A Basic Course in Real Analysis

Prof. P. D. Srivastava
Department of Mathematics
Indian institute of Technology, Kharagpur

Lecture - 12
Compact Sets and its Properties

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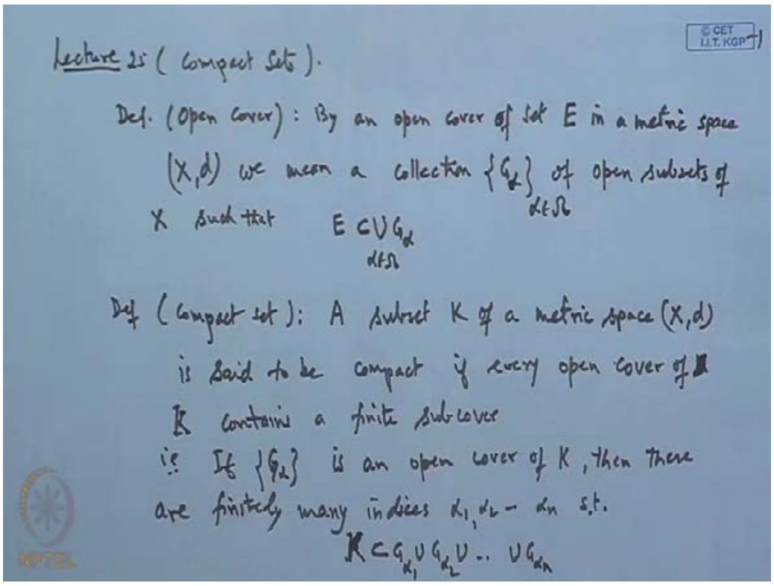

So, today we will discuss compact sets in a general metric space; that require the concept open cover. So, first we see open cover, by an open cover of a set of a set E in a metric space (X,d), (X,d), we mean a collection, a collection of of a set collection G alpha; where alpha belongs to the index set, alpha belongs to the index set omega; of open subsets of X, such that arbitrary union of this open set G alpha covers E; when alpha belongs to omega. So, if an collection of the open set which goes unions arbitrary union covers E, then we said this collection of the open sets in the metric (X,d) is called an open cover for E.

So, we define the compact set. A subset K of a metric space, of a metric space (X,d) is said to be compact, compact is said to be compact. If every open cover, if every open cover of X, of X, every open cover of K sorry, of K; every open cover of K contains a finite sub cover. That is the meaning of this is, that is meaning is, if G alpha, if G alpha is an open cover of K, then there are finitely many indices, there are finitely many indices, indices alpha 1, say alpha 2, alpha n. Such that the finite union of these sets G alpha 1, G alpha 2, G alpha n; finite union of this covers E, covers K, K, covers K, covers K. So, this set is said to be compact.

So, it just like they a compact set requires the finite number of the open sets, which are responsible to cover the entire sets; suppose, we have a security system in our ITs. We are foolishness to apply, to apply the security point vice; means infinite number of people you just put it down the security that is not a voice precision. So, what we do, we put efforts; check post, and then only finite number of check post is basically is sufficient to look after the security. So, that way we set campus, the security system the campus, that forms a compact sets.

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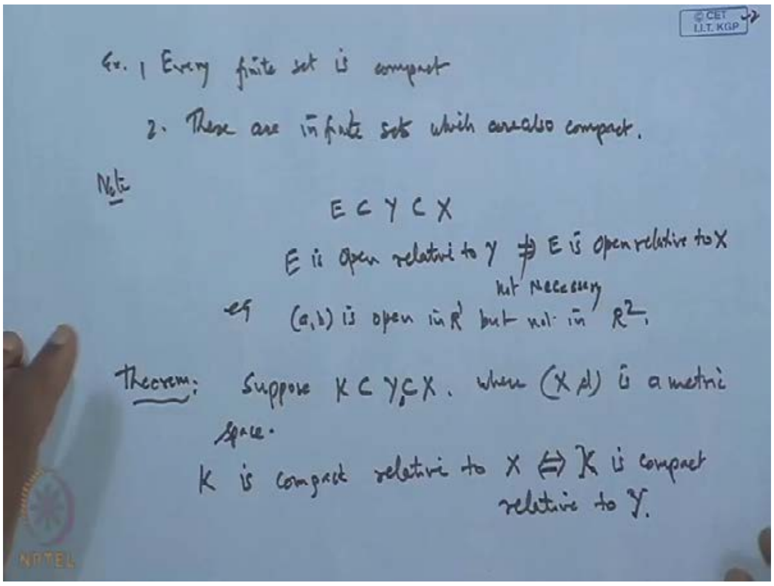

So, it is just like an arbitrary a set E is set to be compact. Whenever any open cover of it has a finite sub cover; that is there are only finite number of open sets who geneon will cover the entire set X. Obviously, finite set is a compact set. Every finite set is compact. There are infinite sets also which are compact. So, we will see that infinite compact sets, we will see the there are infinite sets; which are also compact. Say just I, I take an example of a set of all the real numbers in the interval 0 1. It is an infinite cut sets, but it is compact. Because only we can divide say 2, 3 intervals we can choose such that the union will cover the whole interval a b. So, that will be we will see the others examples.

Now, one thing which we K, that suppose E is a subset of Y, which is subset of X. And if E is open, E is open with respect to relative to Y, relative to E is open, relative to Y, relative to Y. That does not imply that E is open relative to X. It not necessary, not necessarily, not necessary; that if a set is open in a subset, sub metric space by because X t is a metric space, by is a subset of a X. So, by d will also metric space.

