A Basic Course in Real Analysis

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Lecture - 10
Various properties of open set, closure of a set

So, in the last lectures, we have discussed the few concepts like open set, closed set, perfect sets, boundedness of the set in a general material space. Today, we will continue with the same, with certain results related to this sets, and also few properties of this.

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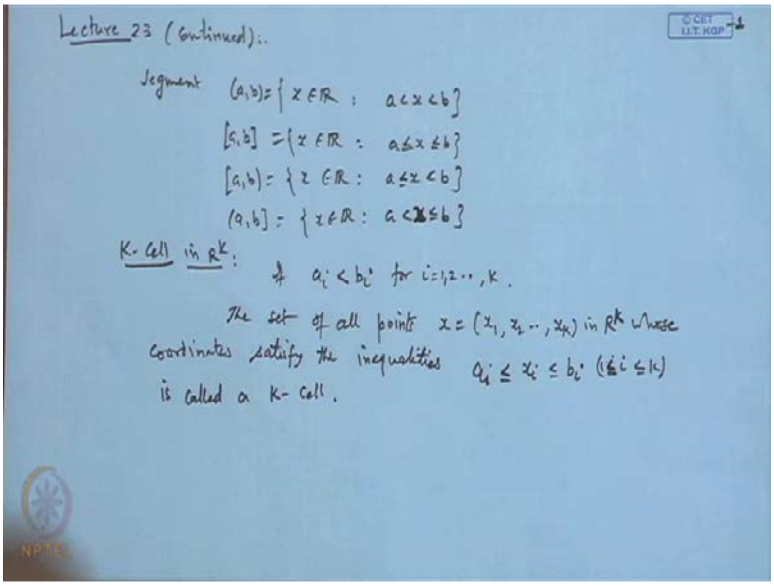

So, before going let us see that is, which is already known the segment a, b be means set of all real numbers x, such that a is strictly less than x less than b. Then this set we call it as an open sets open interval, then segment a b is set of all those real numbers, where a is less than equal to a less than equal to b; this is the close interval and these are the semi close interval set x belongs to r, such that a is less than equal to x less than b, left hand it is bounded by the right hand side is open, while this is a right hand side bounded and left hand side open. So, these are the semi closed intervals x, x less than equal to b.

Now, we define the k-cell in this space R k as follows. Let us suppose, if a i is strictly less than b i, for i equal to 1, 2 and say k, then the set of all points x having the coordinate x 1, x 2, x k in R k, whose coordinates satisfied, the inequalities a i is less than equal to x i is less than equal to b i for i is lying between 1, and k 1 and k is called a k-cell.

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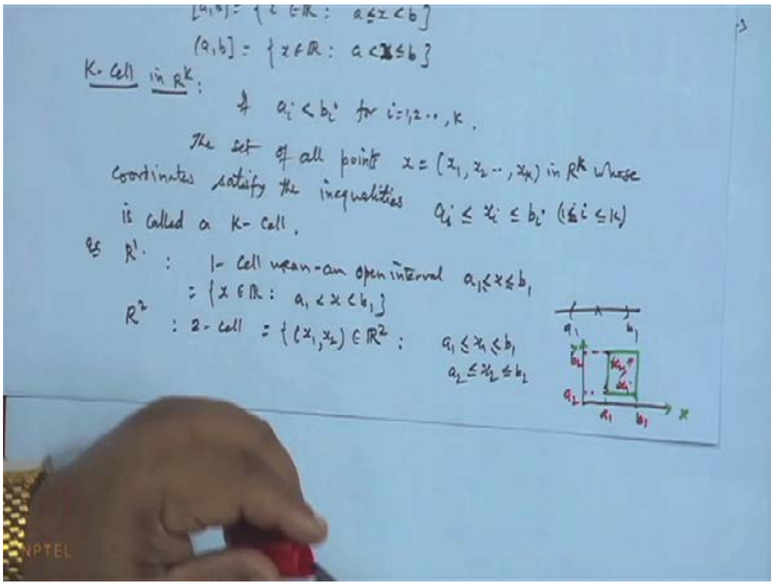

So, the meaning is that, suppose in case of R, 1 say r 1, the 1-cell means an open interval a 1 less than x less than b 1. So, this is a 1-cell a 1 and b 1, and all the points coordinate. So, basically this is the collection of this point x, such that x lies between these. So, it is basically a set of x belongs to R such that x lying between a 1 and b 1.

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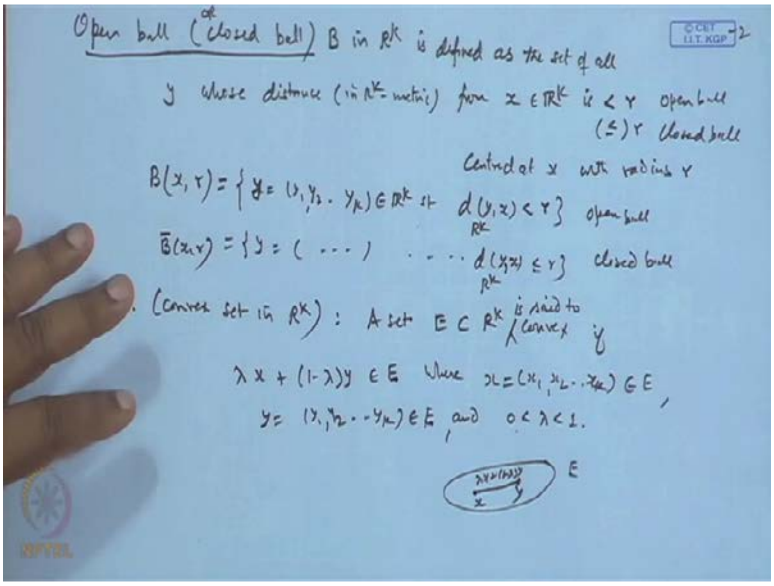

While in case of R 2, the 2 cell means the set of those interval x see, say x 1 x 2 these are the points in R 2 r, such that x 1 lies between a 1 b1, this is equal to also b 1, where the x 2 lies between a 2 b 2. So, it means this is like a rectangle, say this one is our rectangle; this is x axis; this is y axis, and here is this rectangle. So, these are the points, say here is a 1 b 1, and while this point is a 2 b 2. So, x 1 lies here, x 2 lies here. So, it is basically the range inside this a rectangle, closed rectangle bounded by.

