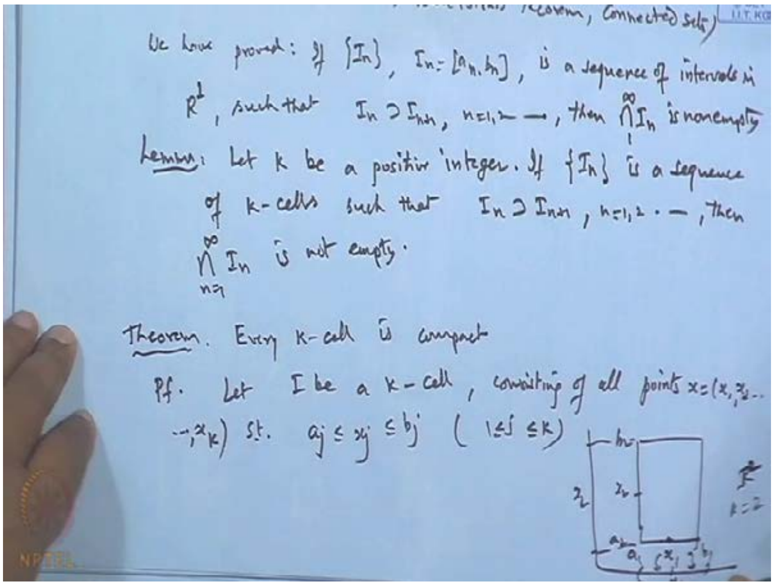


So, before going for this theorem Heine Borel theorem, Weierstrass theorem we require one or two lemmas, which will be needed in the proof of these results. We have seen already, proved the following theorem that if I n, where the I n is say close interval a n b n is a sequence of, is a sequence of intervals in R 1 that is real line, interval of the real line, closed interval real lines such that, such that they are of decreasing natures, such that I n covers I n plus 1 and so on for n is 1, 2, 3 and so on.

Then the intersection of I n, when n is 1 to infinity countable intersection of I n is non-empty. This result we have seen. Now, extending this results we have, we can extend this result to R k space; let k be a positive integer, positive integers if I n sequence I n b is a sequence of, is a sequence of k cells, is a sequence of k cell cells such that I n covers I n plus 1 for n is say 1, 2, 3 and so on, then the countable intersection of the cells 1 to infinity I n is not empty. So, the proof lines on the similar line as we have done it. What we will do we will fix up say j each I and j we can take it, the, for 6 j and apply this one, this proof we will get the results. So, I am just dropping what we are interested which is important part a, that each or every k cell k cell is compact, every k cell is compact. So, 1 cell means close interval, 2 cell is a rectangle close rectangle of course, and third cell, fourth cell and like this. So, every k cell is compact. So, let us see the proof of this first. Suppose I be a, let I be a k cell consisting of all points, all points of the type say x 1 x 2 say x, x 1 x 2 x k, x 1 x 2 x k such that the x j lies between the bond a j less than equal to b j, j is 1 2 k. Let this be a k cell.

So, here this is our set k is 2. So, it is in I 2, this I n in 2 cell means, it is in k is equal to 2. So, we are having the cell like this, where the x 1 vary from a 1 to b 1, where the x 2 vary from a 2 to b 2 a 2 to b 2. Say, this is our here. So, here this is x 1, this is x 1. So, this will vary from a 1 to b 1 and here is x 2 which vary from a 2 to b 2. This is a 1 this one. So, this is a case a, a cell in R 2 space, in the R 2 space. This is in R 2 square like this.

Now, what is a is if in general if I take a k cell then I be suppose k cell in that case it is of the it will contain all the points of the type x 1 x 2 x k, where the each x j will lie between this x 1 will lie between a 1 b 1, x 2 will lie between a 2 b 2, s k will lie between a k b k; so this one. Now let us suppose the distance between that is delta, let delta be sigma b j minus a j whole square j is 1 2 k under root this is our.

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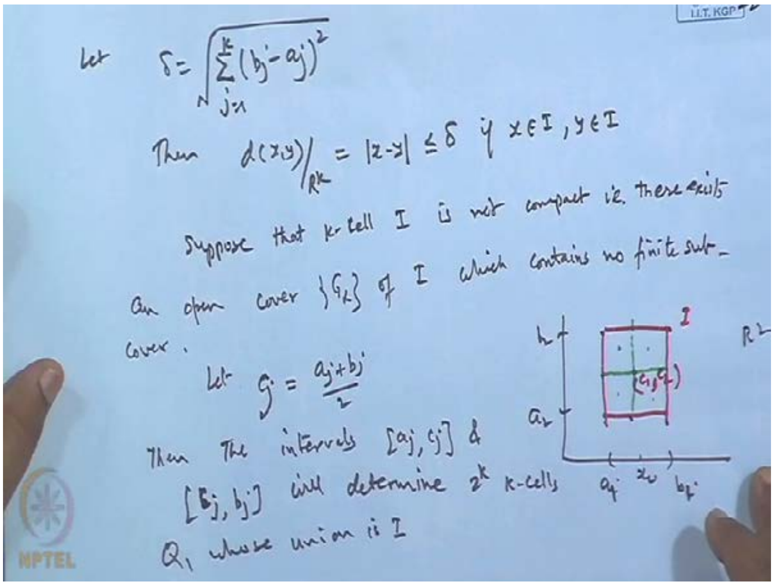


So, if you look this one this is the same as what. So, b 1 minus a 1 square plus b 2 minus a 2 whole square; so, this will be the same as if I picked up the two points x and y in the set, x and y. x is say x 1 x 2, y is say y 1 y 2. If we picked up two points then the distance between x and y in this R 2 is spaced apology, b remains less than this number. So, obviously, then distance between x y in R k space that is we are denoting by same as mode of x minus y will be less than equal to delta when if x belongs to I, y belongs to I. So, x is x 1 x 2 x n, y is y 1 y 2 y n.