So, if a set E is open with respect to this metric space by d, then it may or may not remain open with respect to the metric X. And examples we have seen, the R1 and R2, an open interval (a,b) is open in R1, but not in R2 is it not? So, the openness or the closeness of a set depends under which the set is embedded, but this is not the case. So, for we consider the compactness; if a set is compact relative to the Y, then it has to be compact related to X and Vice Versa. So, this is an interesting result for the (( )) we have this result as follows. Suppose K is a subset of Y, which is subset of X. We are (X,d) is metric space, is a metric space. The result says K is compact, K is compact relative to X; if an only if K is compact, K is compact relative to Y, relative to Y. So, let see the proof of this. So, what we want is that, if K is compact with respect to Y; then it has to compact with respect to X and vice versa.

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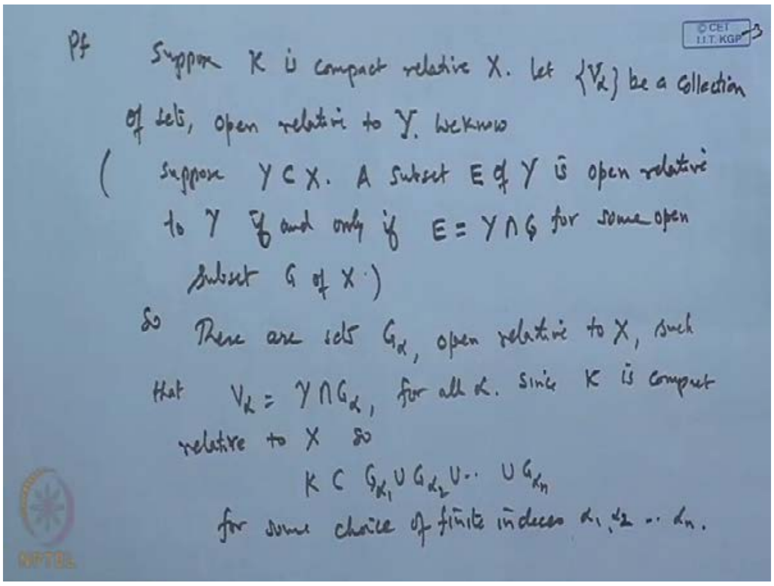

So, let us suppose K is compact, suppose K is compact relative to with respect to or relative to X. It means if we choose a collection of the open set which are relative to X; those countable unions covers K, then there will be a finite subset, finite collection of those open set which can cover K. This every open cover of K in X will have a finite cover so this is known. We wanted to show K is compact related to Y. So, let us consider the open cover of K with respect to Y or related to Y. So, let V alpha be a collection, be a collection of sets open relative to Y, relative to Y.

Now, prior to this we have one result we know, the result says suppose, Y is subset of a metric space X, a subset E of Y is open relative to Y, relative to Y. If and only if, if and only if there is a, if and only if E can be, E is can be written as Y inter section G for some open subset G of X. This result we have already shown. So, using this result we can say, V alpha is given to be an open relative to Y. So, according to this result V alpha can be expressed as, Y intersection G alpha for some open subset G alpha of X. So, we can say that there all the sets. So, there are sets say, G alpha open relative to Y, relative to X, relative to X, such that V alpha is nothing but the Y inter section G alpha for all alpha. V 1 will be Y intersection G 1; V 2 will be though correspondingly we can get the open sets G 1 to G n relative to X.

Now what is given is K is compact relative to X. So, G alpha is an open set relative to X. So, this collection of this open cover will have a finite sub cover. So, since K is compact relative to, relative to X. So, there are so K will be contained in the finite union of these open sets, G alpha 1, G alpha 2, G alpha n. For some choice of finitely many indices for some choice is choice of indices, finite indices, indices alpha 1, alpha 2, alpha n. Like this. Now if I take since, K is contained in since, K is a subset of X; K is contained in Y, this is given, this is given K is a subset of Y; K is contained in Y.

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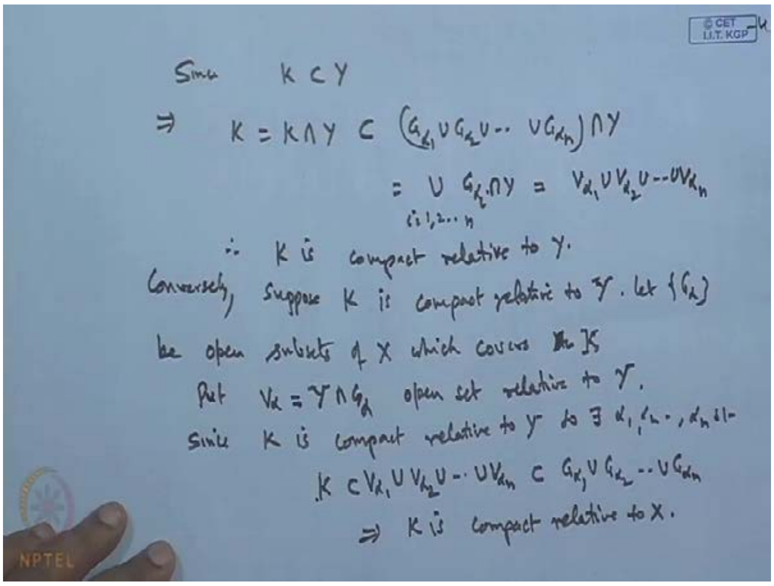

So, K is contained in Y. So, this implies that K; which is K intersection Y is contained in G alpha 1, union G alpha 2, union G alpha n intersection Y. And that is the same is this, that union of, union of G alpha i intersection Y, where i is say 1 2 n; 1 2 n. But G alpha intersection Y is nothing but V alpha is a not. So, this is the same as the union V alpha 1, union V alpha 2, union V alpha n.