So, similarly in case of R k, we have a k-cell concept of this. Now, the concept of the open ball or the closed ball B in R k-cell in R k space is defined as the set of all y whose distance in R k of cos-metric in R k-metric from x; x is a point in R k is strictly less than r, and when it is equal to r, then it is called the closed ball, and this is the open ball centered at x with radius r. So, it means the B centered at x with radius r in R k is placed, where the x belongs to R k is the set of y 1 y 2 say y k belongs to R k, such that d of y x in R k is strictly less than r, then this is the open ball and for closed ball, it is less than equal to x belongs to r, such that d of y x is less than equal to r, this is in R k is a closed bar. So, we sometimes denote also as B bar x r closed bar for this now, we define the convex set in R k, we call a set e sub set of R k said to be convex, if lambda x plus 1 minus lambda 1 minus lambda y belongs to E.

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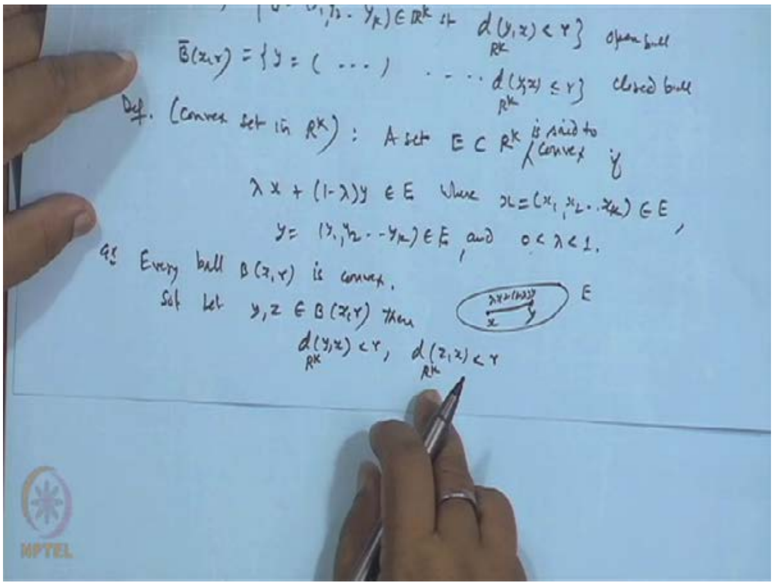

Whenever x, which is x 1 x 2 x k belongs to e y, which is y 1 y 2 y k belongs to E, and lambda lying between a real number lying between 0 and 1. So, this set is said to be convex, it means, this is our set say E, take the 2 point x and y, and if the line segment joining these 2 points, that is the lambda x plus1 minus lambda y, this is the set of all point in between x and y, because lambda lying between 0 and 1. So, entire line segment, if it also lies in E, then we say set E is convex in this set; obviously, this convex set will result or as a examples are every ball centered, say B x r open ball I am choosing open ball is convex every ball is convex, this centered x and radius r, the reason is because if we picked up the 2 elements from this ball say y and z.

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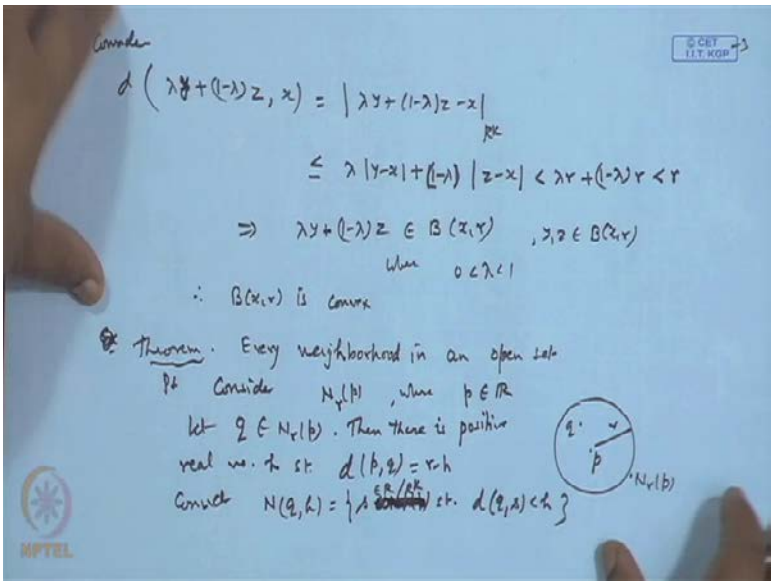

Let y and z belongs to the open ball centered at x with the radius r, then by definition d of y x in R k, because we are choosing in R k off course, is less than r, d of z x in R k is less than r then what is a if I take the linear combination means a line segment joining y and z, if it also these every point on it, this line segment belongs to the ball, then ball will be a convex set.

So, consider this set, where consider the distance of lambda y, y and z lambda y plus1 minus lambda z distance from x is not, that is equal to mod, we are denoting mod of lambda y plus one minus lambda z minus x, and this is in R k. So, need not to write R k, but I am just putting to avoid the confusion, because this mod does not mean here the simply the absolute difference in as, we say in the real line. So, it since the metric is taken from the R k. So, this mod means under root of this coordinate x 1 minus y 1 is square x 1 plus x 2 minus y 2 plus x n minus y n square of this, this is the meaning of this mod.