We wanted to show this I is compact. So, assume that let us suppose I is not compact, suppose that, suppose that I, the k cell I is not compact. It means that out of open cover of this, any open cover if I choose then it cannot have a finite sub cover. So, that is, that is there exist an open cover say G alpha of open sets, open cover of I which contains no finite sub cover, that is the meaning of this if it is not compact.

So, a finite cover. So, let us suppose this is our say here is a I this is b I and here is somewhere x I. This is our say another point. So, when it take choose let us take cover, let us picked up c j as a point c j which is a j plus b j by 2. It means c 1 I am taking as a 1 plus b 1 by 2, c 2 I am taking as a 2 plus b 2. So, if suppose this is in R 2 space, here this is a 1, this is b 1, here is a 2, this is b 2 and this basically we have this, this is our I interval. So, what we are doing we taking a point here which is the point c 1 c 2, c 1 comma c 2, c 1 is the middle point of this c 2. So, once you take that point it will divide this whole I in R 2 in 4 parts.

This will be a one cell, the another cell like this. So, by choosing this way by choosing c here we can get, then the interval, the intervals a j, c j and d and then the interval s is c j and the interval c j c j comma b j, this intervals will determine the cell, will determine 2 k, k cells, k cells suppose Q 1, k cells Q 1 whose union is I, this is what here just like in case of R 2 I have written that if you take this intervals c 1 c 2. So, a I a 1 c 1 then c 1 b 1 similarly, here we get c 1 b 2, c 2 b 2 and like this. So, we get the 4 interval 2 to the power 2 that is 4 k cells we get it here. Now, if I is not compact then at least one of the cell will not be covered by a finite, will not be compact. That is open cover of one of the cell will not have a finite, there exist an open cover of one of the cells here; so at least one of them union.

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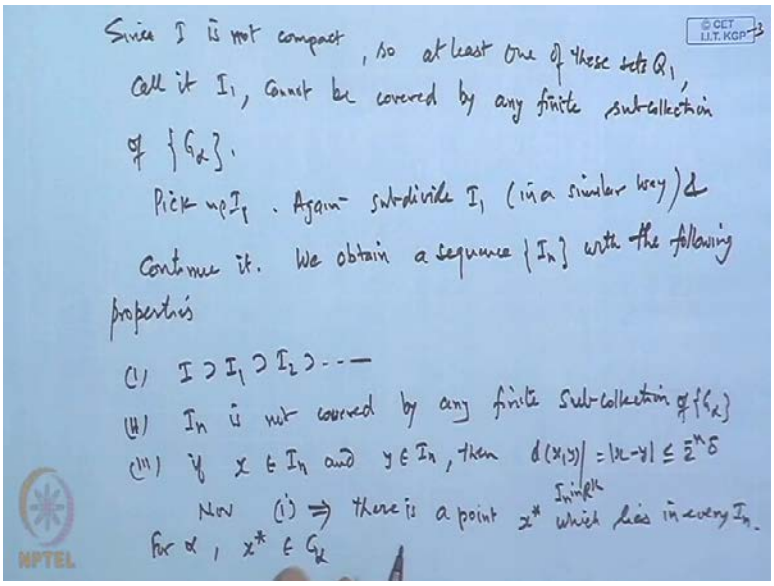


So, what here, since I is not compact; since, I is not compact so, at least, so, at least one of these sets, sets Q 1 set call it I 1 cannot be covered by any finite sub collection of the open cover G alpha. Otherwise if it is so, then I 1 will be compact and contradicts to I; so, that will be next step. Now, picked up now I 1. Now, pick up I 1 now and again divide, again subdivide, this I 1 in a similar way as definitely and continue this process. So, when you divide again you are getting I 1 once a or I 2 is one of the cell, which is not covered by any finite sub collection. So, again I 2 you again subdivide it and like this. So, continue this process. So, when you continue this process, then we obtain continued then we obtained in this process, we obtained a sequence say I n with the following property, with the following properties. The first is this I's which you are getting cell will be satisfy this inequality, I is covers I 1 covers I 2 and so on, there of decreasing nature.

Then second is that I n is not covered by any, I n is not covered by any finite sub collection of the cover G alpha and third is that if x is a point in I n and y is also in I n then the distance of this x y in R k that is mod of x minus y over I n, this is over I n of course, in I n which is in R k. This distance will be less than equal to 2 to the power n minus n into delta.

Because here if this length is divided then what you are getting, this is the one forth basically when n is 2 you are getting the half 1 by delta by 2, delta by 2 and so on and is 3 and like this. So, this will be divided by this I 1, I 2 and so on. Now, what is one? One says so obviously, one will be here, now, one will imply because they are of decreasing nature and any finite sub collection of this will be non empty. So, the intersection of this will be non empty. So, by thus there is a point x star which lies in every I n.

This follows from the results which we have already proved that if I 1 covers I 2 covers I n and if any finite collection of this, that is intersection of this will be non empty just now we have seen that I n’s. Second one is so, if I x star belongs to sum I n. So, I n is a sub is a in k cell, in part of the k cell. So, we can find some alpha. So, for sum alpha x star this will belongs to the, one of the element of, for some element of the open cover say G alpha because we have taken G alpha as a open cover of the I. So, this x star belongs to sum of the G alpha for sum alpha, but G alpha is open. So, this star will behave as an integer point. So, we can find out a neighborhood around the point, this is totally contained in this. So, there exist an r greater than 0.