So, K is contained in this union; that is the finite number of the open subsets relative to Y covers K. This shows K is compact relative to Y; K is compact relative to Y. Now conversely suppose K is compact relative to Y, relative to Y. We wanted to show K is compact relative to X. So, let G alpha be an open cover of X, be open subsets, be open subsets of X which covers, which covers K. It is an open cover, covers X; which covers K, sorry because K is compact with respect to which covers K. K is compact with respect to Y. What we are doing is we wanted to show K is compact with respect to X. So, let us find out an open cover G alpha of K. Now if we prove that, this open cover which is an open cover which are the open sets with respect to X and covers K, if it has a finite sub cover. Then obviously, K will be compact with respect to X.

So, let us put the, put V alpha as the set Y intersection G alpha. Y is a subset of X; G alpha is an open set. So, this V alpha will be open set relative to Y, relative to Y because of the previous result. Now this given, this is given K is compact relative to Y. So, since K is compact relative to Y, to Y. So, this open cover V alpha has a finite sub cover. So, there are these, there exist alpha 1, alpha 2, alpha n such that; the union of these V alpha 1, V alpha 2, V alpha n covers K, covers K. But V alpha 1, V alpha 2, V alpha n these are the subset of what? Subsets of G alpha 1, union G alpha 2, union G alpha n; V alpha 1 is subset of G alpha 1, V alpha n is the subset of G alpha n. So, K is also contained in the finite union of these open subsets, which are open with respect to X. Therefore, every open cover of X, K relative to X have a finite sub cover. So, this implies K is compact relative to X and that is prove the result.

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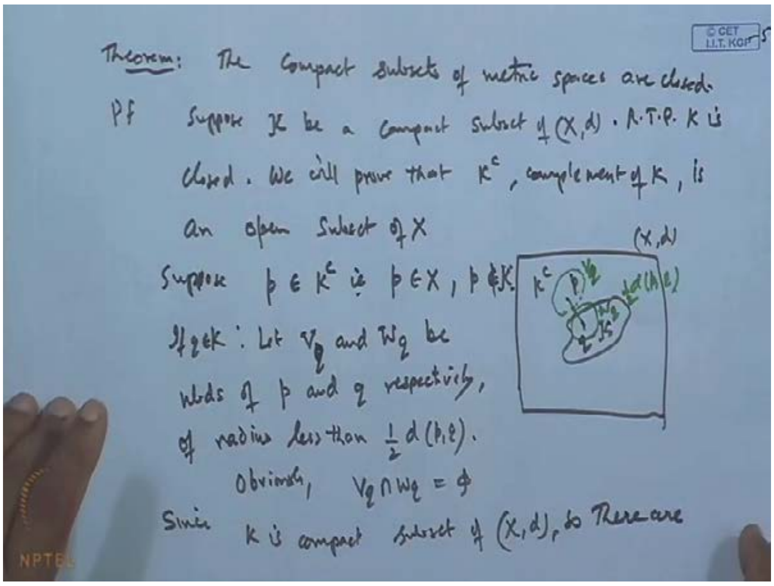

Another result shows that close subset, compact subset. The compact subset of a metric space, of metric spaces, spaces are closed, are closed. Every compact sub set of metric space will be a close set; whatever metric may be is an arbitrary metric space, every compact of set is a close set. Let see the proof. These we will prove by contradiction; suppose, K be a compact subset of X, K be a compact subset of(X,d) of a metrics. So, in order to show K is closed; if I prove this complement is open subset of X then is. So, in order to prove require to prove is, K is closed, closed. So, what we do is be; so we will show, we will prove that it is compliment K c, complement of K this is an open subset of X, subset of X. So, here we have this is our metric space, this is a set K, what we want is the complement of this K c is open with respect to X.

So, it means if I take any point here say p, which does not belongs to K, and if we are able to show there exists a never would around the point p; which is totally contain in K c. Then obviously, this point p will be an interior point of K c, but p is an arbitrary. So, we can say every point of K c; we can draw the never would which is totally contained K c. So, K c becomes o p. So, that is the idea of the prove. So, let us take suppose, p is a point belonging to a complement of K that is K c. But that is in X or you can say p that is p is in X. But but p is not in K; that is the meaning of this. P is in X but it is not in K.

Now let us take a point q, if q belongs to K, if q belongs to K. Then there is a distance between p and q. So, this is the distance between p q. The distance of this is nothing but the d of p q. So, if I take a radius less than half of the distance of p q, and draw the neighborhood around these points; like this draw the neighborhood round this point like this. Then these neighborhood will b disjoints. So, let this neighborhood we denoted by V q and this never would we denote by W q.

