A Basic Course in Real Analysis
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Lecture – 1

Rational Numbers and Rational Cuts

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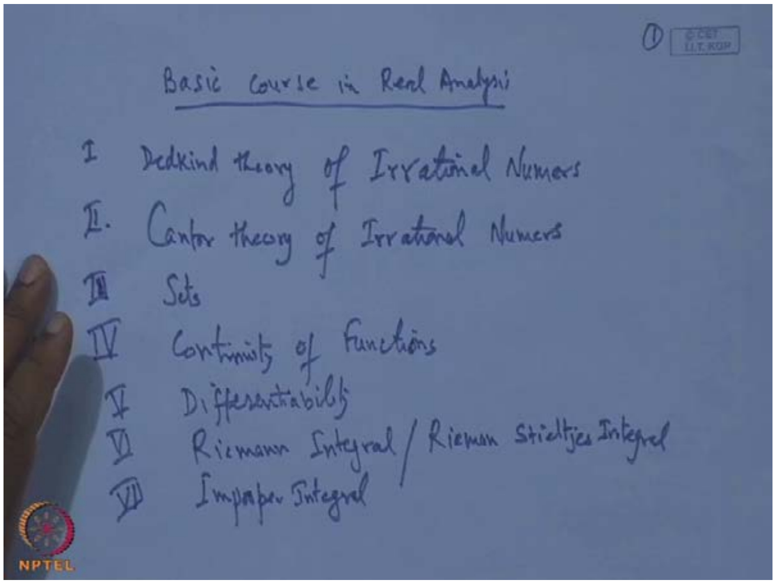

So title shows, it is a basic course in real analysis and the topics which I will cover in this course is Dedekind theory. So, the topic which I will cover, first the Dedekind theory of irrational numbers; the second topic which will give the cantors theory of irrational numbers; the third topic, it is sets which include bounded set and the open, closed and comparative, etcetera; then continuity of the function, of course real valued functions we will take; and differentiability; then we will go for the Riemann integration as well as we will also go for Riemann stieltjes integral; then we will go for, after this if the time permits, then we will go for the improper integral. Though it is not listed in the part, but if the time permits, we will give an idea of the improper integrals, both types whether the upper limits or lower limit is infinity is minus infinity to infinity or 0 to infinity or the case where the limits are finite, proper integral but the function is discontinuous at some point in between, say minus 1 to 1 by 1 x d x, then x at the 0.0 the function is a discontinuous function. So, though the limit is finite but the function discontinuous, it will be treated as a improper integral of the second kind, and the some integral like 0 to infinity and some function f x d x or minus infinity to infinity f x dx, that is also considered to be…And in particular cases, our beta gamma functions which are very useful for those engineering branch. So these we will cover later on.

Now, the first to two chapters which we will discuss the Dedekind theory of irrational numbers and cantor theory of irrational numbers. It is basically in what manner the real numbers has been developed, actually we know the set of sequences, we know the natural numbers is starting from 1, 2, 3 and so on, then when we add two natural numbers, we get a natural number; but when we subtract the 2 natural numbers then we also get some number which is 0 or sometimes negative answer. So, an idea of the generating the whole number comes in picture that is, when the natural numbers is supplemented with 0, 0, 1, 2, 3, this is called the set of whole numbers. And similarly, when the negative integers also included, all the 0, positive integers, negative integers as well as 0, then this is called the integers. And then idea, when the we applied the addition, multiplication, subtraction then any two integers when we add, we get always integer; when we subtract two integer, we always get integer but when we divide the two integers and the denominator is not 0, then we not necessary to always get an integer. So, that gives an idea of the further extension of the number system, and that leads to the concept of irrational numbers. And since always the numbers may not be irrational, that is a number which is square root two or surds, this we will prove, it is not a natural, is not a rational number.

So, again the idea of enlarging this set of rational number again leads to the concept of irrational number and then rational together with irrational will lead to the real number systems. But how this development has been done? That is given by the Dedekind’s in terms of the sections, and cantors also in terms… So, we will give an idea of the development of this number systems in these two chapters and later on, we will go on one each one. So that is idea of it.

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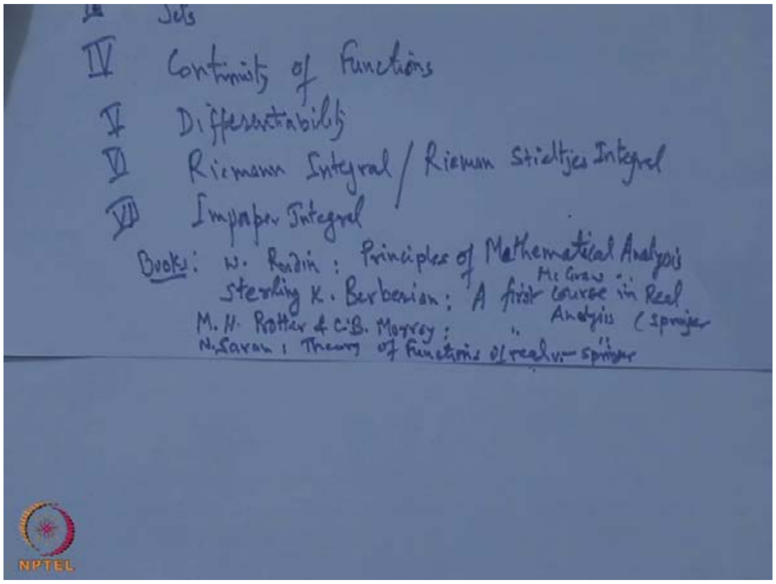