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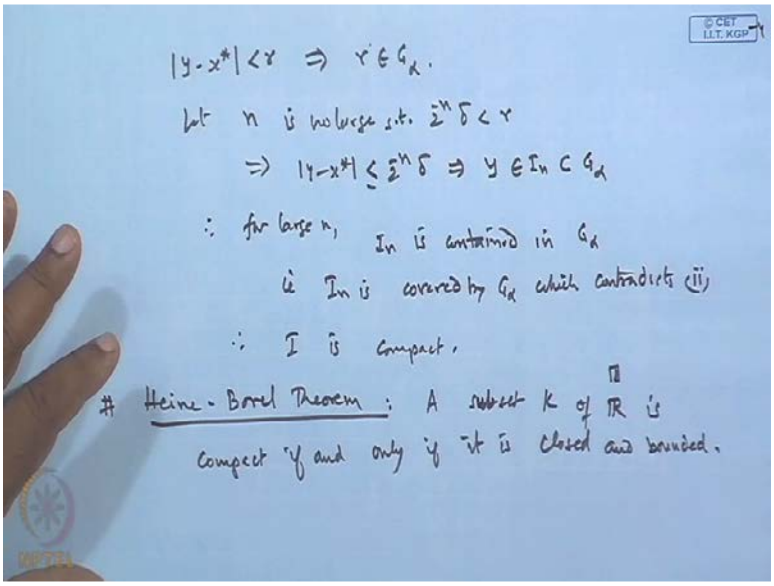


Such that the distance by minus x star is less than r will imply r is in G alpha that is just an integer point that is nothing. Now, r is our own choice. So, let us picked up n, let n is so large, such that this 2 to the power minus n delta is less than r, that is possible because all is once is fixed, you can take n such a large. So, that entire remains less than alpha. So, if it is so then what happens, this implies that by which you are choosing, there basically is in our I n also because I n is satisfy this condition, that if x belongs to y n then this is 2 and this is less than r. So, any element of I n will be the element of r.

So, this implies that the distance by minus x star is less than or equal to 2 to the power of minus delta which implies that the element y which is in G alpha belongs to I n which is subset of G alpha. So, every element of y n I n will be an G alpha for some large n, for some which therefore, for large n of I n is contained in the set G alpha. So, once it contained G alpha, then I n is covered by G alpha. That is I n is covered by G alpha which contradicts the second part of it.

What is the second part says I n is not covered by any finite sub collection, but it is covered by the G alpha, that it is finite sub collection G alpha, contradiction and this contradiction is because of a wrong assumption that I is not compared. So, this so therefore, I is compact. So, this proves the result means every k cell is a compact in particular every close interval is a, in that is a compact set because that is also. Now, this gives a very interesting result which we call it as a Heine Borel theorem.

But basically this theorem says that the statement of the theorem let it be, the theorem is Heine Borel theorem is, if we a subset k, a subset k of R, subset k of R is compact if and only if it is closed and bounded. So, this is a particular when you are choosing R space. So, in general R k space we will prove some results. So, as a particular case we can say and drive the Heine Borel theorem. So, the result is first that is important. The proof of this, the proof follows from the theorem from the next theorem.

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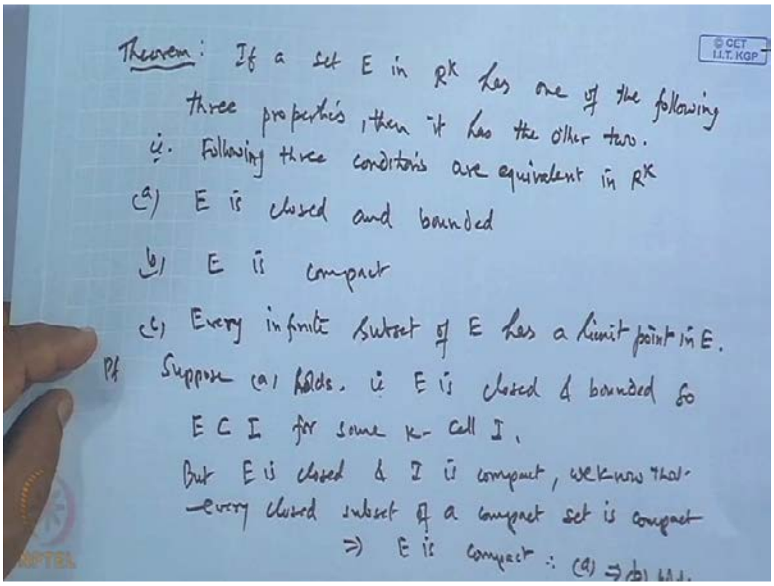

What is this theorem is, the theorem says that if a set E in R k, in R k has one of the following three properties, has one of the following three properties then it has the other two, that is all three are equivalent, that is the following three conditions are equivalent in the space R k. So, in particular when k is 1, it is also true in case of, over the real line what is the condition is, the condition says E is closed, E is closed and bounded.

Second condition shows that E is compact, third condition states every infinite, every infinite subset of E has a limit point in E. So, this thing the proof of this. So, obviously, if it is true then in case of R 1 if E is compact then it is closed and bounded and vice versa. So, the results follows immediately Heine Borel theorem follows immediately. So, lets see the proof. Suppose a holds, it means given E is closed and bounded, but we want so E is compact. That is E is closed and bounded. So, once it is bounded it means it will be enclosed by some cell. So, let. So, E is contained in I for some k cell, k cell I.

