A Basic Course in Real Analysis

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Lecture - 11
Ordered set, least upper bound, greatest lower bound of set

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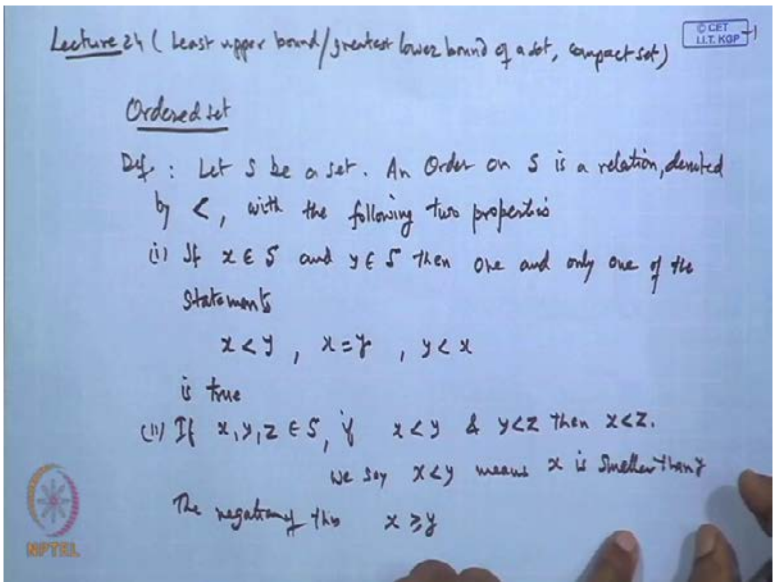

Hello. So, we will discuss the least upper bound; greatest lower bound of a set and compact sets today. So, this requires the concept of ordered set; ordered set first. We define like this; let S be a set, S be a set, an order on S an order on S is a relation, is a relation denoted by sign; this sign which we call later on, we will less than sign or a smaller sign, with the following, with the following two properties. The first property is that if x is in S and y is in S, then one and only one, only one of the following statement, of the statement of the statements, that is x is less than y; x is equal to y and y is less than x is true, is true. And second one is say, if x, y, z they are the elements of S, and if x is less than y and y is less than z, then then x must be less than z.

So, if a set S together with this operation which we call it is together with this relation less than sign, satisfy these two properties, then we say that this is an order on S. This is an order on S, this sign less than sign we normally say; we say x related to the means x is smaller than y, x is smaller than y. And the negation of this, the negation of this is x is greater than or equal to y. This is the negation part of it, means x is greater than equal to y; the negation of this will be x is typically less than y. So, that will be the sign for (( )). Now, obviously when we say the set of rational number or set of the real numbers, then this order is defined; one can identify 2 real number or 2 rational number, one can say which one is low smaller than the other or whether they are equal or weather one is greater than the other, like this.

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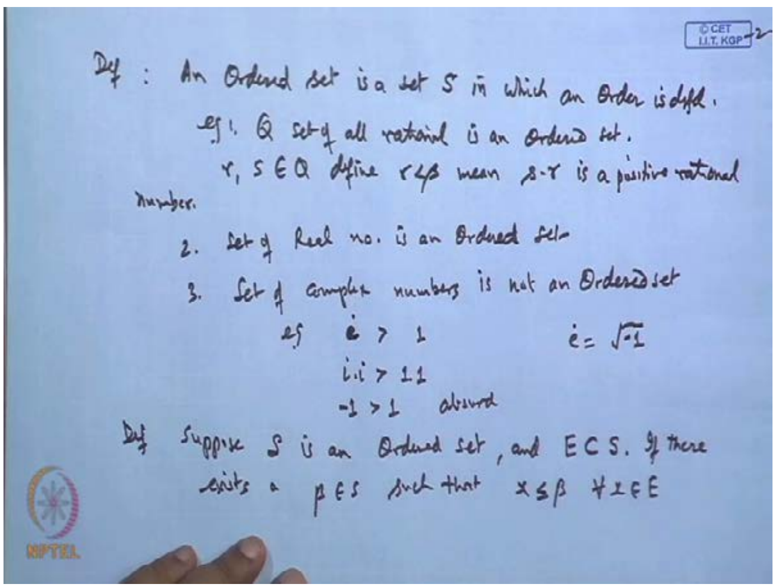

Then the set, order set means, a set, an ordered set S, order set, is a set S, is a set S, in which an order is defined, in which an order is defined; for example, set rational number, set of all rational numbers is an ordered set, is an ordered set, because if we take r and S, suppose these are 2 rational number and define the relation r is less than s means, s minus r means, s minus r is a positive rational, is a positive rational number, is a positive rational number. Then obviously, it was satisfy these two properties, which you, which studied. So, set of rational number is an ordered set; set of real number is an order set, set of real number is an ordered set, ordered set.

However set of complex number, set of complex number is not, complex number is not an ordered set. We cannot introduce the order between the two elements of a (( )) complex number, set of all complex number. Because for example, if suppose I take the complex number e and 1, and when we say e is greater than 1; e is the idle element say square root of minus 1 this is e or i. I will say i, if you i, i, this is complex number.

So, i is greater than 1, it means is the positive we are assuming. So, i into i is still greater than 1 into 1, that i square is minus 1 is greater than 1, which is a absorbed. It means our ordering which we introduce is not correct; similarly, one can show weather if i is less than 1, we can again lead a contradiction and like this. So, we are unable to introduce the order in the over the set of all complex number, that is the set of a complex number is not an ordered field, field we will discuss in algebra or some.