So, this will be less than or equal to lambda times mod y minus x plus1 minus lambda times1 minus lambda times, this is 1 minus of mod z minus x, just adding and subtracting here x and here x, we are getting the same thing then1 minus this, but y and z are in ball. So, this is less than r plus 1 minus lambda into r, and that will be nothing but what is less than r. So, this shows the line, the point lambda y plus1 minus lambda z is in the ball centered at the x, and radius r, and this is true for any arbitrary point y and z, where y and z is an arbitrary point of the ball. So, entire line segment belongs to this, where lambda lies between 0 and 1 therefore, ball is convex set is convex, this we will require it. So, we have given the concepts now, we have seen the concept of the open set, closed set etcetera in the general material space.

So, we also have few results, say in the form of example or may be the theorem, say every neighborhood is an open set. So, proof is let us considered a neighborhood centered at p with a radius r, where the p is a point in r. So, we are taking say neighborhood I am just taking in the form of that suppose circle it may be, but it may be depend on the space and in open set in. So, suppose centered is p with the radius, say r we want this neighborhood to be open, it means every point of this neighborhood, if it is the interior point, then it will be an open set.

So, consider a point q in this neighborhood, let q belongs to N r p then say there is q find out this distance from p. So, one can find. So, there is a then there is a real a positive real number is h, such that the distance from p distance between p and q r minus h q is inside; obviously, the distance will be less than r. So, h can be obtained. So, that the distance is exactly this now, we wanted to show that there will exist a neighborhood around the point q, which is totally contained inside this n r p, then n r p becomes open.

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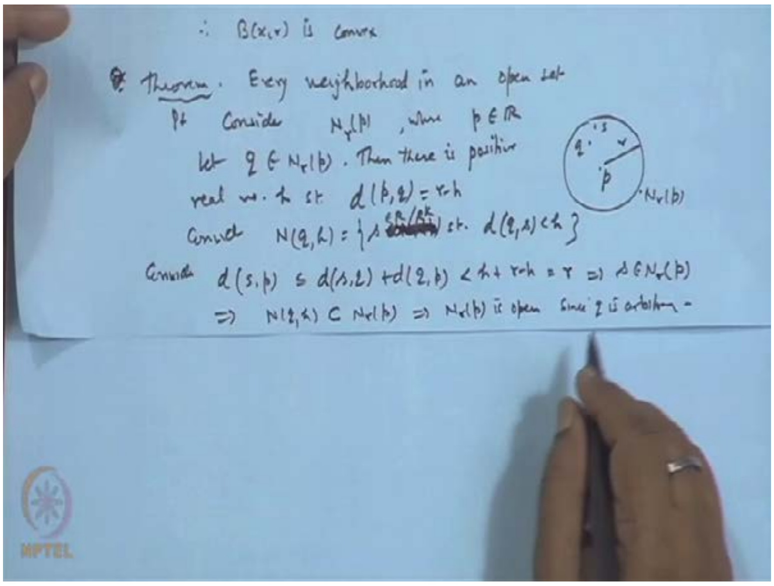

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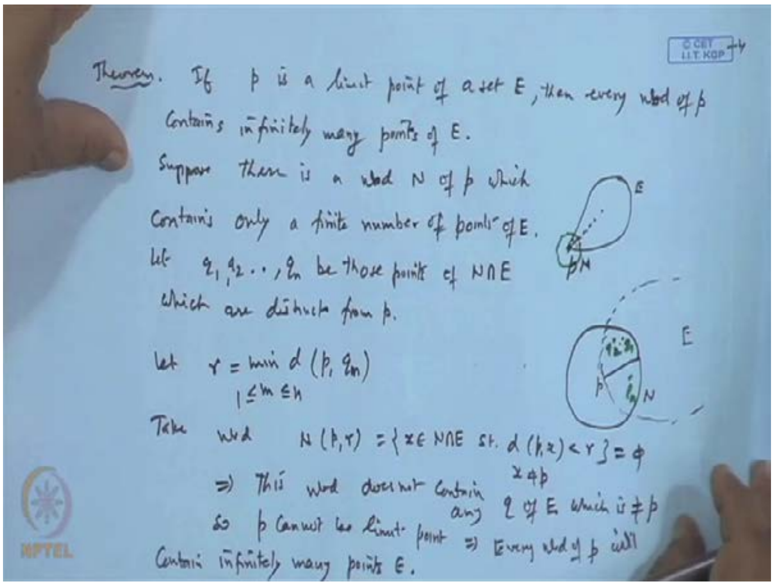

So, consider the set or neighborhood around the point q with a radius h, say this is the set of all s belongs to N r p ah belongs to h, such that not N r p with this, we will show that it is in there, such that distance from this q s is strictly less than h is in R, this is in R real line or R k space, if it so in R k and so on, or R k any one of them clear. So, this, we wanted is entirely containing this. So, let us find out distance consider the distance of this s, s is somewhere here s form p, then this is less than equal to distance from s to q plus distance q to p, but distance s to q is less than h, and this is r minus h.

So, this total is r, it means the distance of s from p is less than r. So, this implies that s is an element in r p, N r p neighborhood of the point p with radius r, and since s is an arbitrary point, basically in this disk this is radius h. So, entire neighborhood the neighborhood N q h is totally contained in N r p, this shows N r p is open, because the point will be interior point and q is arbitrary, since q is arbitrary point. So, every point is an interior point therefore, it is an open set. So, this shows q is a clear, this is open, this subset means q is the integer point as or because q is integer point, and since q is arbitrary therefore, it is completely arbitrary.