The books which I will follow is the Rudin book, Warder Rudin, the principles of mathematical analysis, it is a Mc, Tata Mcgraw Hill book, it is available; then second book which we follow is Sterling k Berberian, a first course in real analysis, in real analysis; then we also follow book M H, M H Protter and C B Morrey, this is a first course in real analysis, same title in the first course analysis; and this is basically, this is Srinjer Ferlag book, Srinjer Ferlag, and this is also Srinjer Ferlag book, and this is Mcgraw Hill, so these are the books which we follow. And Indian authors, you can go something other books are there, there is one book that is by Saran, N.Saran that also a good book, that is a theory of function of real variable, theory of functions of real variables.

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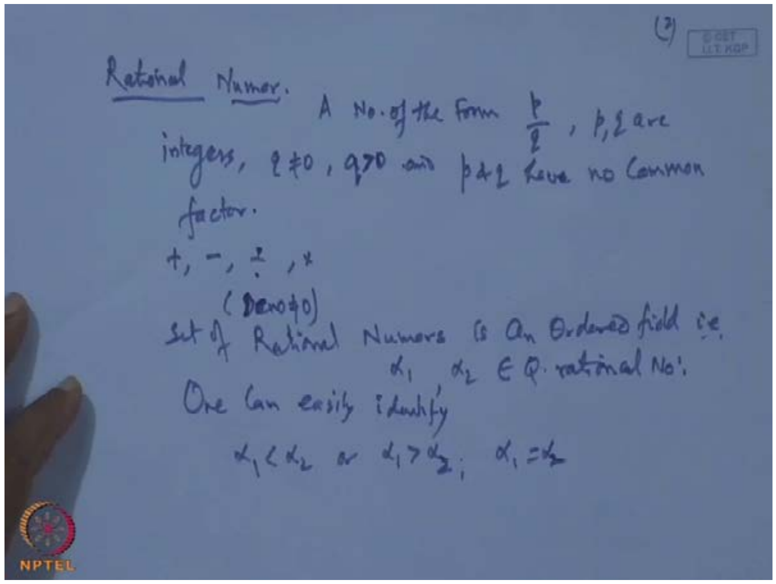

So that is these books is more than sufficient or you can go to many other books, there so many books are available on real analysis. So first, so this is what. Now, let us see the first dedekinds theory of irrational numbers. So, we will start with a rational cuts, first. We know the rational numbers. Rational numbers, a number of the form a number of the form P by Q, here P and Q both are integer, are integers, Q is not equal to 0, and normally we take Q to be a positive, Q greater than 0, and also assume there is no common factor, P and Q have no common factor, P and Q have no common factor. So, the number of form P by Q, where P and Q both are integers, Q is not equal to 0, and normally, we take Q to be greater than 0 and it starts, allow P to vary over the integers then we get all sort of the number, which are called the rational numbers. And these rational numbers, if we add the two rational number, then addition, subtraction, division, multiplication, division means when Q term is not equal to 0, when one term is not equal to denominator, denominator is different from 0. Then division is possible.

So if we apply the operation of addition, that is if two irrational numbers I add, we get again a rational number, if I subtract the two irrational numbers, again we get a rational number; if I divide the two irrational number providing the denominator is not 0, then also we get a rational number; and when we multiply the two irrational numbers, we also get rational number. So basically, it satisfy all the algebraic operations, which are available addition, subtraction, and etcetera. And not only this, this set if rational number is an ordered field, the set of rational numbers is an ordered field. Ordered field, we mean that if you pick up any two rational number from the collection of rational point numbers, then one can easily identify which number is greater than the other or which number is lower, less than the other or they are equal; one can easily identify the ratio between the two number which is less than greater than or equal to. So, any two number, alpha 1 and alpha 2, these are two rational number, Q is the set of rational number, then one can easily, one can, that is one can easily identify either alpha 1 is less than alpha 2 or alpha 1 is greater than alpha 2 or alpha 1 is equal to alpha 2, one can easily identify.