Then we are interested in particular in defining the least upper bound and the greatest lower bound. So, let is see first what is an upper bound and lower. Suppose S is an ordered set, S is an ordered set and E is a non-empty subset of S; now, if there exist some beta, there exist a beta in S. Such that all the elements of E, that is x belongs to E is less than or equal to beta, and this is true for every x belongs to E.

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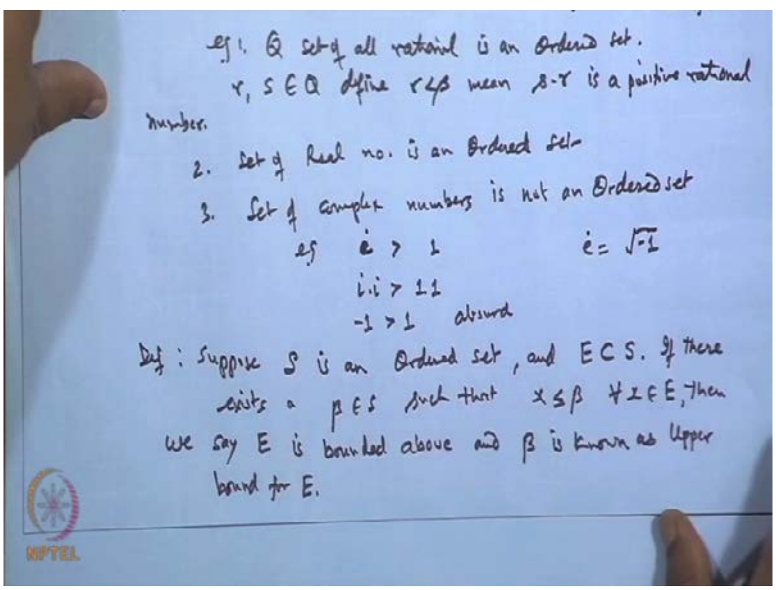

Then we say, then we say E is, then we say beta E is bounded above, E is bounded above, bounded above and beta is an upper bound and beta is known as upper bound, upper bound for E. Now; obviously, beta is an upper bound were there are many infinity many real numbers will be available; which will be an upper bound for E any number; which is greater than beta will act as an upper bound for E. So, we are interested in a get least upper bound for it.

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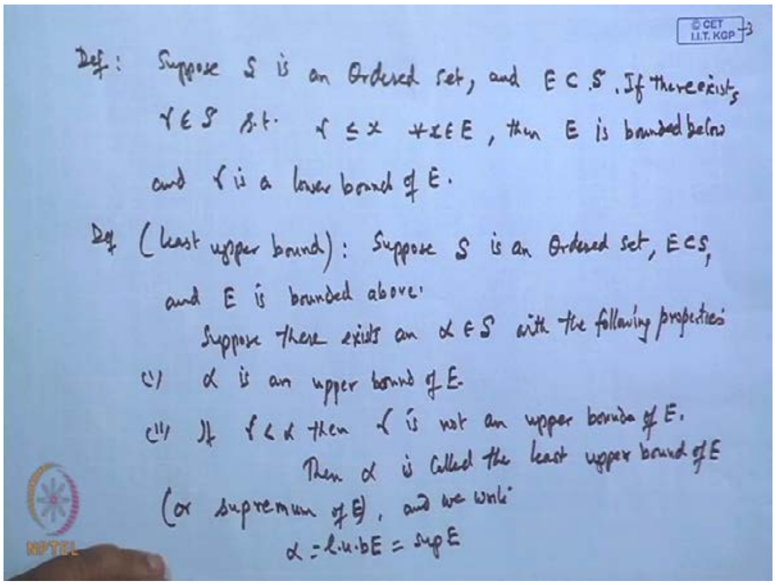

The same case happens if it is lower bound, we define the lower bound in a similar way. Suppose S is an ordered, suppose S is an ordered set, ordered set and E is a sub set of S, non-empty sub set of S. If there exist, if there exist number say gamma belongs to S, such that gamma is less than equal to x, for every x belongs to E. Then we say E is bounded bellow and gamma is the lower bound, is a lower bound, is a lower bound of E. So, their again, there will be many lower bound available as soon as we take any number less than gamma, which will also behave as a lower bound. So, we will be interested in knowing what will be the greatest lower bound of the set E. So, we introduced the concept of the upper bound and lower bound as follow: this is the concept least upper bound; suppose S is an ordered set, S is an ordered set and E be a non-empty subset of S, and also assume E is bounded above, E is bounded above. Now, suppose they all exist, suppose they are exist an alpha belongs to S with the following properties.

Alpha is an upper bound, is an upper bound of E, of E; this is the first property and second one is if, I take a number slightly lower than alpha, then it should not be a as an upper bound, and second is if, gamma is any number less than alpha, then gamma is not an upper bound, a upper bound of E. So, E is a bounded above and alpha is a such a number, which is an upper bound of this, but if we take a number slightly lower than alpha, then that number will not be a when upper it means, alpha is the least upper bound for E. Than alpha is called, then alpha is called the least upper bound, upper bound of E and we do not distinguish and also or some we also say it is a supremum, supremum of E, of E and we write is, we write that least upper bound of set E, each alpha or it is the same as when we say supremum of E.