But what is E, E is closed, but E is closed and I is compact, that we already shown. So, every close subset of a compact set is compact. So, and we know that every closed this is result be a every case compact and every closed and compact cells are compact. So, this is what we are proved earlier. Closed subset of a compact sets are compact. We know that every close subset of a compact set is compact, this we have already proved. So, here E is closed, I is compact. So, E subset of a compact set therefore, this implies E is compact.

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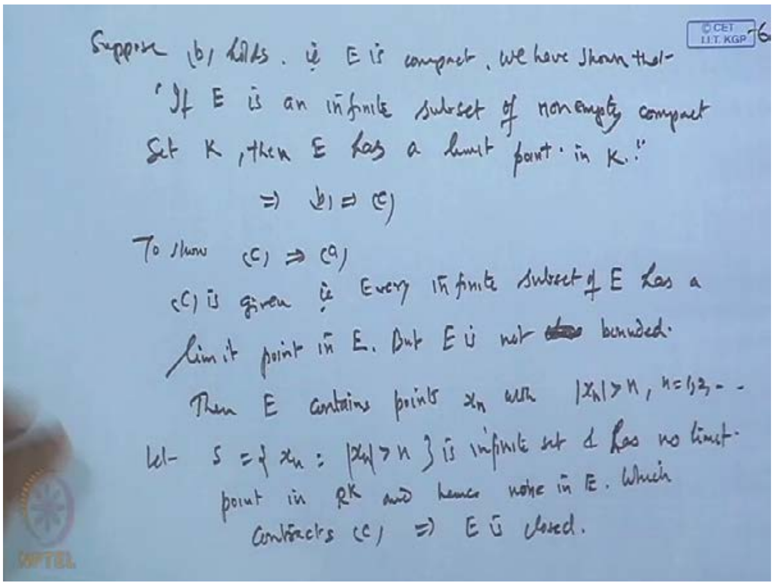

So, once it implies means therefore, a implies b holds. Now suppose b is true. Suppose, b is true then given suppose b holds, b holds then automatacy implies y with the v is mixed is compact, that is E is compact. So, take a sequence, any sequence of the subsets of E. This we result already proved that if E is a infinite subset of a compact set then E has a limit point and we have shown earlier that if E, E is an infinite subset of a non empty compact of non empty compact sets k, then E has, then E has a limit point in k, a limit point in k, this we have shown. So, here E is compact only already given. We wanted that every infinite subset of E has a limit point in E.

So, it what it shows is if this infinite subset of E if I choose and infinite subset of E means we can have with the infinite subset of non empty compact subsets of k then this will have a limit point in it. So, obviously, b implies c. So, that is nothing to prove is this. Now, to show c implies a. This we prove by contradiction. So, given that every infinite set of E has a limit point in it. So, c is given, that is every infinite subset of E has a limit point in E, this is given. So, we wanted to prove a is closed, E is closed and bounded, but suppose, but E is not closed, not bounded let it see first, not bounded.

It means if it is not bounded it means there will be a sequence of the points in E which can whose bound will go to infinity means each x 1 x 2 x n will be there, such that mod of x n will be greater than any arbitrary number n. So, we get than E contains the points x n with the property that mod of x n is greater than n, when n is 1, 2, 3 and so on. x 1 will go greater than 1, x to the. So, the limit of this x n will not exist. So, let s be the set of or such x n such that x n is greater, mod of x n is greater than x.

Then obviously, this is an infinite set, is infinite set because if it is finite then we cannot, we can get the bound for x n. So, it is an infinite set and clearly if a no limit point and has no limit point in R k. This is the set we are choosing in our s. So, once ... So, it has no hence has none in E and hence not in none in E. In this sequence we will not have any limit point in (( )), that is one thing thus is bound. So, what is this? This contradictory, which contradicts c, because c says that if you take any infinite subset of E, it must have a limit point E, s is an infinite subset of E, but it does not have a limit point c.

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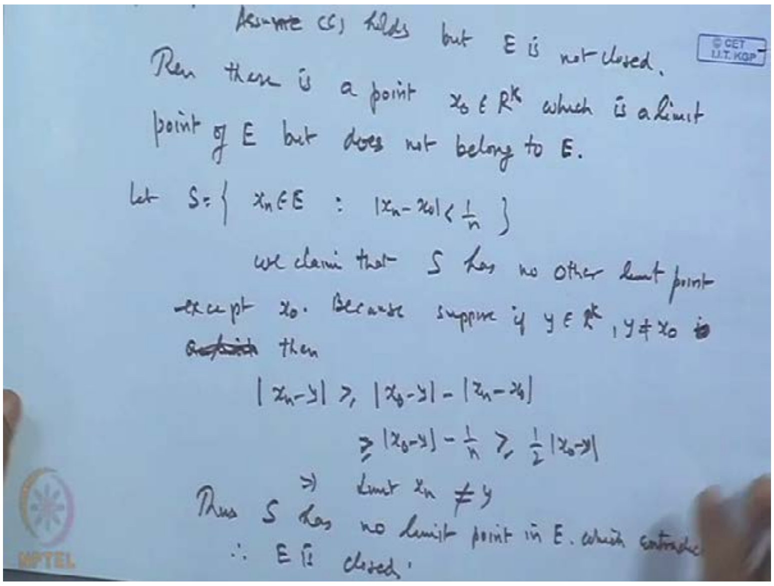

So, this implies that E is closed. Again to show E is bounded, so support further assume c holds, assume c holds, but E is not closed. It means the limit point of the E, all the limit points is not an E. So, thus we can get the limit point then there is a point, then there is a point. Say there is point suppose x naught belongs to R k, which is a limit point of E, which is a limit point of E, but which is a limit, but not a point of, but does not belongs to, does not belong to E.