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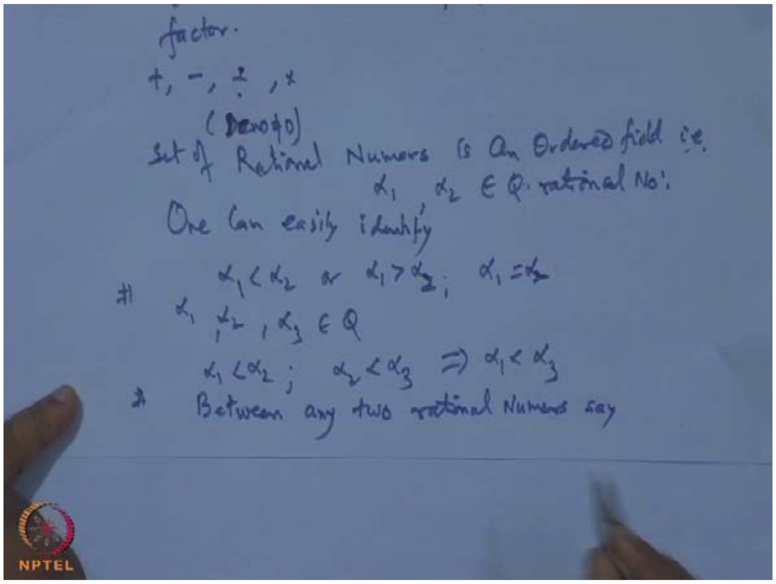

And not only this, if there are three numbers, then suppose alpha 1, alpha 2, alpha 3, these are three rational numbers, and if alpha 1 is less than alpha 2, alpha 2 is less than alpha 3, then easily one can show alpha 1 is less than alpha 3, that is the associative property in case of this, and that’s alpha 1 less than alpha 2 is less than alpha 3, and we get the tangetic property, is it not? Then alpha 1 and alpha 3 are related with less than sign. So, this is what we are… A very interesting property of this lesson of a number is that between any two rational numbers, between any two rational numbers say alpha and beta.

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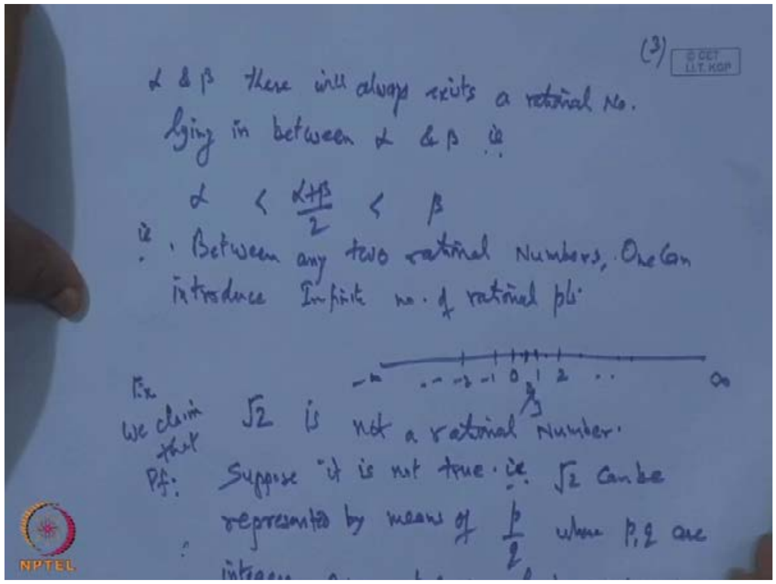

Say alpha and beta, there will always exists a rational number, a rational number lying in between alpha and beta, that is if alpha is this number and beta is this number, one can always, get a rational number, which is alpha plus beta by 2, lies between alpha and beta, that alpha is less than this, this is less than beta. So, we can always find out the number in between alpha and beta. And this is true for a… This is also a rational number, so in fact between two rational number, one can introduce infinite number of the rational points, in between this. So basically that is between any two rational number, rational numbers, one can introduce infinite number of rational point, rational point, is it clear? Now, suppose I represent these rational numbers by means of a point on a straight line by choosing a fix position of 0, and then fix the length of the identity 1; and then on this scale, one can find out a location of a rational points on this layer lines, on this layer line, is it or not? Depending on the length, how much distance is from the 0, say 2 by 3, it means its distance from the origin is two third. So, you can grab this point 2 by 3, say minus 2 by 3, so on, like this. So, the rational point can be arranged, can be located on the, identified by a point on the, on a straight line. Now, will it complete the entire real line? Will it absorb all the points of the real line? Or in other words, what I mean is that can you say because between two rational number, we can always get a infinite number, with this rational points; so and if I arrange this rational points on the some straight line, then can you say that entire straight line minus infinity to infinity can is filled up by means of rational points? The answer is no, that is, there are the points lying on this straight line which are not at all rational point not at all rational point. So, those points which are not at all rational point, we cannot be put it in the set of rational number. So it means, the set of rational number is not at all complete, is it not? So we need the further extension of the rational point. Now for example, if we take a point under root two, square root of two, we claim that, we claim that square root is not a rational number, is not a rational number, this is our claim. Suppose, I prove it by contradiction. Suppose, it is a rational number, then it must lead to a contraction. So suppose, it is not true, that is the meaning of this is, that is under root two can be represented by means of P by Q, by rational point where P and Q are integers, Q is greater than 0, not equal to 0 and P and Q are not have any factor common, any common factor, are not having any common factor, is it? Suppose, I prove it by contradiction. Suppose, I assume that root two is a rational point, it means root two can be expressed in the form of P by Q, where P and Q both are integers, Q is not equal to 0, Q is positive and P and Q is not having any factor common, is it not? So that is it.