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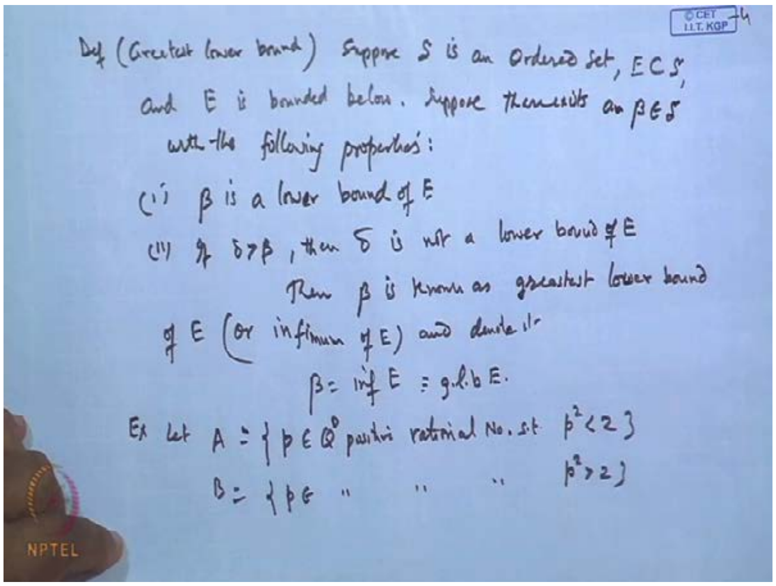

The similar way, we define the greatest lower bound, in the similar way we can introduced the concept of the greatest lower bound of E. So, what we assume is suppose S is an ordered set, S is an ordered set; E is a sub set non-empty sub set of S and E is bounded below, bounded below. Suppose they are exist and say beta belongs to S, beta belongs to S with the following property, with the following properties; the first is beta is a lower bound, is a lower bound of E, is a lower bound of E. Second one is if, a number if I choose, if a number say delta which is greater than beta. Then delta is not an is not a lower bound of E, is not a lower bound of E. If we take any number delta slightly I have done it will not be lower bound; then this beta, then beta is known as, as known as greatest lower bound of E or we can also say it is the infimum, infimum of the set E, and we denote this as, denote it as, beta is the infimum of the set E or is the same is the greatest lower bound of E.

So, this is the way we define the greatest lower bound and upper and the least upper bound, greatest and least. Now let us take an example, suppose I take the set A, let A be the, A be the set of all positive rational number p, p is positive rational number. So, it is positive rational numbers such that, p square is less than 2 and let B is the set of all positive rational numbers; such that p square is greater than 2, p square is greater than 2.

Now if we look that A and B; A is the set of all positive rational number who's square is less than 2 and B is the set of all positive rational number is square is greater than 2; obviously, A is bounded above. In fact, all the elements of B will be the upper bound, will be act will act as an upper bound for A, but A does not act the least of upper bound because we cannot get the rational number which is for, which we can say is a really a number, which is an least upper bound for A.

Similarly, for the B; if we look the, all the elements of B, satisfy this condition, then it has a lower bound, and all the elements of A behave as a lower bound of this plus all rational number, which are negative or 0 will behave as a lower bound for this, but neither A nor B has an upper bound and the greatest lower bound. So, this will show that A contains no largest number and B is clearly, A contains no largest number; while the B and B contains no smallest number. So, in this case the greatest lower bound A has no greatest, B has no greatest lower bound and A has no largest number.

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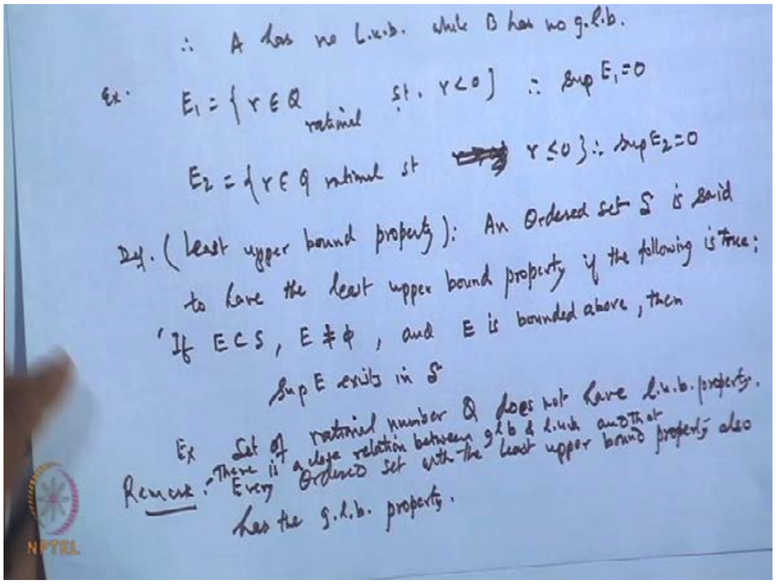

So this, so therefore, we can say A has no least upper bound, least upper bound; where the B has no greatest lower bound that is obviously true. let us take an another example, suppose I take the set of rational number, let E 1 be the set of all rational number r belongs to Q; such that r is stickles than 0, and E 2 be the set of all rational number Q rationales, such that r is greater than equal to zero, r is sorry less than equal to zero; same such that r is less than equal to 0. Suppose I take this thing, than the set of rational number which is less than 0; so obviously it has an upper bound 0, here also is upper bound 0. It means the supremum value of E, so therefore supremum of E 1 will be 0; supremum of E 2 will also be 0, but you see that supremum value of E 1 does not belongs to E 1 by the supremum value of the E 2 belongs to it.