So, this would be another result also which is also useful, if p is a limit point of a set E, then every neighborhood of p contains infinitely many point of E, what he says is that suppose, this is a set E, and p is somewhere here ,say p this is p now, this p is a limit point of the set E, if we draw any neighborhood around the point p, suppose I draw any neighborhood around the point p, then this neighborhood definitely will contain infinitely many points of p, then if p is a limit point then any neighborhood around the point p or every neighborhood of the point p will definitely include infinite number of points, if it is not, then p cannot be a limit point, that we will show by.

So, in order to prove this thing, we will suppose there is a neighborhood of the point p, which does not include the infinite number point, but includes only finite number of points then we will lead a contradiction. So, let us suppose, there is a neighborhood, say capital N of p, which contains only a finite number of points only a finite number of points of E say these points are q 1 q 2 q n. So, let q 1 q 2, and say q n be those points be those points of N intersection E, because this N, this is our set N. So, n intersection c will be this set.

So, this contains only finite number points q 1 q 2 q n, which are distinct from p. So, now, what we do this is our point p here, this is say neighborhood, and here is something like e this is our E. So, here is the point q 1 q 2 q n and so on. So, these are the points say q 1; these are the point q 2; these are the point q n; these are the points now, what we do is, we find out the distances of these points from p.

So, let us find the distance is d of these points q m, where the m varies from 1 to n, and then find out the minimum distance from this p, that is m 1 is less than equal to n, find out the distance from q 1 to p q 2 to p q n to p, and among all these distance, find out the minimum one, find the lowest one say, and this distance is suppose r, let r be this now, if we construct a neighborhood, take the neighborhood centered at the point p with the radius r.

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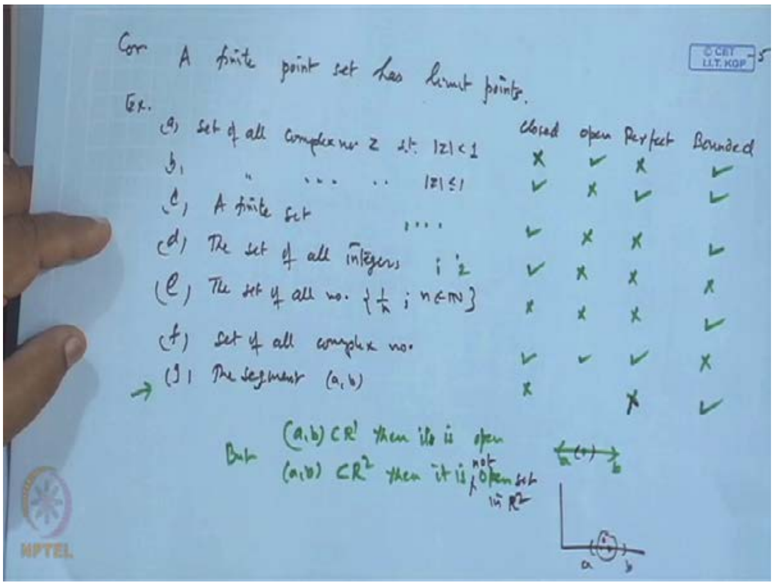

Then this neighborhood means set of those points x belongs to the neighborhood belongs to N intersection, say E such that distance of p to x is strictly less than r so; obviously, once you get this thing, then what is this, this set will be empty set, because x should be different form p, because there are in this neighborhood, this is the neighborhood and intersection E, it contains only the q 1 q 2 q n s only N point, and the distances are taken from minimum distances r. So, if we find a neighborhood of point, whose distance is still lower than r, then we do not get any point except p itself, but we want the p, the point should be different from p. So, basically there is no such point an empty set. So, p is not element.

So, this neighborhood does not contain a point of E, this is different from p. So, this implies. So, this neighborhood does not contain a point q of E, which is different from p does not contain any q, which is different from p. So, p cannot be a limit point, because by definition of the limit point, if p is a limit point, then every neighborhood of p must contain at least few points of a, at least some point of E, which is the.

So, this contradiction and contradiction is because our wrong assumption that a neighborhood, there is a neighborhood, which contains a only finitely many points, only finite number points. So, this is all this implies that the every neighborhood of p will contain infinitely many points of E, this proves the results now, as a corollary of this, we can say a finite point set has no limit point, because again in a similar reason, we can give that has no limit points, let us take few examples where this.

Suppose, we have the sets like set of all complex numbers Z, such that mod Z is strictly less than 1, then set of all complex number Z, such that mod Z is less than equal to 1, then c is a finite set, d is say set of all integers, E is the set consisting of the number is like this the set of all complex numbers of the form, say 1 by n, where the n belongs to N natural number, all numbers of this, and E f set of all complex numbers, and then the segment a b.

Let us see this example, see whether these sets are closed which are open, whether they are closed, whether they are open, whether they are perfect or whether they are bounded because this concepts, we have already seen now, you see the first one; obviously, it is not closed, because closed means the set is closed, when the mod of z is less than equal to one is the point, which is the limit point of this. So, it will not be a closed set. So, this will not be a closed set no; so here is no.

Then it is open because mod z less than every point will be the integer point, it will be an open, then perfect set, the set is said to be perfect, if it is closed, and every point of E is limit point say mod Z is less than 1 is not an closed set then; obviously, it will be not be a perfect set, and then bounded, yes it is bounded by 1 means all the point of Z is less than equal to 1. So, it is a bounded set, this set, yes it is closed because all the points which are this limit point all belongs to E the limit point of the c is a point of this limit point is there it is not open, because of the closeness it is not open perfect.

Though it is closed, and then what all the every point, whether every point is the limit point or not, yes it is inside that this more than less than equal to 1, every point is a interior point, when mod Z less than 1, and 1 is also the limit point in a. So, it will be a perfect set, then bounded, it is bounded because 1 is the finite set; finite set, it does not have any limit points. So, we can assume all the limit points inside it are there. So, we can say it is closed, but it is not open, why it is not open because the finite set, when you choose these are the sets. So, if we draw the neighborhood around the point p, then we do not get any point other than this. So, this will not be an interior point. So, it will not be an open set.