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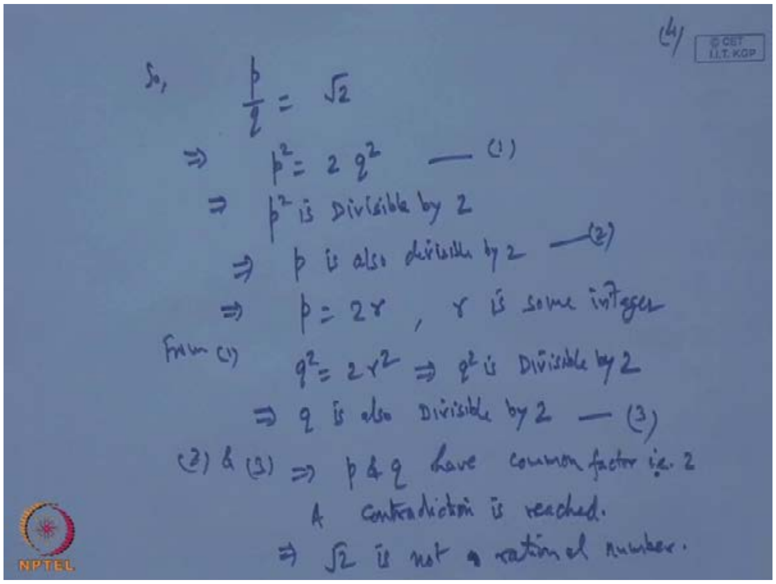

Let us see. So, we get basically. Now so, what we get is that P by Q is equal to root 2, correct? Square it; so, this implies P square is equal to 2 times of Q square. Now P square is 2 times of Q square, it means what? The P square is divisible by 2 with the Q is integer, Q is integer, so Q also be integer. So, if I take P square by 2, you are basically getting an integer. So what can you say? So that P square is divisible by 2. Now, if P square divisible by 2, can you say P is also divisible by 2? If P square is divisible by 2, then P, can you say it is divisible by 2? Divisible by 2, because the reason is if it is not divisible by 2. So, if it is not divisible by 2. So, what it means the P by 2 is not a integer, is not a integer. So, if it is not a integer, when you square it, what will happen? The square of this will not be integer. So, you are not getting the Q square, you follow me? You are not getting integer. So, if P square divisible by 2, the P has to be divisible by 2. It means when P is divisible by 2 then P can written in the form of 2 times R, where R is some integer, is this clear or not? Or I will explain again, suppose when we say P square is equal to Q square; will not imply P square is an even integer? Even integer, what? Integer which are multiple of 2, these are even integer. So, P square is an even integer, so square root of this P, even integer means it is divisible by 2 so square root of P will also be even, because if P is not even, the square of this may not be immune, because say P is 3, what is the square? It is nine, it is not a even integer.

So, if P square is even, the division of this, square of this will be even. So, that is why we say P is also divisible by 2. Once, P is divisible by 2, P can be expressed in the form of 2 R, where R is some integer, then only you can divide by 2, you are getting integer. Otherwise, no; so, substitute it here. So, what we get from equation one? So, from one, if I substitute then what you get? Q square becomes 2 times of R square, is it clear? Again, the same argument, it means what? Q square is divisible by 2, is divisible by 2. So, Q must also be divisible by 2, Q is also divisible by 2. It means P is also divisible by 2, this is 2, Q is also divisible by 2, its 3 equation. So, what you get? Common factor 2. So, two and three implies that P and Q have common factor 2, that is 2 which leads a contradiction, because if root 2 is a rational number and express in the form of P by Q, then P and Q should not have any common factor.

So, a contradiction, a contradiction is reached and this contradiction is because our wrong assumption that root 2 is a rational number, it means this implies root 2 is not a rational point, is not a rational number, is this clear? And in fact, this is the one example, I have taken, you take any square root of a positive integer, any square root, a square root of any positive number, say under root 5, under root 6, under root seven, this all will be a irrational number, we call it this later on, which are irrational point, not rational points.

So, it means the set of a line, the set of points a rational point on the line is not complete, there are gaps, which gaps can be filled up by means of these irrational points. How to introduce this irrational point? That is done by the Dedekind’s. How did it introduce this rational point concept with the irrational point? That is done by Dedekind’s, same in the cantors.