So, it is not necessary when we say their least upper bound or greatest lower bound, then it is not require; it is not necessary that the that supremum value will be a point of the set, it may or may not be the point of set, that we have observed here. Similarly, but both are having the same; similarly, now another interesting property which is a connection with the order set is this is known as the least upper bound property, least upper bound property.

What is the least upper bound property? An ordered set, an ordered set S is said to, is said to have, is said to have the least upper bound, least upper bound property. If the following is true, is true, following is true, but therefore if, if, E is a non-empty sub set of S, is not empty, is a non empty sub set of S and E is bounded above, bounded above; then the supremum of E, that is least upper bound of E will exist, an exist in S, exist in S. So, this is the least upper bound property of a set; an order set as a said to have a least upper bound property if, the following is true. That is if we take any sub set non-empty sub set of which is bounded above; then supremum we exist in S, then we say this set S is a least upper bound property. If, for any set this supremum does not exist, then the set will not have a least of upper bound property; for example, set of rational numbers, set of rational numbers that is which is noted by Q, does not have, does not have least upper bound property, bound property and (Refer Slide Time: 13:33) that we have seen already with this example.

Because, basically A and B these are the two sub sets of the rational numbers, and neither the A nor B has an upper bound. It is not neither A has does not have a upper bound, B does not have the lower bound for it. So, basically the set of rational number you can say does not have a least upper bound property. Now, there is a relation between the greatest lower bound, least upper bound and the least upper bound property. In fact, it is shown that, if the set is having the least upper bound property, then it must have a greatest lower bound property also and that can be judged in the next theorem.

The relation between the, relation between the least upper bound, greatest lower bound and this; so theorem says or you can before this, you can write the remark; I would write the remark. Here is every ordered set, there is a relation between the greater lower bound, that every order set, order set with the least upper bound property, upper bound properties with the least also has there is their remark is there is a close relation, close relation between, between greatest lower bound and the least upper bound and that, and that, this that every order set with the least upper bound property; also, has the has the greatest lower bound property.

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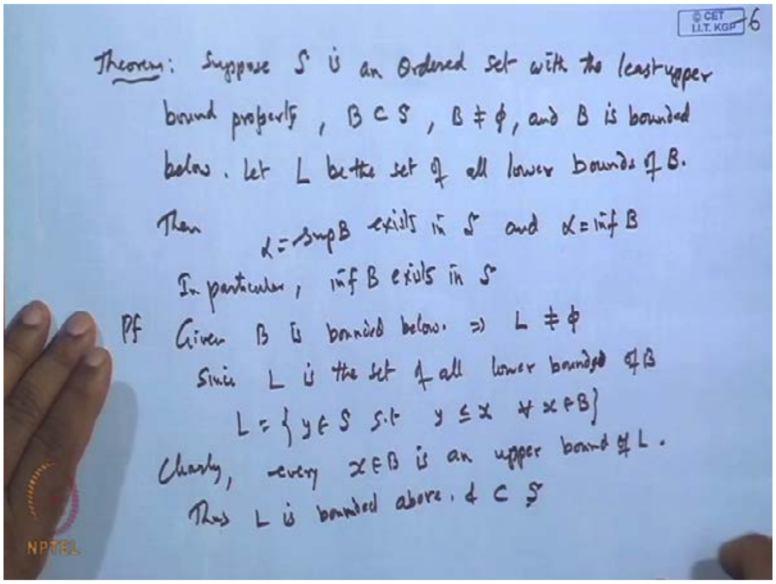

This can be seen with the help of this result the theorem says that if suppose, suppose S is an order set, ordered set with the least upper bound property, with the least upper bound property, upper bound property and suppose B is non empty sub set of S, B is a non empty sub set of s having and B is bounded below, bounded below. So, S is an order, order set which has a least upper bound property and a set B has a property which is bounded below. Now; this together will implies the least relation between the greatest lower bound property. So, what he say is if B is bounded below and let L be the set of all all lower bounds of B, lower bounds of B; then the supremum of B, that is the least upper bound of B; that is alpha will exist, exist in S and and this alpha will be the infimum value of B; that is it will the greatest lower bound for B and in particular infimum of this B exists in S.

If you want let us see the proof of this. What is given is S ordered set which has a upper bound property; upper bound property means if B if any set is there, which is a sub set non empty sub set of this and if has in upper bound; then the supremum of this will exist in this. Now here we are assuming that S has a upper bound property and a non-empty sub set B is bounded below. Then; because of this upper bound property and this condition we will saw the B will have the greatest lower bound and infimum of B will exists in S; that is what it is. So, since B is given B is bounded below; this is given and what is level L? L is the set of all lower bound of B. B is bounded below it is already given is means there is a bound available. So, L is non-empty.

So, so this implies L is non-empty, now L is the set of all lower bound of B. So, what is L? So, clearly since L is the collection of, L is the set of all lower bound, bounds, lower bounds of B. So, basically L consist of those y, it means L is the set of those points in S, y in S, such that y is less than equal to x for every x belongs to B, because L is the collection of the lower bound. So, the y is set less than equal to x and y will be the lower bound for B and or such why we satisfy this condition, will come in the class L and this will be a non-empty set; this one thing is clear, now every x. So, if we look the L. L is the collection of those points which are low less than equal to x for a B. It means every point of B behave as an upper bound for x.