Because all the elements point does not belongs to E, because E is not closed. So, if it is not, then we can get this sequence of the points in E that is so, the set x n belongs to E such that mod x n minus x naught can be made less than say 1 by n. This collection will be there is inverse a set of all points E which satisfy this. Some a sequence will be obtained we are this.

Now, let us find s with the set of those points. Let s with the set of those point of a is which satisfy this condition. We claim that s cannot has x naught as a limit, s has no other limit point, s has no other limit points, s with no other limit point except x naught because if suppose y is a another point of this. Suppose because suppose if y belongs to R k and y is different from x naught, is a limit point suppose then we get, then the distance between x n minus y because y is a then a limit point we will just (( )) because suppose y is an another point which is different from x then we can say x n y n is greater than equal to x naught minus y minus x n minus x naught this y, but x naught minus y because x naught and y naught is different so, we can just put it as it is, x naught minus y.

Now, x n minus x naught is less than so, minus of this is greater than (( )). Now, n we can choose in such a way so, that the whole thing, this is greater or equal to, the whole thing is greater than or equal to half of this. Now, y and x are different. So, this is fixed point it means that as n tends to infinity x n does not go to y. So, this implies the limit of x n is not y. So, it cannot have a point other than x naught as a limit point, but x naught is a point which does not belongs to our set E. So, this showed y is not a limit point of (( )) thus has a no limit point in E, thus s has no limit because x naught is in not in E. So, x s has no limit point in E, hence contradicts our assumption three, which contradicts c therefore, our E is closed.

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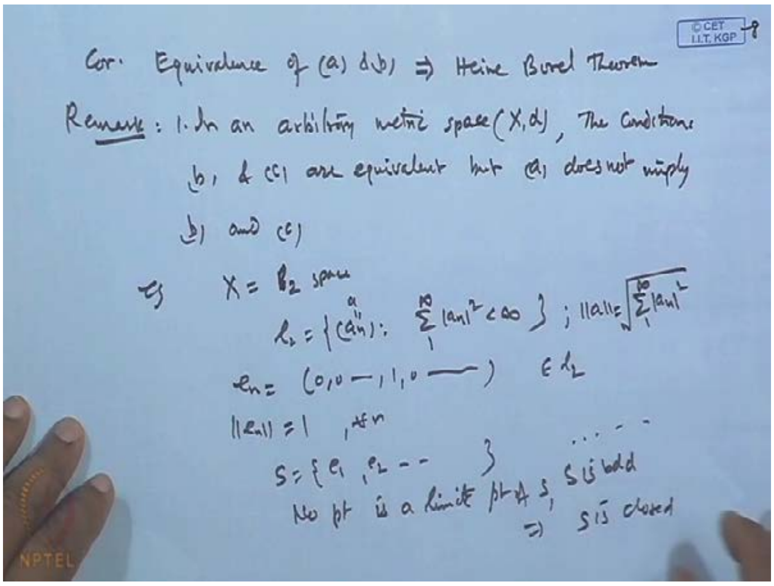


So, equivalence of one and two will implies the Heine Borel theorem, b will imply Heine Borel theorem. That is the proof for the Heine Borel theorem. Now here we put some remark. The remark say that in an arbitrary metric space, in an arbitrary metric space metric space X d, the conditions b and c are equivalent, b and c, but a does not. It means in general b and c a does not imply, but a does not imply, b and c in general. It means for arbitrary metric space the compact set and the infinite subset of E has a limit point are equivalent, but if a set E is bounded and closed then you cannot save it, the it remains compact or it will have a finite infinite subset of E, will have a limit point in it. It is not, may not be true for example, if suppose I take x as a l 2 space, l 2 space means set of those sequences a n such that sigma mod a n square 1 to infinity is finite, in l 2 space and if I take the sequence E and n 0, 0, 0, 1, 0, 0 this is the points belonging to l 2 space. Now, if you take the norm of E n.

Now, here norm of this if suppose this I denote by a, then norm of a of course, this a case of functional (( )), but I will. So, we are not much going in detail, but this is the norm. So, norm of n is 1 for each n therefore, e 1 e 2 E n this set is there, e 1 e 2 n and so on. Then each point having norm 1, is it now. So, it is bounded. So, s is bounded. And none of the point is a limit point, no point is a limit point because it is a point set only, limit point of s. Therefore, we can say all the limit points of s belongs to E. So, this shows s is closed also. So, it is a closed (( )), but it is a infinite set.