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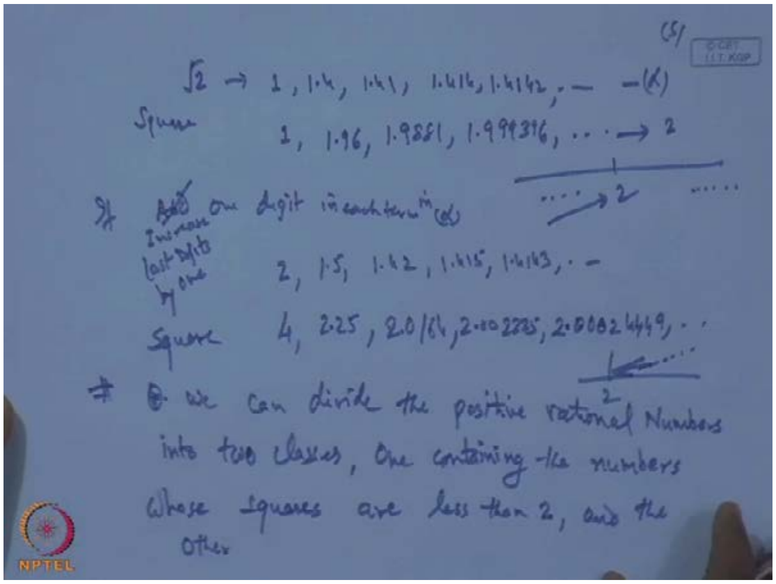

Now, let us look another thing from root 2, if you say root 2, and suppose I just calculate square root of 2 by using our calculator, then what value you are getting? Suppose first 1, then 1.4 only first decimal place, then 1.41 second decimal place, and if I continue this, then what you get? 1.1.414, then 1.4142, and so on. Just take that square root using your calculator with the first decimal place, second decimal place, third and continue, is it not? Now, if I square this, so square of 1 is 1, square of 1 point 4 is nothing but 1.496, square of 1.41 will be 1.9881, then 1.9993 9 6 and so on, this is square, is it not?

Now, if I plot this point, what is this? This sequence of the square does it not tending to 2, from the left hand side this is our 2, you are getting the points which are approaching towards 2, is it not? So, and these all are rational points, because you can write this 1.961, 1.9161, 96 by 10 square, this you can write this divided by 10 to the power 4 and like this. So, these are all rational points. So, a sequence of the rational number is obtained from the left hand side which is approximating 2, is it clear? Now, if I add the, add 1 digit in each term or increase 1 digit, increase 1 digit in say, this is alpha, in alpha. So, what we get? Here you are getting 2, 1 digit more 2, here you are getting 1.5, here you are getting 1.42, 1.415, 1.4143, just increase the last digit by 1, increase the last digit by 1 one digit, increase last digit, digit by 1. So, you are getting. Now, if I square again, what you get? 4, this is something like 2.25, this is something 2.0164, 2.00225 and so on.

This is 2.00024449, what you get now? You are getting a sequence of these points, this is two; you are getting a sequence of the rational number which is, which are approaching to two from right side. So, 2 is sandwiched with each. You are getting a sequence of rational number which is approximating 2 from the left, you are also getting a sequence of rational number which is approximating 2 from the right hand side and limiting point is coming to be this root of 2; the sequences are rational but limit point is not a rational point, is this clear? So that gives the idea of the irrational number, it means the irrational numbers, they are basically a number which can be approximated by means of the rational points, by which can be approximated by means of a rational points. And it lies always in between two, any two rational number they are, we can always get a rational number ss well as irrational number, in between two

So that’s it. In this any doubt please? Now, one more thing is you can divide the whole rational number in to two classes; suppose, I say the set of all rational number whose square is less than 2, and the set of those rational number square is greater than 2; if I say, the class set of, suppose I say one can or we can divide, we can divide the positive rational number, rational numbers in to two classes, two classes, one containing the number, one containing the number, numbers whose squares whose squares are less than 2 and the other, and the other, and the other one contain the other whose square, and the other who those, other those whose squares are greater than 2, greater than 2. So, this our, so this our 2. We have taken those rational number whose square is less than 2, then that class we will denote by L, that first class this one containing the number who square are less than 2, this will be denoted by L. And that those rational number whose squares are greater than 2, we denote by R. So, all the rational number will be either in L or will be in R, any rational number you pick up whose square is greater than 2, you write in R, if it is less you write in a L.

So this way, we can entire rational number, we can the separate out in the form of the two classes one L, lower class and one is the upper class. And then, what about the 2? 2 neither belongs to L nor belongs to R, is it not? Because it’s not at all rational point. But suppose, I take the set of those rational points which are less than equal to 1 and thus, those which are greater than 1. So, all rational number which are less than or equal to 1, it will include all the points less than equal to 1, all the points which are lying towards left of 1, and the points which are excitingly greater than 1 lies towards the right of the 1, and 1 itself is a point in lower class, because lower class I am choosing those rational numbers who which are less than or equal to, clear?

So and one is a rational point, so it means the entire real line, we can entire this number system, we can break up in to the classes, lower class and upper class, and these classes are also of two types where the point, which break up the two classes, the point lies in the one of the class or may not lie in the one class, point which lies in the one of the class will correspond to the rational number like say one lies in there. So, the entire class, we say it corresponds to one, but the point which does not lie, that corresponds to the irrational number. So like under root. So, that will give the classes, is it correct? Now these classes have certain property.