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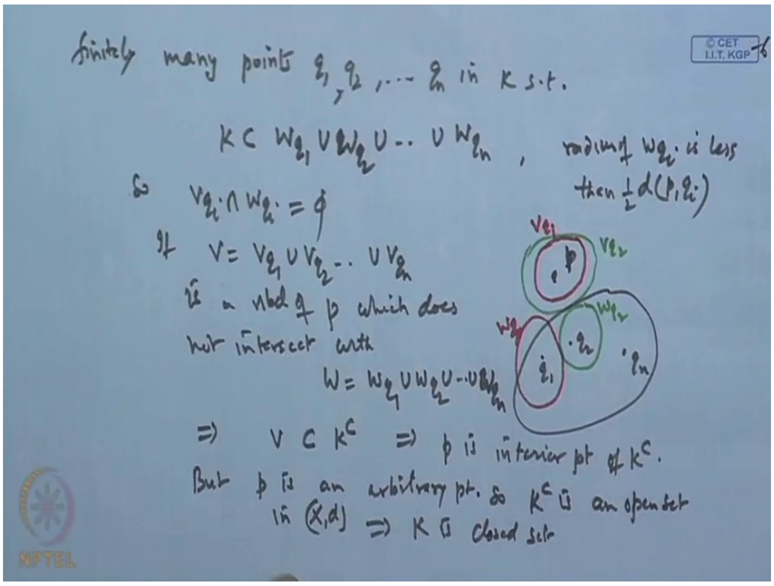

So, let us take this a let V p, V p, V q, V q and V q and W q, V q and W q be neighborhood of p, be never would of p and q respectively, p and q, p q respectively, of radius, of radius less than, less than half the distance between (p,q). Then obviously, that V q intersection W q will be an disjoints. That is true. Now let us take since K is compact, K is given to be compact, is a compact K is compacts subsets of metric space (X,d). So, every open cover of K will have a finite sub cover. So, there are the points q 1, q 2 in q; such that, that open ball drawn q 1, q 2, q n finite number of involve, will be sufficient to cover K.

So, let us there are finitely so there are finitely, there are finitely many points say q 1, q 2, q n; K 1 K 2 in K, such that, such that the open balls or never would drawn at this point with the radius half of q1 p etcetera, will remain. Such that the never would drawn at these points with the radius I will say after. So, W q 2 union W q n, W q n; where the radius of this, radius of W say q i is half of less than, half of the distance from p to q i, p to q i.

Now this finite number of this neighborhood will cover K because K is compact. Now since these neighborhoods you are drawing this is our K, here is q 1; q 2 say, q n here is p. So, what we are doing is we are drawing a ball say, this ball and here it say this ball. So, here it is W q 1 this is nothing but V q 1 which is disjoint. Then we are taking say q 2. So, again p 2 we are taking say this is our say V q 2 and here it is say W q 2. Again they are disjoint because the distance does not match there no, is less than half of the radius is less than half of p of q 2. So, obviously, they all this V q 1, V q 2, V q n will be disjoint with that. So, let if so V q 1, V q i intersection W q i is empty. That is... So, if we take V as finite union of V q 1, V q 2, V q n suppose this finite union; then this will be a neighborhood, is a neighborhood of p, is a never would of p.

And this neighborhood does not intersect, which does not intersect with W, with W which is W means, W q 1 union, W q 2 union, this one say W q n, W q n. This does not intersect with this because; this is the smallest one which is disjoint with everyone. So, we have a totally a neighborhood. So, this implies that this neighborhood V is totally contained in the complement part of K. Therefore, p is an interior point, interior point of K complement, but p is an arbitrary, arbitrary point. So, this shows that interior of this is an open set in (X,d). So, once it is open then K must be closed. So, every compact of set or metric space is a close set and that is what to be proved.

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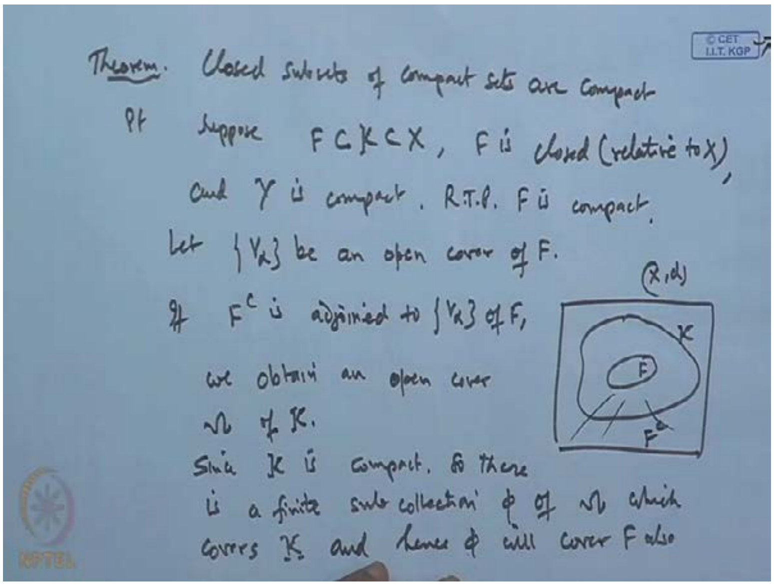

Another results close subsets of a compact set, of compact sets close subsets of compact sets are compact. So, this we wanted to show proof. So, let us take an F which is a subset of K, which is subset of X; where F is closed, relative to X and Y is a compact set. What we want to show this close subset that is F, which is a subset of a compact set, is compact. So, every close subset of a compact set is compact that is what so so require to prove is F is compact, F is compact means that it will cover, every open cover of F will have a finite sub cover. So, let us take let V alpha be an open cover of F, open cover of F. Now this is our scenario so here this is our (X,d) here this is a set K and here is somewhere F.