Since, it is closed, but no limit point. So, we can say this set is not perfect because every point must be the limit point, perfect set is closed and every point of p is a limit point, but every point is a not a limit point. So, it is no, and then off course, it is bounded because you know only finite number of z, we can identify a point find out its distance from this particular point.

So, you can find an m upper bound. So, that is bounded. Set of integers; this is closed set again, the same thing set of integers, the point every point is not a limit point, because again, there is a gap between 1 and 2, there is a gap. So, we can draw the neighborhood around the point 1, which does not include any integer. So, one cannot be limit point, similarly all other integers they do not have a limit point.

So, set of integer does not have a limit point, but we can assume all the limit points inside it. So, we can say it is a closed set say d, but it is not open because the point are not interior. So, it is not open then it is perfect, again every point is not a limit point. So, it is not a perfect, and whether this is bounded or not; obviously, this is 1 2 3 is the unbounded set, you cannot find at n. So, that all the point is less than equal to n. So, this is not bounded then set of all numbers 1 by n, 0 is the limit point for this except 0, no other point is a limit point for this.

So, this set is not closed, E is not closed because closed means all the limit point belongs to this, because 0 is not the point in it. So, 0 is a limit point, which is not n. So, it is not closed, it is not an open set, why because the point 1 by 2 1 by 3, they are not the limit points, again they are not a limit interior points therefore, it is not open, it is not a perfect set; however, it is bounded because the bounded for the bound is 1. Set of complex number is an entire set, we can say all the limit points are inside, as well as the every point is the limit interior point, because we can draw the ball around the each point, which is totally contained in the complex plane.

So, it is open, closed is perfect also, because all the points are limit point every point is a limit point. So, we can say it is perfect, and then bounded set is no, because is un-bounded plane. Segment; segment is not an closed, because it is an open segment, it is an open set, but when it stay the segment a b, then the openness depends on the topology on the sets, suppose I take a b as a subset of r 1, then it is open, but if we take a b as a subset of R 2, then it is not open why, because it is not open because every point of this set is not a interior point the reason is.

In this case, we can choose this is a set a b, whatever the point you can choose, you can draw the neighborhood around that point, which is totally contained in, but in this case the a b interval is this, this is our a b and the neighborhood, when you take the neighborhood will be something like this. So, here the points are available here, which are not the points belonging to the set. So, we can say this point cannot be an integer point. So, similarly as that so cannot be an open set its, then it is not open set in R 2. So, that is important for this. So, that is why we are not, and further the perfectness is not there.

So, we can say cross, and then bounded, yes because it is bounded a b, whatever the topology you take it is bounded. So, you can this. So, this way we can identify now, here this example shows that openness or closeness are relative concept, say the set which is a subset of a set, which is the set of which it is a subset, say with respect to that, whether it is open or not, if we consider a b as a subset of R then; obviously, it is a open, but when we consider a b as a subset of R 2 then it is not open. So, it is a relative concept. So, we can further give a definition of the openness or closeness with respect to the set that is called the relatively open or relatively closed sets.

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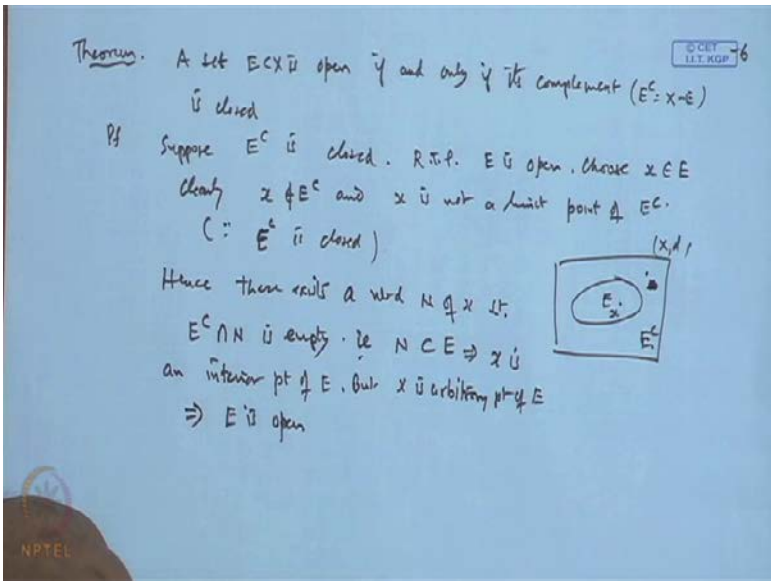

So, we can now the results are further, we have another results theorem, the theorem is. A set E is open, if and only if its complements, complement is closed, complement means that if x is a subset of x, x be the a metric space, then complement of this means denoted by E c, which is x minus E, complement of the set is closed. So, proof is. So, let us suppose that E c is closed, suppose the complement of the set E c is closed, then what we wanted to show E is open required to prove is e is open, it means every point of E is an interior point.

So, choose an element x belonging to E, if I prove that there is a neighborhood around the point x, this is totally contained in E, then e becomes open now, since x is in E, and E compliment is the set of those points which are not in E. So, clearly x is not an element of E compliment and. So, x is not a element of this now, further x is not a limit point of E compliment and x is not a limit point of E compliments. So, is why it is not the limit point of the compliment, because the reason is E compliment is closed.