So, clearly clearly every x in B is an upper bound, is an upper bound of L. So, it means L is bounded above, thus L is bounded above, bounded above. So, L is a non-empty set which is bounded above, it is a sub set of S, L is a sub set of S. So, bounded above an is a sub set of S, is it not? Is it not? So, we can apply the property because S is an ordered set, having the least of upper bound property. So, by the property since S has a, since S, since S has a least upper bound property. So, so by this L will have a supremum value.

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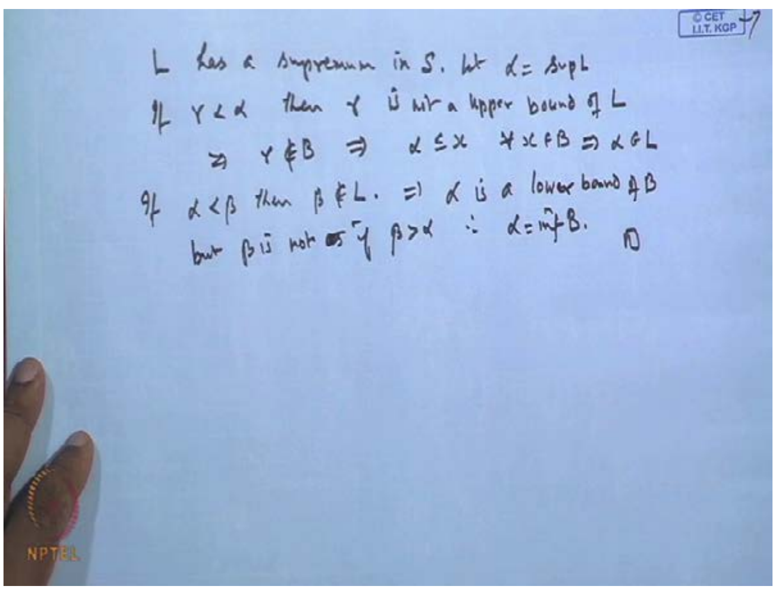

So, L has a supremum value, supremum in S, in s exists, supremum value will exists and let it be let alpha is that supremum value of L is alpha. Suppose supremum value of this alpha. Now if we choose gamma; if gamma is any number less than alpha, then gamma is not an upper bound of L, upper bound of L, because alpha is the least upper bound. So, if we take any number lower than gamma, lower than alpha, than that cannot behave even upper bound for it. Otherwise gamma will be the least upper bound. So, gamma if it is less than alpha, it cannot be upper bounded L and what is our B? B is the set of those points which are for such that every point of this, is an upper bound for it and here gamma is not coming an upper bound for L. So; obviously, gamma cannot be a point in B. Because all the points B must be an upper bound, is an upper bound which we have shown, but gamma is not an upper bound of L therefore, gamma cannot be a point of B. So, what the this follows it implies, that it implies that alpha is less than equal to x, alpha is less than for every x belongs to B, because any number less than alpha cannot be a point of B. So, alpha will be the least number and then alpha will be less than equal to x. So, this source that alpha belongs to L, that is would alpha belongs to L; that is the supremum will exists and it is in L.

Now if any number which is less than alpha, any number beta which is greater than alpha, then beta cannot be in L, because alpha is the least upper bound and all the beta is greater than. So, again it will not be in L. So, once it is not in L, alpha is not in L, beta in L, then what happened; that this beta which is greater than alpha, in other words that alpha will be the infumum value of B, because then beta will be the, in which beta will be in B. So, it is a alpha beta. So, this shows alpha is a lower bound, lower bound of B, is a lower bound for B; but beta is not, but beta is not as if beta is greater than alpha it will not the lower bound for. So, it is the least. Therefore, alpha will be the infumum of B and that proofs the (( )). So, this shows the (( )). Now have having proof this thing, we will come back again to the sets we are, we are discussing the open sets and close sets etcetera, we are the supremum value, infimum value will be required. So, we need basically we wanted to saw that result, we are the supremum concept and infimum concept is required.

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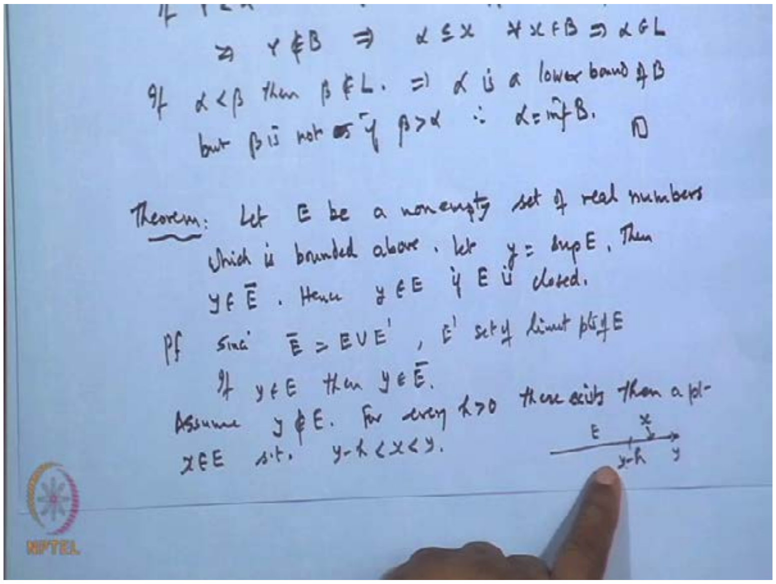