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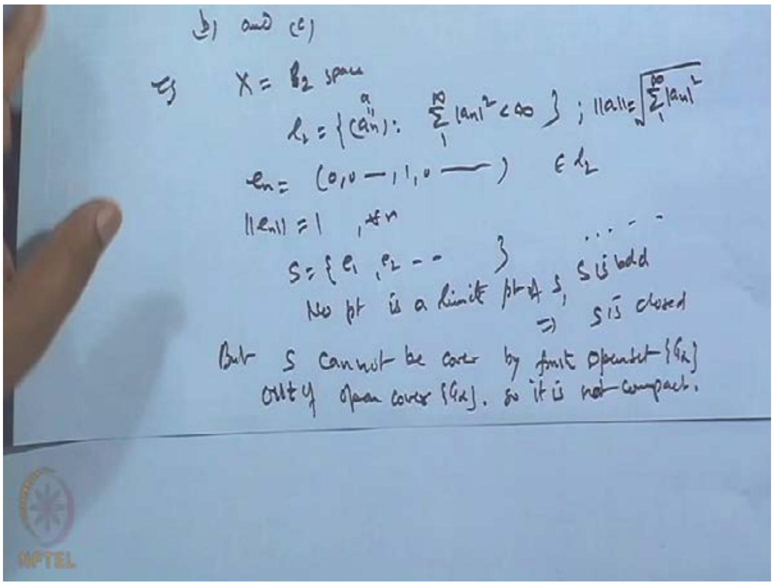


So, we cannot cover it by means of a finite sub cover, but s cannot covered by finite open sets say G alpha out of open cover G alpha means many open covers here. We cannot choose the finite cover which comes because it is infinite. So, it is not compact, it is not compact. So, this all contradicts our. So, that is what, we are not going detail for this because it is a part of the function.

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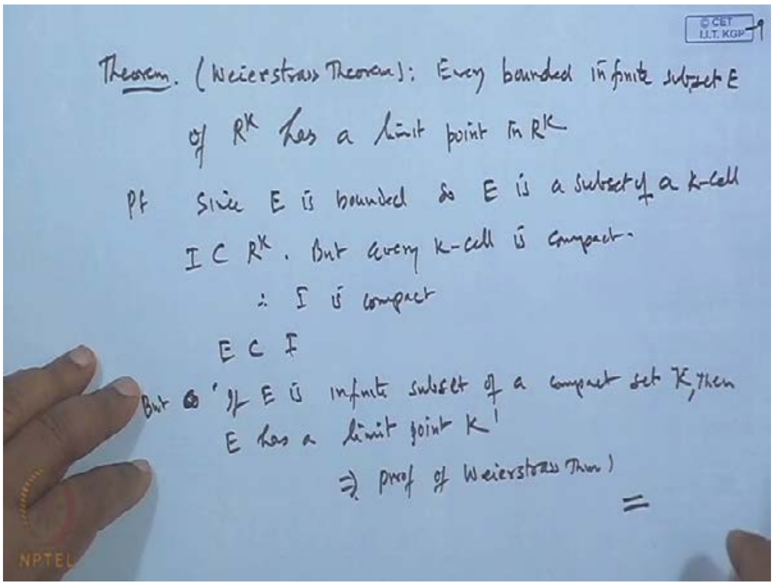


The next result which we say the Weierstrass theorem, the theorem says every bounded infinite subset, infinite subset of R k, every bounded infinite subset of R k has a limit point in R k, the proof is very easy. Since E every bounded infinite subsets say E of R k has (( )). Since E is bounded which is infinite also a set is a subset. So, E is a subset of a k cell I which is contained in R k because it bounded means it will cover by some k cell, but what is the earlier theorem says that every k cell is compact, but every k cell is compact therefore, I is compact. So, E which is contained in I, which is a compact set.

And what this result says the one result which we have already shown that every, if you choose a finite subset of a compact set then E has a limit point in k; so, every infinite subset of compact set. So, if E is an infinite set of a compact set k then E has a limit point in k this we know, but we know this result. So, using this implies that our Weierstrass theorem the proof of the Weierstrass, the Weiersstraas theorem we wanted to prove a infinite bounded infinite subset of a has a limit point of an R k. So, this completes the proof of this. Now, we have one more concept of a set which comes in (( )) is that connected set.

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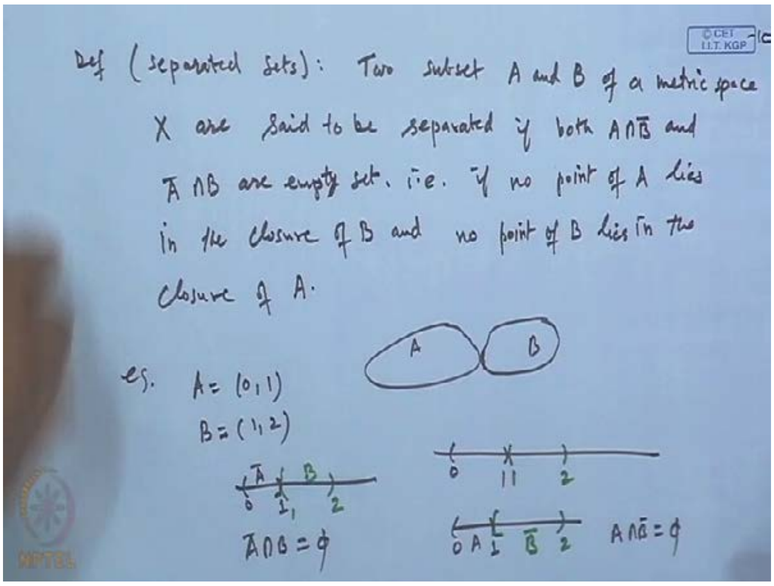


So, let us define connected set. So, first we have a separate set, separated sets. Two subsets A and B of a metric space, of a metric space capital x are said to be separated if both A intersection B closure and A closure intersection B are empty set, are empty sets, are empty set, it means that is if no point of A lies in the closure of B and no point of B lies in the closure of A, then we say this two sets are separate, it means this is the one set A and here is another set B, B not this one.