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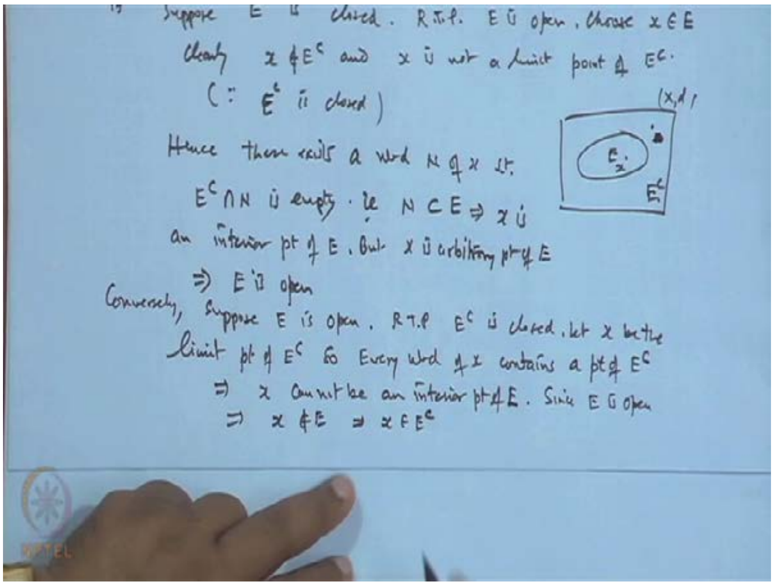

So, closed by definition, all the limit point must be the points of the set, if set is closed it includes all of its limit points. So, if x is not in this. So, x cannot be limit because if it is a limit point, it must be the point of E c. So, here is this E, and this is our space x d, and here this is E c. So, we are taking a point x here, which is in x, we are taking x here now, what we say is x is neither in E c nor is a limit point of E c. So, there will be some neighborhood around the point, we can draw which is totally away from E c or does not intersect with E c and lies in it. So, there exists, hence there exist, because it is not a limit point, it is not a limit point. So, we can find out a neighborhood around the point, if which is totally away from E c or intersect of E c is empty. So, there exists a neighborhood N of x, such that the neighborhood N of x, such that intersection with E c is empty, because it is not a limit point, and if it is limit point, then every neighborhood of x must contain some point of E c.

So, since it is not a limit point, we can identify a neighborhood, whose intersection with this empty, we do not have any point inside this neighborhood, that is the neighborhood is totally contained in E because it contains only the points of E. So, what they show that x is a point around, which we have a neighborhood n this is totally contained in E, this shows that E is open, this shows x is an interior point of E, hence an x is arbitrary, but x is arbitrary point of E. So, every point has is an integer point therefore, E is open clear. So, this shows the concept conversely, suppose E is given to be open. Conversely, let us assume E is open; we wanted to show that E c is closed. So, to show E c is closed so.

We wanted to prove that all the limit point of the E c is the points in E. So, let x be the limit point of E c, (( )), once it is done limit points, every neighborhood of x will include the point of E c. So, every neighborhood of x every neighborhood of contains a point of E c. So, once the every neighborhood contains the E c it means x cannot be an interior point of E c, because if it is an interior point then there will be a neighborhood around the point x this is totally contained in E, but every neighborhood contains the point of E c also.

So, this shows that this neighborhood, this x cannot be an interior point of E cannot be an interior point of E. Now, further what is given is E is open, further since E is open. So, what do you mean? It means, if x is a point in E, then because E is open. So, x must be an interior point, but here we have shown x is not an interior point therefore, this implies that x can not be in E. So, this implies that x will be E compliment. So, it will be an E compliment, this follows that E compliment is closed, that is what is proved.

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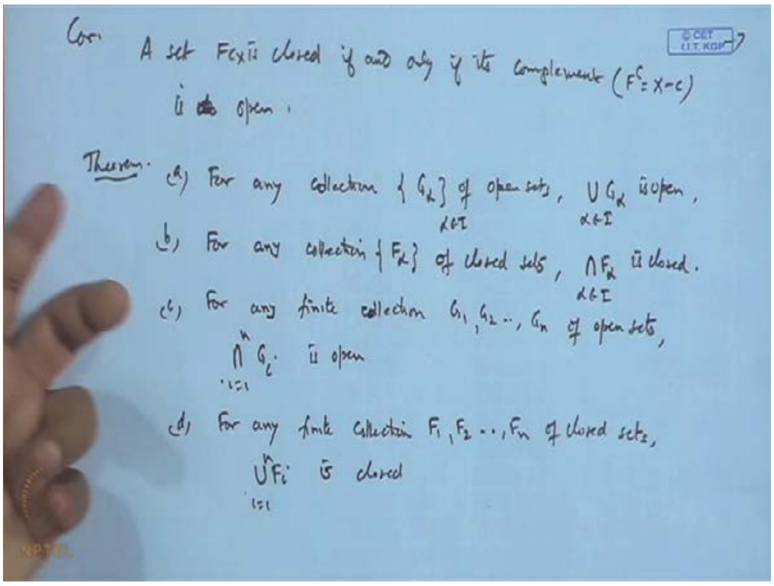

Now, as a corollary of this, we can corollary a set F is closed, if and only if its compliment is open, compliment means, F it is a subset of x, compliment means F c which is x minus c is open now, this proof goes just a previous form the previous theorem, we can drive this corollary easily, then no point of mistake. Now, following result shows for any collection G alpha of open sets, where alpha belongs to the index sets the union of G alpha, when alpha belongs to i; i is a index set, alpha is in i; i is a index set, then G alpha is open, this is one; second result says for any collection F alpha of closed sets, the intersection arbitrary intersection, this is a arbitrary union, arbitrary intersection F alpha belongs to i is closed.