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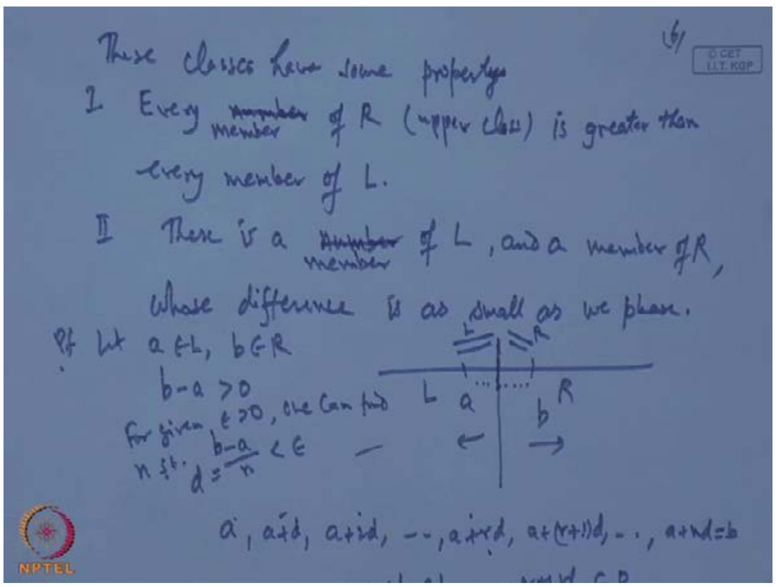

The property is these classes have some property, properties. The first property is that the every number of R, every number of R, R means upper class, remember this we call it as a upper class, every number, member of, every member, every member of R is greater than, is greater than every member of L, which is obviously 2. The second property says that there is a number of a L, number of a L is lower class, and a member, there is a number of one member of one, write member is a member of L and a member of R, and a member of R whose difference, whose difference is as small, as small as we phase, what do you mean by this? Suppose, this is our lower class, this is upper class; what he says is, there is the member of lower class say suppose a is the member of lower class and a member of the upper class say b, such that which difference is as small as we phase, this we wanted.

Then so let us say, let a belongs to lower class, b belongs to the upper class. So b minus a will be positive, let a belongs to the lower class, b belongs to the upper class, why a minus, b minus a is greater than 0, because the elements of the upper class will always be greater than the elements of the lower class. So, b minus a greater than 0, is it? Now, this length b minus, given any epsilon, given epsilon greater than 0. One can find n, one can find n such that b minus a by n is less than epsilon, because b minus a is positive and epsilon, suppose 10 to the power minus 2. So, you can choose your n in such a way, so that the b minus a by n is less than the 10 to the power minus 2; let it be d. So what we get? You are starting with a next term will be a plus d, then a plus 2 d and so on, a plus R d, a plus R plus 1 d, and like this, and last term is a plus n d, that is equal to b, is it not? These are the class, is it correct? Like this.

So, we are getting these things a plus n d is b. Now, a is in the lower class, this is our, a is this side b is this side, you all introducing the point in between a and b, by this; this is a, this is the new point, this is another point, this is another point like this. So, as soon as you introduce the point, there will be a cross, this point will cross this line, is it not? Which line that divides lower and upper class. So, this is the line, here is L, here is R, this is L, this is R. So, in R will come, a number R will obtain, so that a plus R d, a plus R d belongs to the lower class where a plus R plus 1 d belongs to the upper class, for some R, is it not? For some R, so for some R, some R this will happen, because you are continuously dividing and then approaching a. So, when you go slowly here, a point will here, some point will be here and immediately it will jump and go to this. So, for certain R, this point is a plus R d, this point is a plus R plus 1 d. Now, what is the difference of this? So, difference of this will be of these points is nothing but what? Is nothing but d; you see, this one, and d is always be less than epsilon. So, but what our claim is that there is a member of L and that member is a plus R d, and a member of R that a plus R plus 1 d whose difference a can be made as small as we phase, epsilon is given choice; so you can choose any epsilon accordingly, you can identify the two points, the difference can be made like this. So, this is thus second property.

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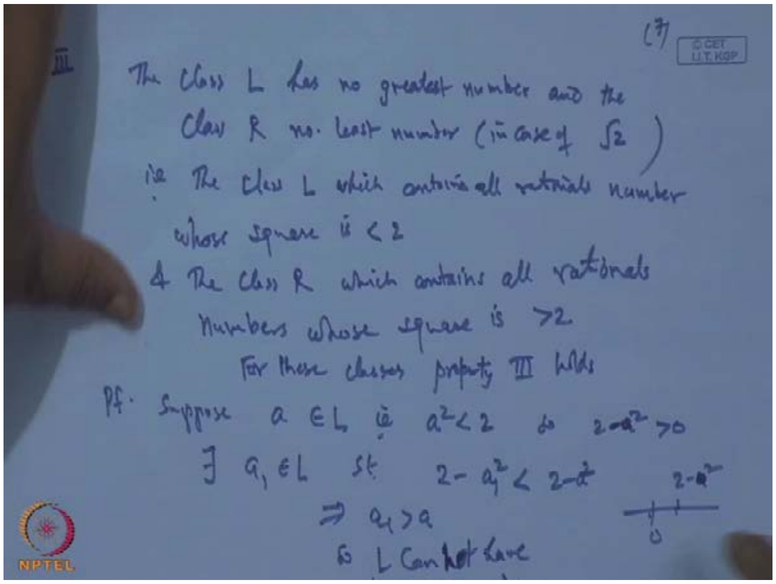

Third property says the class L, the class L has no greatest number, greatest number and the class R no least number, R no least number. In case of square root 2, remember when the class is produced by square root, I am not saying rational point; in case of the square root which is set of all rational number whose square root is less than 2, is a lower class, set of all number whose square is greater than 2, for that class lower and upper class, lower will not have any greatest number, upper class will not have any that.