Now, if I take the limit points of A, all the point of A then it should not belongs to the closure of B and if it take any point of B all its limit point, if it does not belongs to A then we say A and B are separated. For example, if I take this set say 0 1 and 0 1 suppose I take the set 0 1, B is the set say 1 2, if I look this. So, this is the 0 1 and here it is 1 2, this is 1 2, this is. Now, if we look what is the closure of B, the closure of B is this. So, this is our 0 1 is the set A and when you take this then it becomes B 2 that is the closure of B 1, B 1 which includes 1 and open at 2, but 1 is a point which belongs to only B 1 does not belongs to A. So, neither the point of B nor its limit points belongs to A.

Similarly, if we take the others say this is 0 and 1, this is the closure of this and the open sets this one is 0 2, 1 2, 1 2 is our B. So, A 1 intersection B is empty and A 1 intersection B is empty, then here A 1 A intersection B 1 is also empty. So, A and B are separated set. There is a difference between the disjoint set and separate one. Separated set of course, are disjoints but disjoint sets may not be separated.

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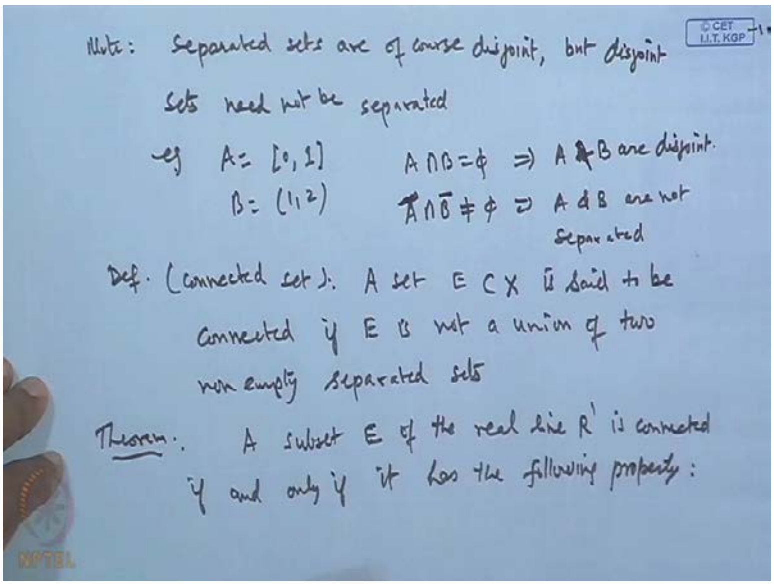


Separated sets are of disjoints are of course disjoint set but disjoint set, disjoint sets need not be separated for example, if you look that interval 1 2, A is the set suppose I take the interval close interval 0 1 and B is a set which is an open interval 1 2. Now, A and B are disjoint. A intersection B is empty. So, this implies A and B are disjoint, but A bar intersection B is not empty, A bar intersection B because no A 1, A intersection B 1, sorry. A 1 B, A intersection B 1 is not empty because when you take the closure of this 1 is the limit point, 2 is also limit point. So, intersection will include 1, but the separate set said if both these are empty set. So, here at least this is not empty. So, this shows A and B are not separated. Now, we have defined the connected set now. A set E which is subset of x is said to be connected if E is not a union of two non empty separated sets. So, this now one result we have and this connectiveness over real lines, what it says is a subset E of the real line R 1 is connected if and only if it has the following property.

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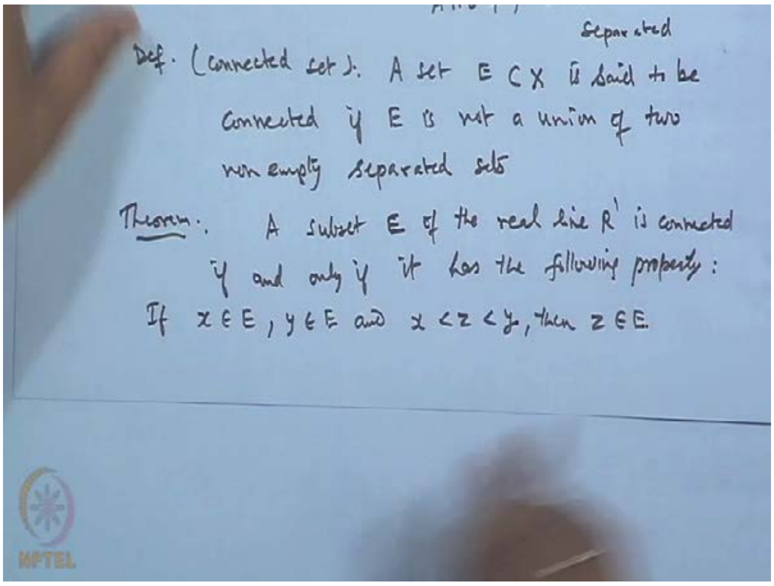


The property says if x, if x belongs to E, y belongs to E and x is less than z less than y then z is also an E, the proof is very simple.