Now, we are taking an open cover of F. What is F c? F c will be the complement of F. So, here somewhere we have a F c, this is our F c. Now if we take an open cover of F. Then some of the open cover will intersects F c also because these are F c is F closed. F c will be complement will be an open set. So, it will be an adjoined to V alpha. So, if F of c is a is adjoined, adjoined to the open set V alpha; open cover V alpha of F, then V obtained an V obtained an the open cover of an open cover, V obtain an open cover omega of X of K, omega of K. This is the open cover of V alpha now some of them will, will definitely adjoined with this, now we are taking the open cover of K; now this open cover omega may contains F of c also, somehow. So, since K is further let us see, since K is compact. So, every open cover will have a finite cover. So, there is a finite sub collection of omega.

So, there is a finite sub collection, finite sub collection of, finite sub collection of pie of omega, which covers, which covers K by definition of the compact sets K; another possibility. Now once this omega, which is an open cover of K and since K is compact so it will have a finite sub cover; so it means that pie will cover F also. And hence, hence this pie will cover F also, finite sub collection will cover F also now, in this pie if the pie c is F c is also member, then we can drop that.

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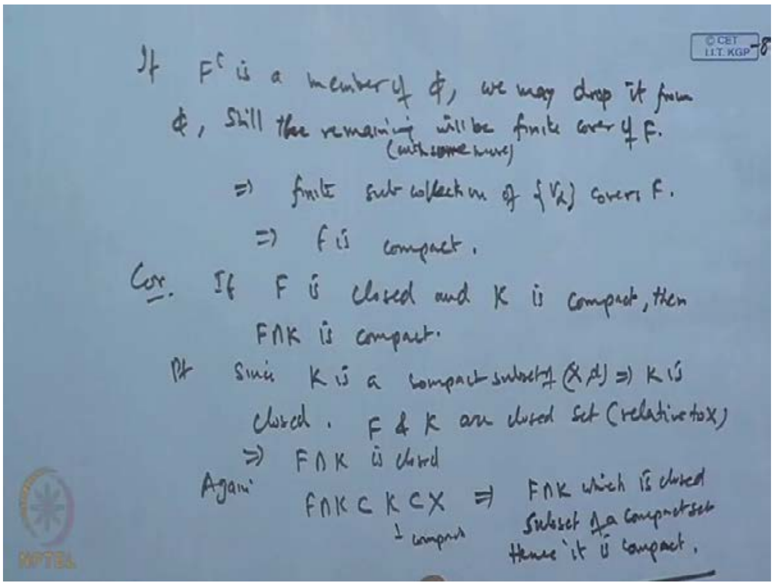

If, if F c is a member of this, is a member of pie; then even if we drop, we can then we may drop, we may drop it from pie. Still, still the remaining will be the finite cover of F, finite cover of F, still retain an open cover of F. Remaining with some, with others, with some more, with some more will be finite cover of a. So, but this shows, that this shows, that this implies, that this sub collection of this a finite sub collection of this open cover, V alpha covers F, so this shows F is compact. So, that is the very interest now, as coronary to this is if F is closed, F is closed and K is compact, K is compact, then F intersection K is compact.

Now proof photo is very easy, what is F? F is closed K is compact subset of X; every compact sub set of F is closed. So, since K is a compact sub set of say metric space (X,d) so it implies that K is closed. Every compact sub set of this further F and K is all close set. So, relative to X, relative to X so this implies that intersection part of this intersection of two close sets is closed. Again this intersection F, intersection K is totally contain in K, which is contain in X? K is compact, this is compact. So, every close sub set of a compact set is compact so this implies F intersection K; which is a closed subset of a compact set hence it is compact, hence it is compact. So, that proofs the hence it is compact.

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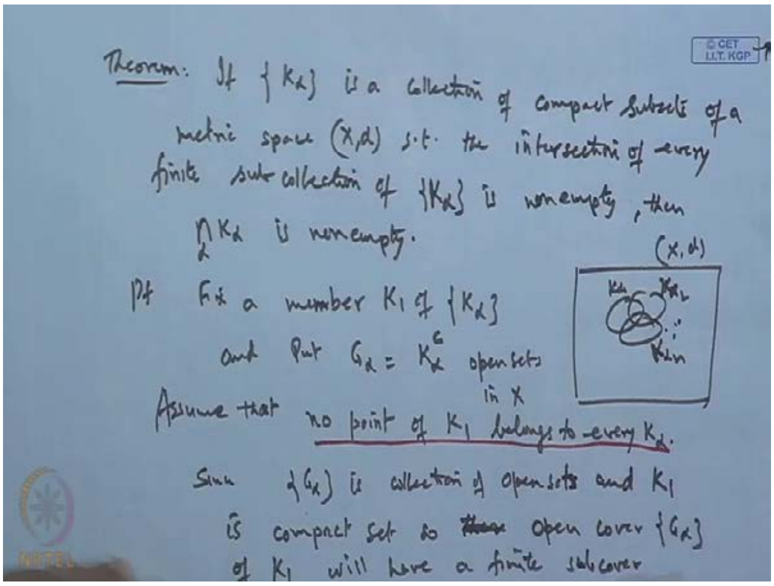

So, this shows that so next is a in this (( )) this is also interesting result if, K alpha is a collection of, if sequence K alpha is a collection of compact subset, compact subsets of a metric space, of a metric space say (X,d) and such that the, such that the intersection of, intersection of every finite, every finite sub collection of K alpha, sub collection of these K alpha, sub collection of K alpha is non empty. Then the arbitrary intersection of K alpha, K alpha is non-empty.