That is, that is the class L, lower class L which contains all rational numbers whose square strictly less than 2, and the class R which contains all rational numbers whose square is greater than 2. Now, these classes, for these classes, these classes property, property third hold, that is in this case, the lower class will not have a largest number and upper class will not have a least number. Suppose, I have, for a proof is, suppose a be a, suppose a belongs to the lower class that is the a square is less than 2, is it not? So 2 minus a square will be positive. Once it is positive, you can identify the number a 1; so there exists a number a 1 belongs to L, there exists a number a 1 such that 2 minus a 1 square which is less than 2 minus a square, is it not? Because there was a gap 2 minus a square is greater than 0. So basically, this is our 0, here is some where think number 2 minus a square. So, again there is gap in between, we can identify some rational numbers, is it not? Which like, so you can find a 1 such that 2 minus a square is less than this, but what do you (( )) This shows a 1 is greater than a 2, two gets cancelled, minus a minus a square is less than, so when you multiply by minus order reverses, so it is greater than a. It means if a, if I claim a is the largest number which is not true, because you are able to get another number a 1, which lies in L, which is greater than a. So, L cannot have a largest number. Similarly, one can show for the upper class R. Similarly, so L cannot have, cannot have largest number, and cannot have a largest number. Similarly, R cannot have least number, is this? Now, based on this our discussion, we can now divide in to the concept of cuts of rational points.

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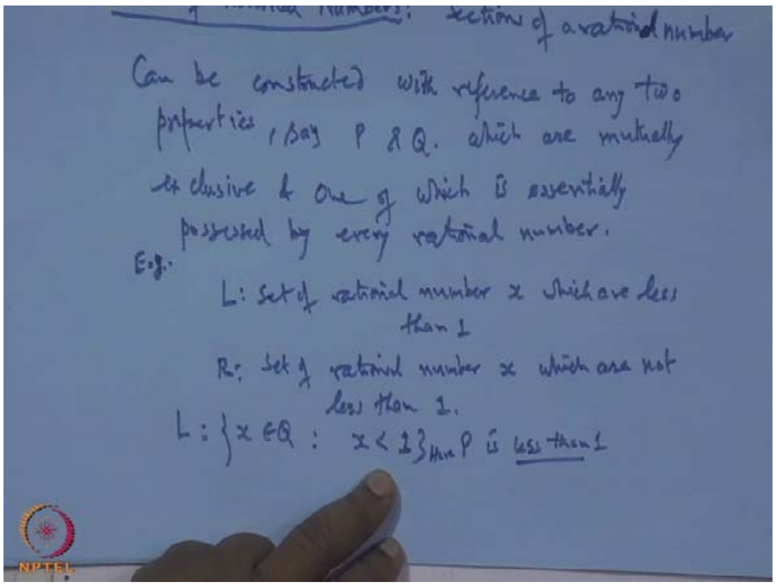

So, that is also called as sections. So based on this, now we introduce the sections of rational number, of sections of rational numbers. So, what we here is, sections of a rational number, sections of a rational number, section or cut of a rational number, section of a rational number can be, can be constructed, constructed with reference to any two property, to any two properties, any two properties say teach say P and Q, P and Q which are mutually exclusive, exclusive and one of which is essentially, and which are mutually exclusive, and one of which, and one of which is essentially possessed, essentially possessed by a rational number, by a rational number. Let us see what is the meaning of this; When we say the section of a rational, rational number then we construct a with the difference to any two properties, say P and Q which are mutually exclusive and one of which is essentially possessed by rational; say for example, when we say the set of those rational numbers, set of rational number, rational number, set of rational numbers x which are less than one, and the set of rational numbers x which are, which are less than 1 and similarly, which are not less than 1.