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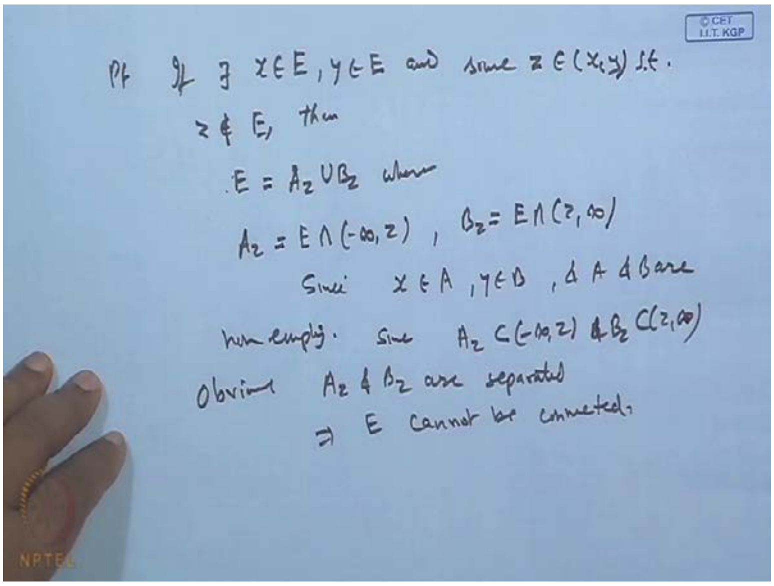


So, you just I will just give the outlines. If there exist suppose if there exist a point z, if there exists x belongs to E, y belongs to E and some z belonging to the interval x y such that z is not in E. Then we will reach a contradiction. Then E can be expressed as the union of the set, where A z is the set of E intersection minus infinity z, B z is the E intersection z infinity. So, since x and y are in way, since x belongs to A because it is in

E and y belongs to B and A and B or non empty and A and B are non empty or non since this A is an are non empty then, since A is the A z which is subset of minus infinity z and B z which is subset of z infinity. Therefore, these are two separated set then obviously, A z and B z are separated. So, once they are separated then E cannot be connected set because E is the union of this two sets.

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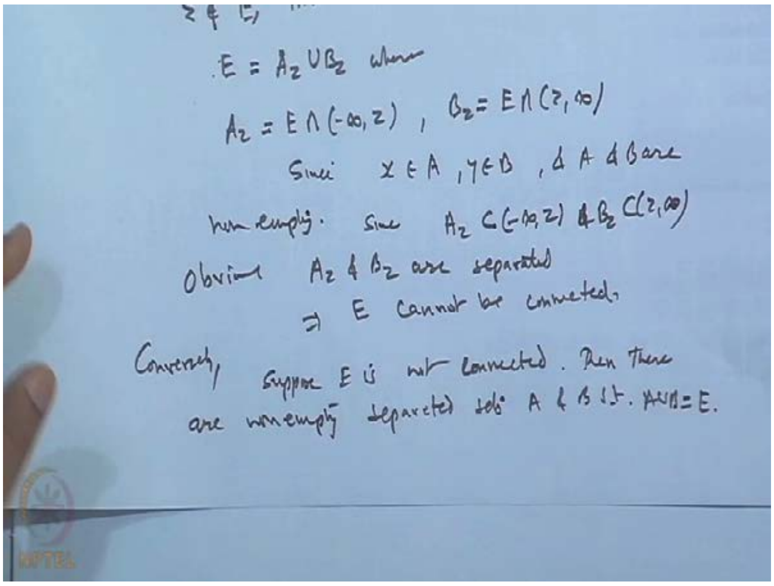


So, it is not a, so, contradiction therefore, this result may not be true, this our, conversely if suppose E is not connected, then there are non empty separated set A and B, such that their union A union B is E. Now, picked up x and y.

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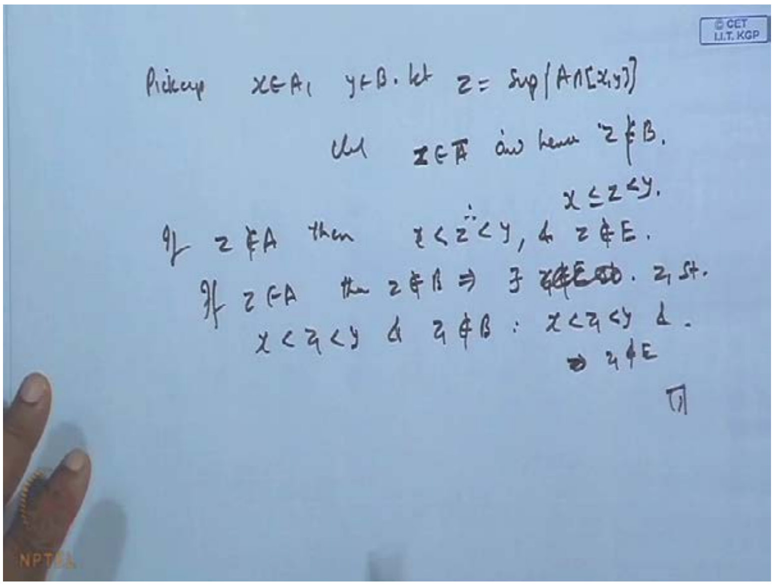

Now, pick up x belongs to A, y belongs to B and let z is the supremum value of A intersection this closed set x y. Now, obviously, this set A, obviously, clearly x belongs to the A closure, x closure and hence, and hence this z sorry z belongs to A closure, and this z cannot belongs to B. So, what we get? In fact so therefore, x may be less than equal to z and is strictly less than y.

Now, if z does not belongs to A then in case, then we have x less than z less than y and z is not in E. If z belongs to A, then we have z is not in B. So, we get that exist at z 1 belongs to, does not belongs to E, but such that z 1 there exist a z 1 such that z x less than z 1, less than y and z 1 is not in B. Therefore, x less than z 1 less than y and this implies that z 1 is not in y and z 1 is not in E. So, this completes the proof.

Thank you very much.