So, this was the finite intersection property basically if, K alpha is a collection of compact subsets of a metric space (X,d). This is our (X,d) and K 1, K 2, K n these are the compact subsets of this, K alpha 1; K alpha 2; K alpha n; these are the compact subset of this (X,d) and if we take the finite intersection of these finite intersection of K alpha1, K alpha 2 say K alpha n like these. If we take the finite each finite intersection is non-empty set, then arbitrary intersection will be non-empty.

So, this we will prove by contradiction, how will you say? Suppose, one of the sets prove, what we will do is we will pick up one of the sets out of K alpha say K 1, such that, that no element of the K alpha, that set K 1 belongs to each K alpha. It means when you take the intersection of K 1 to K alpha or intersection, Some at least, some of the points of that will be out of it, will remain out of it means, there are compact the not every point of K 1 is or none of the point of that set belongs to each alpha, that each K alpha that is what here.

So, fix a, fix a member K 1, then we will reach a contradiction, then fix a member K 1 of this sequence of compact set K 1 from the sequence and put, and put, say G alpha as the complement part of K alpha c. Now K alpha is a compact set, set so it is a close set. So, G alpha will be an open set in X; yes or no? Now what we assume that, assume that this K 1 assume that, no points of K 1, no point, no point of K 1 a K 1 belongs to, belongs to every K alpha, every K alpha. So, this is our assumption. This is very no point of K 1 belongs to K alpha, it means when you find a set pick of the point one, one point of K 1. Than at least one of the alpha K, alpha will be available where that point does not belongs to like that; so no point of K 1, is no point of K 1 belongs to every alpha, K alpha.

Since, G alpha is an open sets and K alpha since this collection, G alpha is a collection of open sets, collection of open sets and K 1, and K 1, and K 1 is a compact set. Because it is one of the set, which you choosing out of K alpha, compact set. So, by definition so K 1 so there are all so G alpha is a collection of open set and K 1 is compact so the open cover, open cover G alpha of K 1 will have a finite sub cover, will have a finite, will have then so open cover G alpha of K 1 will have a finite sub cover.

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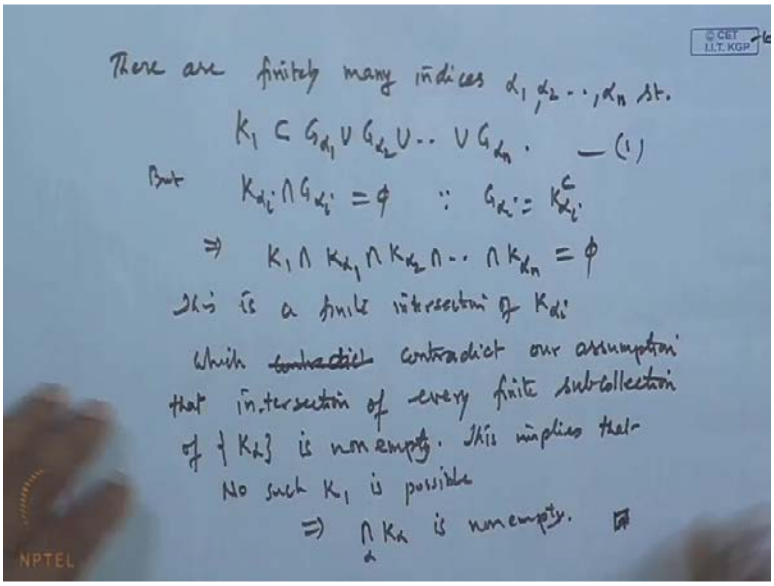

Finite sub cover means, that is, that is there are finitely many, there are finitely many indices say alpha 1, alpha , alpha n such that the K 1 is contained in G alpha 1, G alpha 2, G alpha n. This but fine now, if we this is say 1, but K alpha i intersection G alpha is empty because because what? Because the G alpha i is taking as the complement of K alpha i. So, if I take the inters K 1 intersection this so this implies K 1 intersection with K alpha 1; intersection with K alpha 2; intersection with K alpha n; then this will be empty set. But what is this? This is a finite intersection of K alpha, but this is, this so this is a finite intersection, finite intersection of K alpha. Is it not?

That is if I picked up a finite collection of this set K alpha, then this are the finite, but what is the condition? If K alpha is a collection of the compact subset of metric space, such that intersection of every finite sub collection is non-empty. So, which contradict which contradict, contradict, contradict, which contradict, our assumption that, that intersection of, intersection of every finite, every finite sub collection of K alpha, of K alpha is non-empty. Let us coming to empty, so its contradiction is because our wrong assumption, that K 1 is one of the member out of K alpha is there which has a property, then no point of K 1 belongs to every alpha. So, this list this implies, this implies that no that, that no such K 1; is possible.

It means that, if we picked up it means, whenever you take the points it is belongs to at least one of the point, at least it will belongs to all of that alpha, that is what it say ok. So, this shows, this is only possible, that this implies that the arbitrary intersection of K alpha, over alpha is a non-empty set. Because our assumption is wrong, assumption is that there is a some K 1, which has a property assuming that, no point of K 1 belongs to every alpha. So, if you picked up one point, that is one of the alphas are, they are where that point does not belong so that this leads to a contradiction. It means whatever the point you choose, it will belongs to every alpha, K alpha; hence the intersection will be non empty so that is proofs the result.