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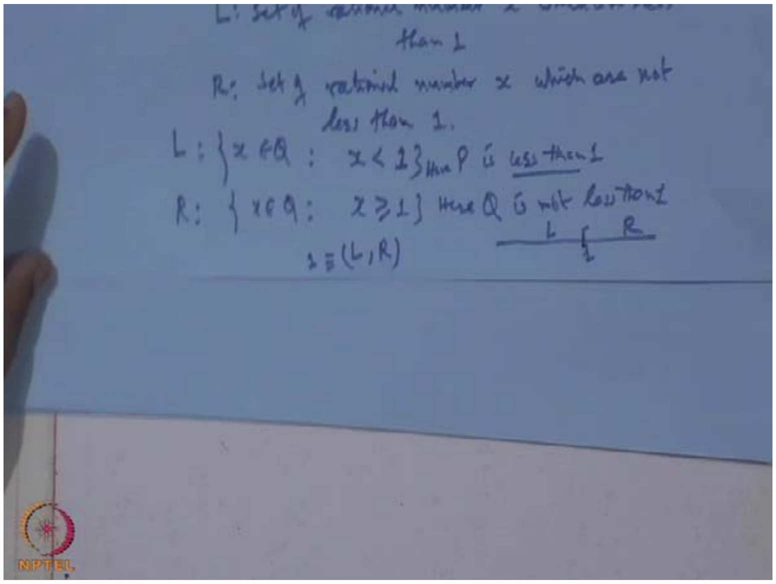

Suppose, I give these two statements set of those rational numbers which are less than 1, I denote say L, this is R, it means what is L? L is the set of those rational number Q such that x is strictly less than 1. So what is the property? Property P is less than, so here, here the property P is less than 1. This is the property which you are imposing, is it not? Set of those rational numbers, set of those point x such that x is less than 1, x satisfy this property and what is R? R is the set of those rational points such that x is not less than 1, it means what? It will be greater than or equal to 1 that is here Q is not less than 1. So, this is the property.

So what he said that every rational number can be constructed with reference to two properties, P and Q. This is the property P, this is Q, clear? So, that one is the number which can be separated out, which can separated by rational or rational number into 2 classes, one is the one class, another one is R class, and R contains 1, R contains 1, clear? So, this R which in upper class and L is the lower class, and this is represented by 1 if I say the section L R is represented by 1, then what you mean by this? It means the L, lower class contains those rational numbers which are less than one, and upper class R contains those rational which are not less than 1, then 1 is a number which divides the whole rational number into two classes, or every rational number, or any rational number will be either part of a L or part of r.

So this will be a cut, this known as the cut. Q is the collection of rational number, section Q is the collection of rational number. So, from this Q, we are picking up those rational numbers, Q indicates the collection of rational number. Here also Q is also collection of rational number in the right side Q indicates, this right side, this one, this property, yes, yes, I am, so let be this P and Q, so should I write this Q dash.

Q 1, this is Q, rational, I am taking rational. Let it be denoted by something, sir here I think you're in the right side, Q rational, rational, is it? Sir, now Q rational, set of rational, this property is something, this two are not, this is the property, is it? Similarly, when we go for the square root of 2, square root of 2, then it say it also divides the rational point into 2 parts, 2 classes, those rational number whose square is less than 2, those rational number whose square is greater than 2. So, entire rational number, you can divide into two, two classes; one is the lower class, where it the whole rational number.

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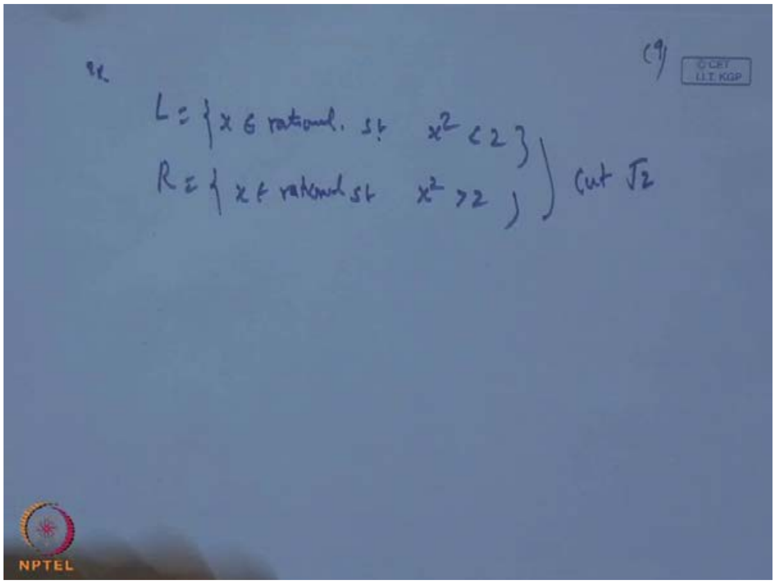

Similarly, second example is suppose L, I said set of those rational Q, irrational point x belongs to rational, such that square of this is less than 2, and R is the set of those rational such that square of this is greater than 2. So, this the lower class, this is the upper class and 2 is the number digit, square root of 2 dividing the number; but 2, square root 2 neither belongs to L nor belongs to R. So basically, this is, this cut, this will give a cut or section corresponding to root 2. There is a, this was a cut corresponding to one, but one was one in, one of the class, here root 2 is none of the class. So, there is two types of cut, one is the rational cut another one is irrational cut; and combination of this, gives the continuum in the real line; when a collection of all these cuts gives the entire real line, that is known as the continuum, and that this way the real line, the system of the real number has been defined. So, this is the story given by dedekinds introduced in terms of the cuts.

Thank you very much. Thanks.