So, that is why all these things were taking. So, this result we wanted to saw the result is let E be a non-empty, non-empty set of real numbers, non-empty set of real numbers which is bounded above, which is bounded above and let y is the supremum value of E, that is least upper bound for E. Then the result says that y will be a point of closer of E, closer of E, means set E together which is limit point is the closer set. Hence y belongs to E, if E is closed. So; obviously, when each close set E bar is equal to E. So, this is a second part follows immediately, they nothing. The first part we wanted to saw first. So, that E is a non-empty set of the real number which is bounded above and supremum of E is B suppose y, then y will be a point in E bar. Now since E bar is a basically, E bar is the union of E and E dash, where E dash is the set of all limits points, set of limits points of E. Collection of all the limits points of E denoted by E dash, now if y belongs to E, if y belongs to E, y belongs to E then; obviously, y will be the element of E bar because it is union of this and y bar, so nothing to prove.

So, let us suppose y is not in E, but y is a limit point of E, we will saw then y is a limit point. So, assume y is not in E, but we wanted to saw y is in E closer it means, y must be a limit point of E so that we wanted to prove. So, let us see for every h greater than 0 their exists, their exists, their exists, then a point there exists, then a point say x belongs to E, there exists a point x belongs to E. Such that, such that y minus h less than x less than y force. Why? Here this is our set say y. E, this is the set E; the point y is not in E is outside of it. Then we can find for each h greater than 0, we can find at least some point which lies in the interval say this is y, in between y minus h to y, this point x will always be available, otherwise if it is not so. Then y minus h will behave as a, y minus h will behave as a upper bound for E, if it is not so.

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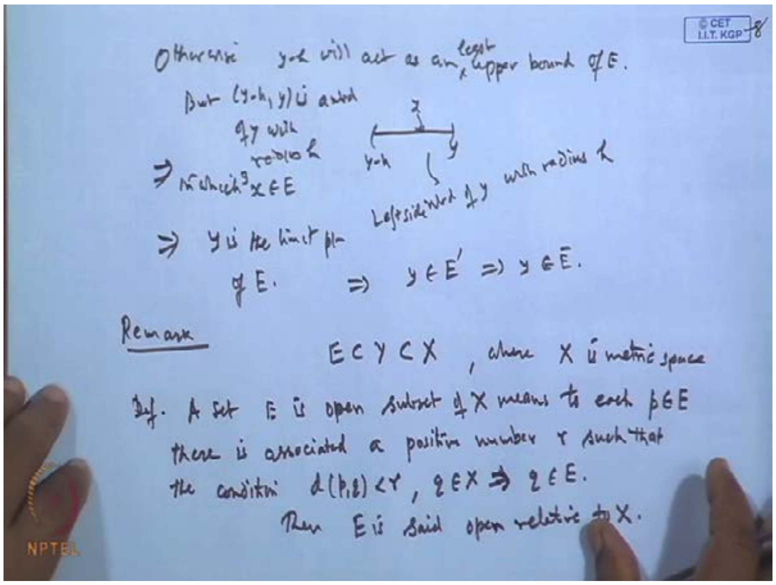

And this is true because otherwise y minus h will act as an upper bound, least upper bound for E, as a least upper bound of E; which is not true. Because y is given to be the upper bound. So, as soon as you take a number slightly lower than this, then this number must be available. It means in between y minus y n and y one can always get a at least one number of x, E which is available, but what is the y minus y n. But this interval by y minus h, is it not in real x lies. So, is it not a neighborhood of y, with radios says h it is the left hand neighborhood left side, left hand left side neighborhood of y in h. So, they are exists of neighborhood of y, which includes the point of x. It means every neighborhood of this y will inputs at least some point of E.

So, this shows that, but this is the neighborhood, is a neighborhood of y with radios h; in which the point x is in E, in which they will exist a point x in E. So, this shows that y is the limit point of E. Because the definition of the limit point of the set in means every neighborhood around the point y, every never hood to y possible is small radios may B must includes the points of E and this is true here that if we take any never hood of y, at least one point x is available. The otherwise, if it is not available then it will contradict to the fake that y is the supremum value of

E. So, if it is a limit point then y must be the point in the closer, E dash hence y is in closer of this set. So, this proves the result.

Now remark we can see; we know if E is a sub set of Y, is a sub set of X, suppose we are X is a metric space. Then we have seen that a set may be open in X, may be open in Y and may not remain open in X. This we have seen just like a open interval a B which we have seen it is open in r 1, but it is not open in r 2. So, in case of the open set or close set the space which in closes the set is important. Here the set is E is open. So, that is why we introduced the concept of an open set relative to the space, relative to Y or relative to X.

So, we introduce here two definition; that one is, a set E is open, a set E is open, E is open sub set of X. Means means to each p belongs to E, belongs to E there is associated associated a positive number, a positive number r such that such that the condition, such that the condition d of (p, q) is less than r, where q belongs to X imply that, implies that q is in E. Than we say E is open with respect to X, open sub set of X means then E is said to be open open relative to X, relative to X. Similarly we can say Y, see Y is also may take a place under the same may take topology d. So, E may also be open to respect to Y.