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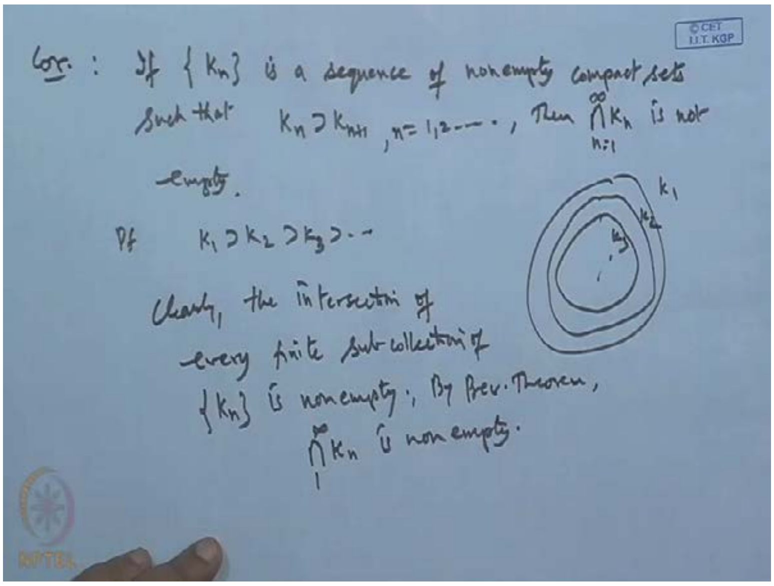

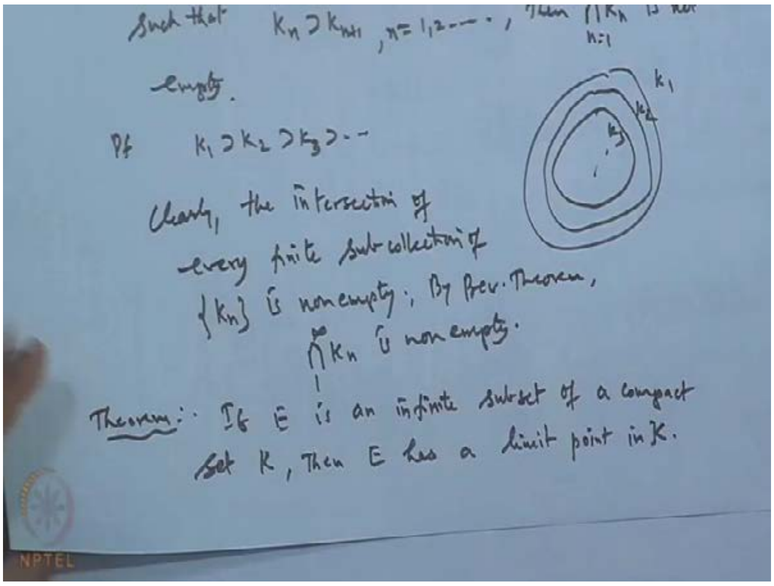

Now, consequence of this is the following result as a coronary, the coronary is what is says the if, if K n is a sequence of, sequence of non-empty compact, non-empty compact sets K, if a non-empty compact sets. Such that, such that K n covers K n plus 1 and so K n plus 1, where n is 1, 2, 3 and so on. Then the arbitrary, then the countable intersection of K n, 1 to infinity is not empty. The proof is follows from there, because what is given K 1? Contains K 2, contains K 3 contains and so on. So, this is our K 1, here is K 2, here is K 3 and like this. So, if I take any finite collection of K ns, then there intersection is non-empty. So, clearly, clearly the intersection, intersection of every finite, every finite sub collection of K ns of sub collection is non-empty and K ns are the sequence of compact sets so from the so by previous theorem, the arbitrary the countable intersection of this 1 to infinity is non-empty, that is proved. Now, we have another results, this result also be used if E is, E is an infinite sub set, infinite sub sets of a compact set, of a compact set K, of a compact set K then E, then E has a limit point in K, limit point in K.

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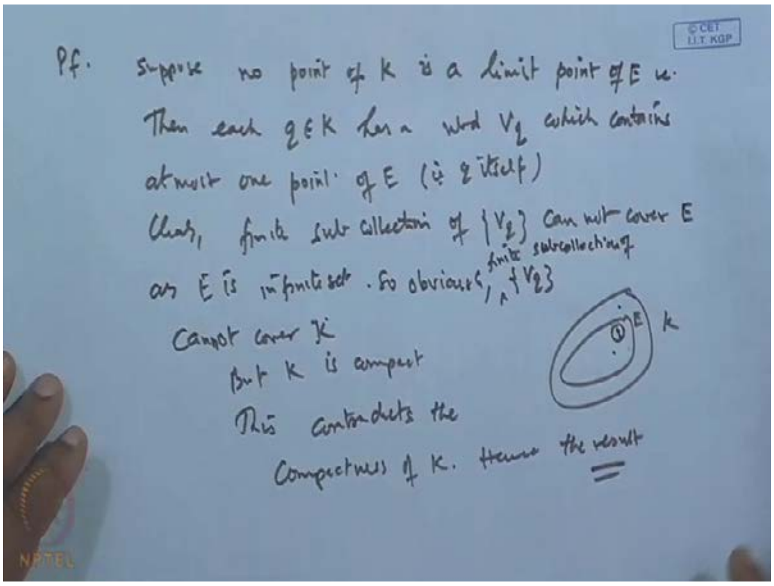