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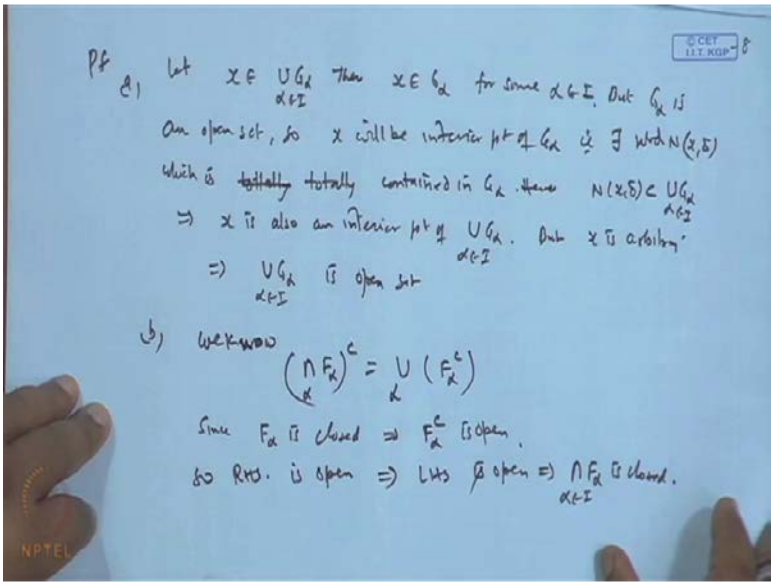

Third, for any finite collection G 1, G 2, G n of open sets, the finite intersection of these open sets is open, d is for any finite collection F 1 F 2 F n of closed sets, the finite union of these F i is closed. So, here this c and d is not valid for an arbitrary or infinite collection or countable intersection of g I, and countable union of F i, it is not valid may not be true.

So, only for finite cases is true, the proofs again follows like this, it is given that each G alpha, for each alpha is a open set, we want the arbitrary union is open. So, let us take a point, let x belongs to the arbitrary union of G alpha belongs to this, it means that then x will belongs to G alpha, for some alpha belongs to i, but G alpha is open is an open set. So, there is just a neighborhood around the point G which is totally contained inside it.

So, x will be an interior point of G alpha that is there exist a neighborhood n with, say suitable radius delta such, which is totally contained totally contained in G alpha, hence this neighborhood N x delta will also contain in the countable union of G alpha. So, this implies x is also an interior point of union of G alpha belongs to i, but x is an arbitrary point.

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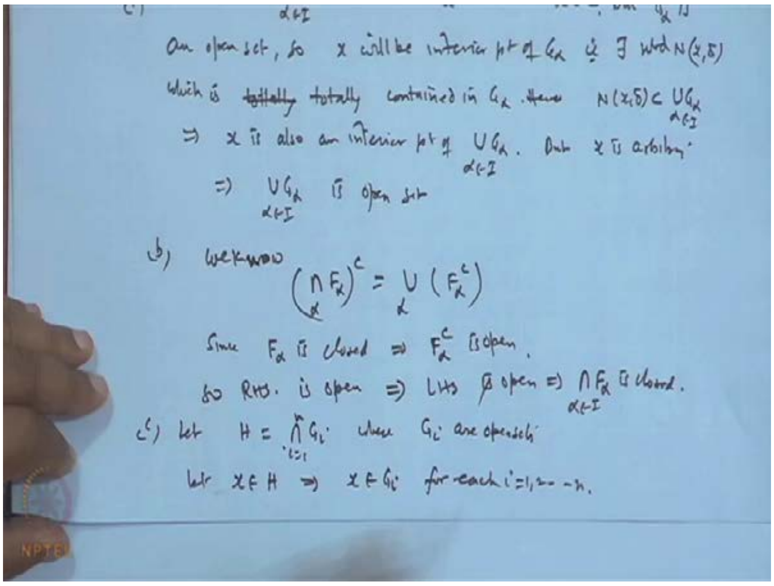

So, what we say that countable union of G alpha is of N, because every point becomes the interior point. So, this implies that countable union G alpha belongs to i is an open set, similarly for the part b follows from this result, we know this is simple say theoretical result arbitrary intersection of this, if i find the compliment of this becomes the arbitrary union of the compliment of this F alpha c now, what is given is F alpha is closed. So, F alpha c will be open, since F alpha is closed. So, this implies that compliment of this F alpha c is open, and right hand side is the arbitrary union of the open set, right hand side is open therefore, this implies the left hand side will be open. So, this implies the left hand side is open.

So, compliment of compliment is closed. So, this implies that arbitrary intersection alpha belongs to i is closed, that is the proof, then second c part, we wanted to show that finite intersection of the open set is open. So, let us take H as the finite intersection of G i, i is one to n, where G i s are open sets. So, let us take the point x belongs to H, let x belongs to h. So, this implies that x belongs to G i for each i 1 to n, if g i is open.

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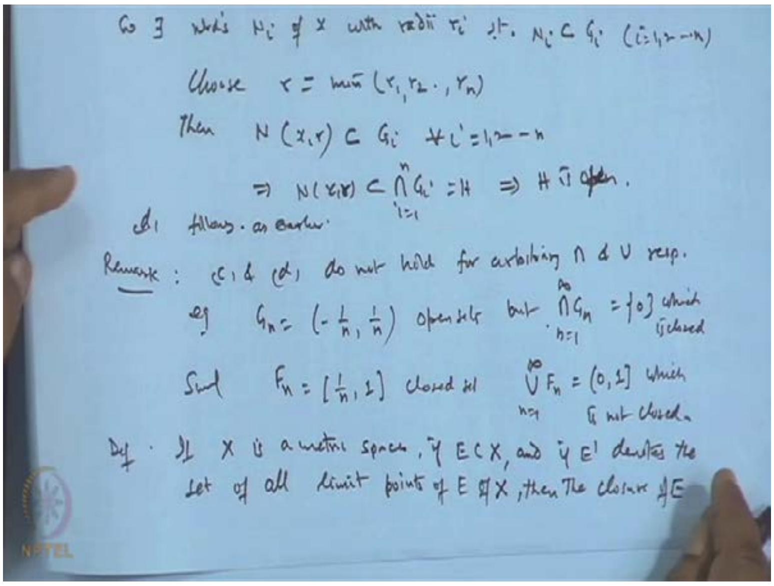

So, there will be a neighborhood around the each G, around x with a suitable radius r i. So, we can say. So, there exist neighborhoods N i of x with radii r I, where i is 1 to n, such that the N i this neighborhood is contained in G i, when i is 1 to n, with each N x, x is in all Z. So, for each i there will be a neighborhood N I, which is totally contained in G i. Now, choose the radius r choose r as the minimum of these r 1 r 2 r n.