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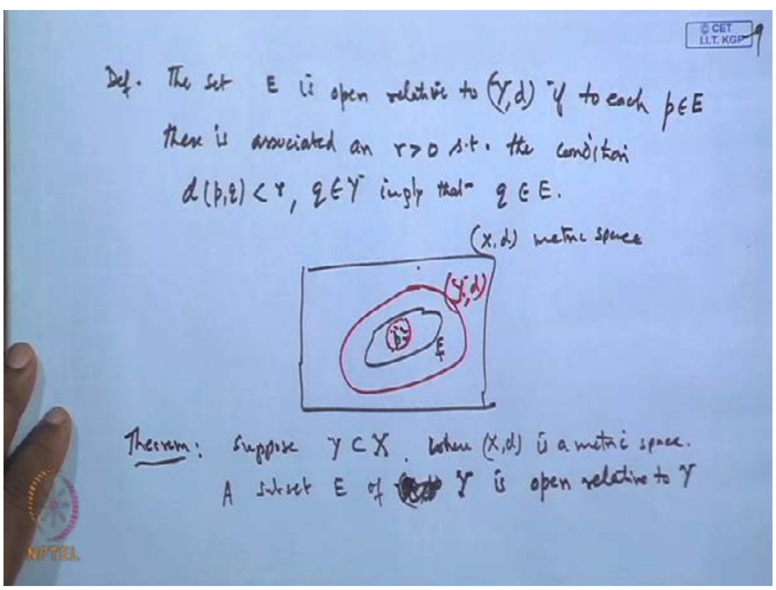

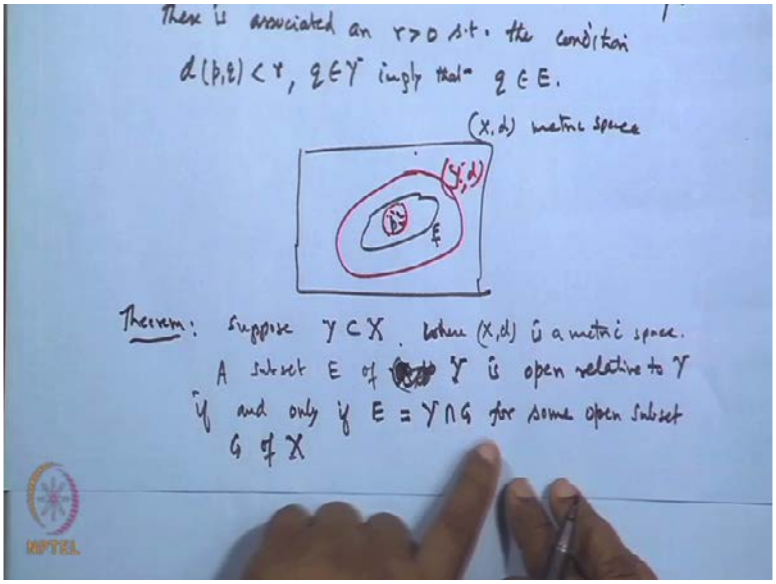

Then we say define, we define that a the set E is open, the set E is open open relative to Y, relative to Y, by d, by d or metric a place by d if to each, if to each here also (X, d) you write X d is a matric a place. Relative to X Y, relative to y if to each p belongs to E there is, there is, there is associated, they are each an associated an r greater than 0; such that, such that the condition condition d of (p, q) is less than r, q belongs to Y imply that, imply that q is in E.

Then we say it is a, it means what, suppose we have the set X d which is a metric space a set E this, is a set E we say it is open in X means, that if we take any point p; In E then one can always find out neighborhood around the point p or a boll centered at p with a suitable radios say r, such that all the points inside this is a point of E, is a point of E. Then we say that E is open in all the point q, q which are of X, all the point q which are in X if they are there are the point E then we say it is a opinion. It means that every point is interior point with respect to (X, d) but when we say this is our Y, Y is a sub set of X. So, it is also a metric space with respect to d, then we say that E is open with respect to Y.

Now here when you draw the neighborhood around the p, then the point q which you are choosing, will must be a point of Y. Because you are not getting the one off course the point of Y is also the point of X, but there may be some point which are in X, but not in Y. So, this relation when the distance of p q is less than r and q belongs to Y implies it q is in E. Then we say E is open in Y. So, as if there is no X, only E is a sub set of Y and E will be the open set in Y every point of p is an interior point with respect to Y; that is all. Then we say E is open relative to Y similarly E is open, now this has been shown that a set E may be open with respect to Y, sub set may not be open with respect to the large set and that example we have seen; however, in case of the compact set, we will see this result, this restriction is not there. So, that is more fluently than our open set or close set. So, that is cover.

So, we will see that, before going for the compact set definition we have one more results, that result shows, what will be the form of the open sets in the relative case. Suppose Y is a non empty sub set of X, (X, d) be a metric space where, (X, d) is a metric space. Let (X, d) be a metric and by be a non empty sub set of X. A sub set E of X, a sub set E of (X, d), of (X, d) is open sub set E of (X, d) a sub set E of Y I am sorry. So, I was sub set E of (Y d), (Y, d) let it be (Y, d), a sub set E of Y, a sub set E of Y is open is open relative to Y, relative to Y if an only if, if an only if, if an only if E can be expressed as or E can be written as Y inter section G for some open sub set G of X.

So, what this results says is, let (X, d) be a metric space and Y is a non empty sub set of X, So, Y under the same metric d will also be metric space and suppose E, is a sub set of Y. Then we say a sub set E of Y will be open with respect to Y or related to Y if E can be expressed into this form, for some open set G of X. If an only that is if E is of this form then E will be a open set, sub set of Y and if E is open, then it can be express into this form. So, vice versa; let us see the proof of this.