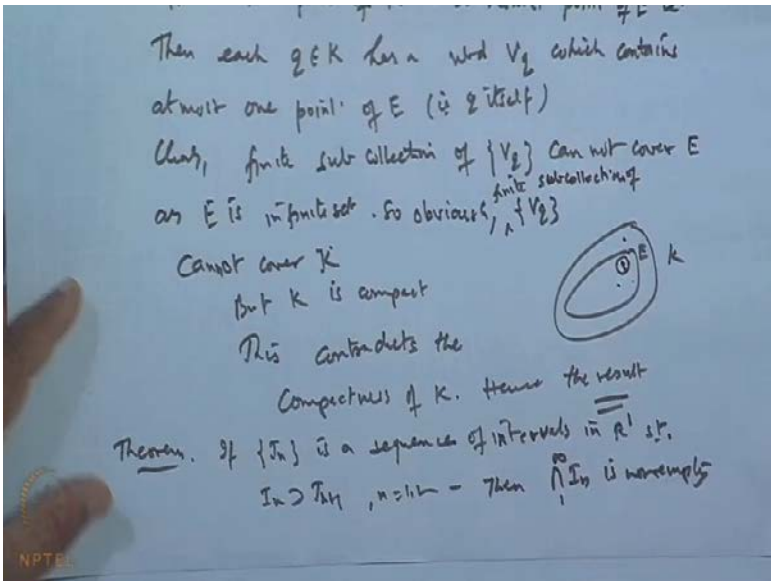

So, let us suppose contradiction, we prove again by contradiction so let us say, proof suppose no point of K is a limit point of E, so suppose no point of K is a limit point is a limit point of E. It means what? If no point of K is a limit point of E it means, if we take each K, that is then then each at each, each q, each point q belongs to K, have a, has a neighborhood, has a neighborhood say V q which contains, which contains at most one point, at most one point that is the q itself. One point of E that it q itself center because if suppose it contains infinitely many point, then q becomes the limit point of it. So, that this the contradiction so suppose it contains at most one point now, if I take the finite sub collection of this, then clearly the finite sub collection, finite sub collection of these neighborhood V q will not cover, cannot cover, cannot cover say E, cannot cover E because this is our, this is our K and E is this.

Now, what we are assuming is no point of K is a limit point of this. Suppose so suppose if I take a q here then this neighborhood q, this neighborhood q will contain a neighborhood, it does not include the other points of the key. Because q is not a limit point at the most it will contain only the point, but since E has a infinite collection of the points. So, a finite number of the disk open cannot cover E because this will only have finite number of points available on this, cannot cover E as E is infinite set. So if this neighborhood cannot cover finite, it can also not cover k.

So obviously this neighborhood, obviously this neighborhood cannot cover finite, finite sub collection, finite sub collection of this V q cannot, cannot cover K. But what is K? Is compact, but K is compact. So, every open cover must have a finite sub cover now, V q we are taking an open cover for this. So, it cannot cover V q. So, finite sub cover must cover q, which is contradiction, because does not cover q. So, this contradiction, this contradicts the compactness of K, hence the result, hence the result follows the result. Now, another two question theorems are there, of course a just one more theorem and then proof is immediate. If a I n is, if I n is a sequence of intervals, intervals in R 1, R 1 such that the I ns cover, I n plus 1, where n is 1, 2, 3 then the intersection of I n 1 to infinity is non-empty.

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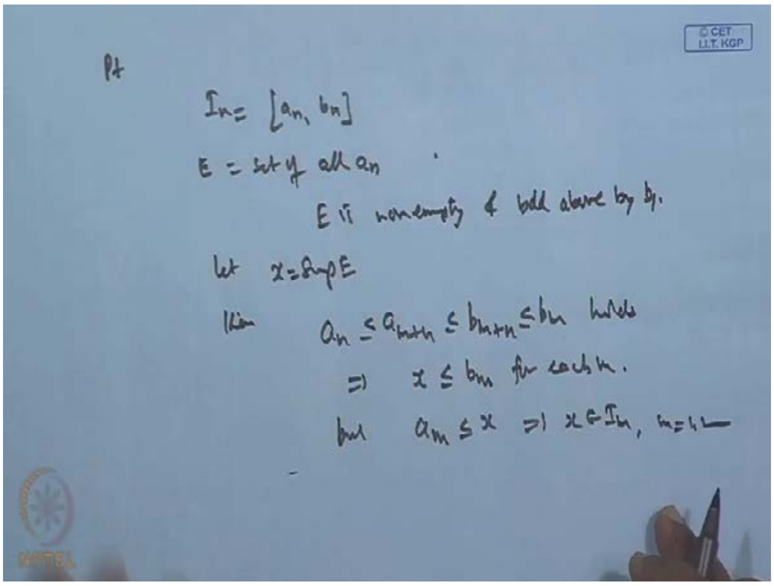

The proof is very simple, suppose I n I take a n, b n, a close intervals and let E be the set of all a ns, then obviously is obviously, E is non-empty and bounded above by b 1. And so let supermum of E is suppose X. Then clearly, then clearly a ns which is less than equal to a m plus n; which is less than equal to b m plus n; then this is less than b n holds. Therefore, when you take the supermum of these, we get x is less than equal to b m; for each m, for each m, but a n is already less, get less than equal to x. So, what they show? This shows that X belongs to I n, when m is 1, 2, 3. So, we get the finite intersection of these non-empty if, n sub collection of I n intersection of this is non-empty; therefore, the arbitrary intersection will also non-empty. So, this proves the result.

Thank you very much.