So, if we draw the neighborhood, then the neighborhood with the centered x and radius r; obviously, will contain in G i for each i 1 to n therefore, it will contain this intersection contained in the intersection, which is H. So, this shows that H is open; the second part follows in the same d that is a finite collection will like. So, a b c d; d follows as earlier now, let us take the case, when there are an infinite intersection.

So, c and d does not hold for arbitrary intersection and union respectively, the examples are suppose I take the G n as minus 1 by n 1 by n, then these are all open sets, but the intersection of G n, n is 1 to infinity is the single term say 0, which is closed similarly, if we take the say our F n as, say 1 by n and 1 these are all closed sets the arbitrary union of these F n, and is 1 to infinity is the semi closed interval 0 to 1 which is not closed. So, this shows the arbitrary intersection they are not true now, we define the concept of the sets, if x is a metric space, and if E is a subset of X, and if E dash donates the set of all limit points of E of X, then the closure of E is the set denoted by E bar, which is E union E dash.

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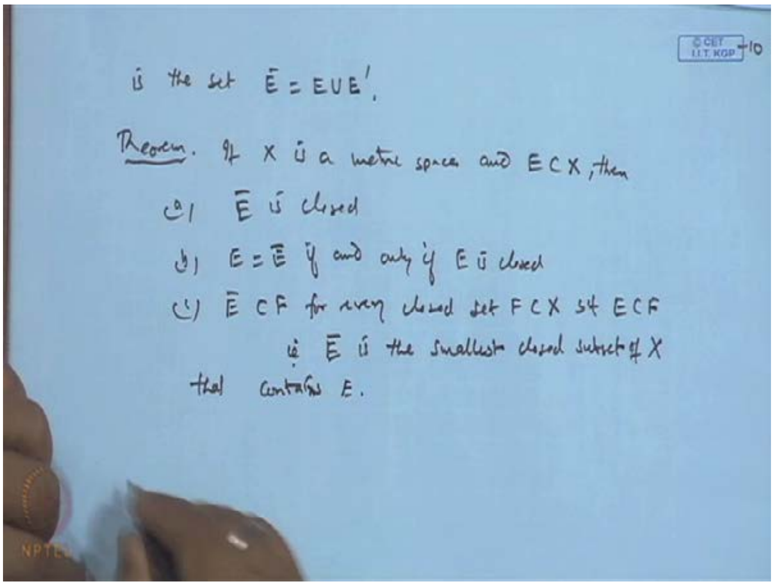

So, closure of this. So, what is this is closure means, suppose a set E is given, then set of all limit points, if I included in E, then the collection will be known as closure. So, closure of the set is the set, which includes all of its limit point as well as the point of E off course, then there is a result theorem is, if X is a metric space, and E is a subset of X, then the closure of this set E bar is closed set; b if E is equal to E bar that is, then if and only if and only if E is closed.

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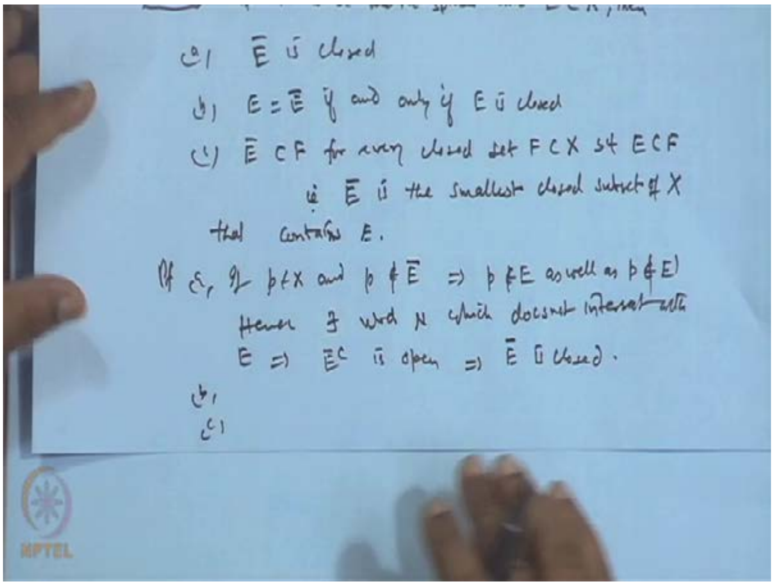

And third is, if E bar is contained in F for every closed set F, which is contained in X, such that E is contained in F, it means that is E bar closure of this is the smallest the closed set subset of X that contains E, the proof is simple, just I will give the first proof a E bar is closed is given to be, it is then closure of this is close. So, let us say, if p belongs to X, and p is suppose not in the closure, then p is neither a point, p does not belongs to X, as well as p is not a limit point of this. So, once it not a limit of this, there exist a neighborhood N, which does not intersect with E. So, if does not intersect with E, then compliment of E bar is therefore, open therefore, e bar is open the compliment of this is open, E c is open then the compliment E c of E bar c is open compliment of E bar c is open, and this shows E is closed; this shows e bar is closed; So, that is second and third part follows very easily. So, we are just dropping and continue.

Thank you very much.

Thank you.