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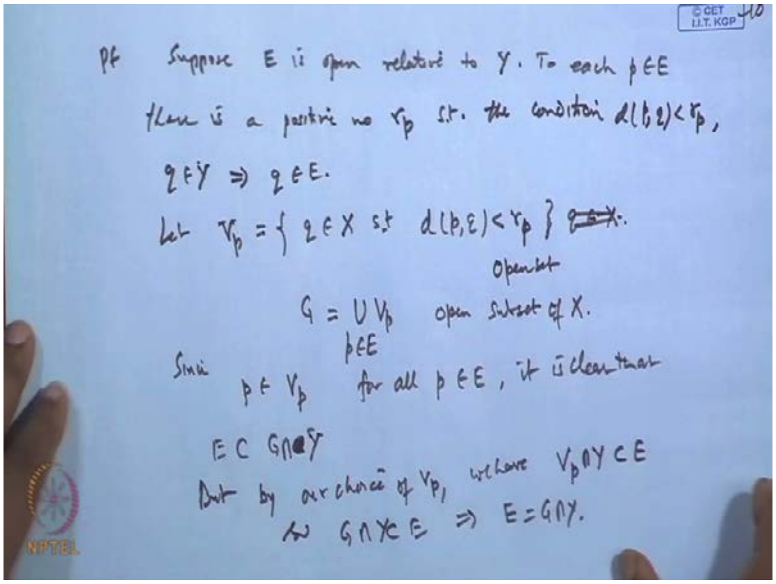

Suppose E is open relative to Y, E is open relative to Y, Y. We wanted to show E will be of this once. So, why the definition of the relative to Y means to each p belongs to E, there is a, there is a positive number, positive number say our r p such that, such that the condition condition d of (p, q) less than r p, where the q belongs to Y implies implies that q will be in E. This is by definition when E is open relative to Y. Now let us consider V p is that collection of all such q belongs to x, such that distance from p q is less than r p, less than r p where q is in the elements of Y; let us q is in X, q is in X. So, this is already there, (( )) Let us see now; obviously, this is a neighborhood. So, once in neighborhood, it is an open set, it is an open set and G if I take the union of all these V p where the p belongs to set E then this collection of the open set will also be able. So, it is an open set is an open sub set of X. Clear? Nothing to...

Now, since p is in the neighborhood V p with centered p and radios say r p, p is in the for all p belongs to E. This is while construction V, because p is the center of this neighborhood. So, it is clear that, than it is clear, that E will be contend in G, which is con G inter section Y. G inter section y. Because this V p, p is a set in E and all the points in E belongs to V p and G is the union of V p. So, E every point of E is in y as well as G. as well as in G. So, it is intersection of this thing is obvious. By our choice, but by our choice of B p we can say, we can say that we have that V p intersection Y, is a sub set of E. By our choice means, because we have V p constructed like this way, set of all that. This is There is a positive such that this one is, so when restrict q to Y. Then all these points basically they are the points common to E; intersection with this and contending E. So, by our choice, because this will be the set to each p there is a (( )) because E is open, because E is an open set. So, this entire thing is available in E, because E giving to be an open set related this. So, his is our choice therefore, for our for V p. So, that. So, the G intersection Y, take the union of this G inter section Y is contending E. Enhance E will be equal to G intersection Y. So, one result is complete.

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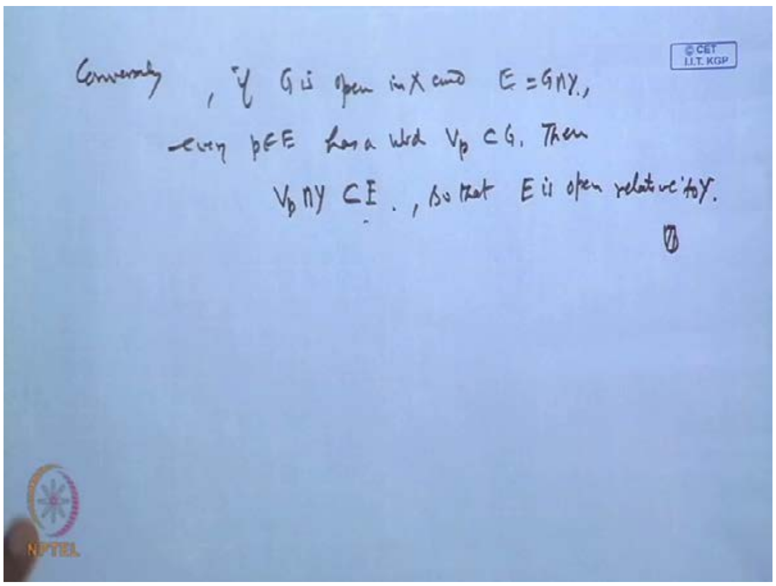

Conversely just one movement, conversely if G is open, G is open in X and E is of the form G inter section Y, then every p belongs to E, every p belongs to E has a neighborhood V p this is totally contend in G. Because G is open and E is of this form. So, E will also be for any p is belongs to E means, it will be in G and G is open. So, never hood must be available in G. Then the G, V p intersection Y, neighborhood inter section Y will be contend in E; because p is in there and G is in this form. So, intersection will be available in E. So, that E is open. So, that E is open relative to Y. Y, relative to Y and that is proves the results.

Thank you very much.

Thanks.