Probability and Statistics

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Module No. #01
Lecture No. #22
Transformations of Random Vectors

So, we have seen the distributions of several random variables. Many times we are not interested in the original random variable itself, but certain function of it for example, sums of random variables, or say, difference, or any linear function of those random variables.

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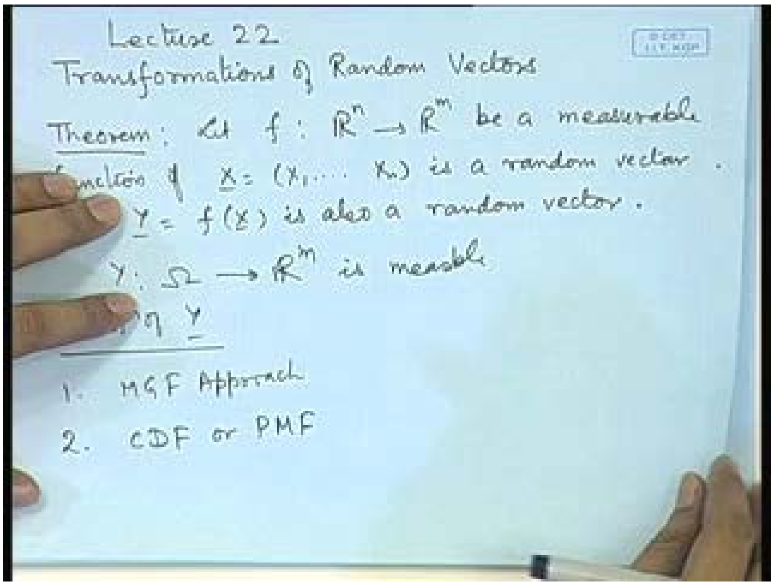

So, in general if I have a measurable function of random vector X1, X2, Xn, then it will also be a random variable. So, we state it in the form of following theorem: let f from say Rn to Rm be a measurable function, so, if X is equal to say X1, X2, Xn is a random vector, then let us call it say, Y, Y is equal to fX is also a random vector. This is so because random variable X is a measurable function from omega into Rm, and a measurable function of a measurable function is measurable function. So, Y becomes a measurable function from basically omega into Rm, so, this is measurable, and so this is a random vector.

So, now, the methods of determining the distribution of Y. So, one is the mgf approach. We have already seen application of this approach in determining distributions of sums of certain random variables. So, if we are having certain independent random variables and we want the distribution of the sum, then it is the distribution, it is the product of the individual mgfs, and in many cases where the mgf, the product of the mgfs can be determined in an explicit form as an identifiable mgf then the distribution of the sum can be determined. It can also be used for the distribution of difference, etcetera, where the forms are well defined. In the case of discrete distributions, or in certain other cases where the cdf can be directly used then we can use directly the cdf or the probability mass function.

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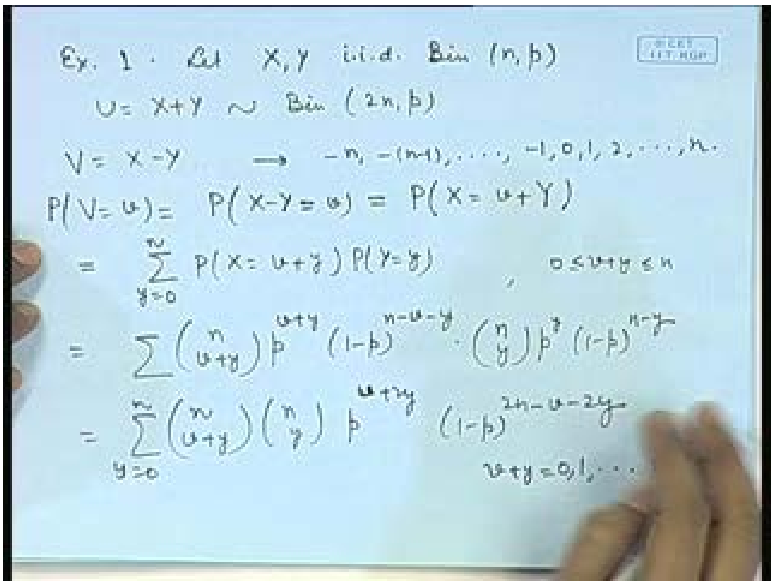

Let me give an example of this. Suppose X and Y are independent and identically distributed binomial n, p variables. Suppose we want the distribution of U, that is X plus Y then from mgf approach we are able to determine it as binomial 2n,p. Now, suppose we want the distribution of say V, that is X minus Y then, let us look at the set of values of V, this will vary from minus n, minus n minus 1, minus 1, 0, 1, 2 up to n because each of X and Y can take value up to 0,1 to n. So, probability of V is equal to say small v, that is probability of X minus Y is equal to v, this we can write as X minus, is equal to v plus Y; now, Y can take values using a binomial distribution n p, so we can use the theorem of total probability here and write it as probability X is equal to say v plus y into probability of Y is equal to y- this is because of independence I can split- for y is equal to 0 to n, now, this is subject to the condition that v plus y is also lying between 0 to n.

So, this is equal to ncv plus y p to the power v plus y 1 minus p to the power n minus v minus y ncy p to the power y into 1 minus p to the power n minus y; so, this is equal to sigma ncv plus y ncy p to the power v plus 2y and 1 minus p to the power 2n minus v minus 2y, where y is equal to 0 to n subject to the condition that v plus y is also taking values 0, 1 to n because v plus y denotes a value of the random variable X here.

So, this shows that in the case of discrete random variables, directly the probability mass function can be used to determine the distribution of a function.

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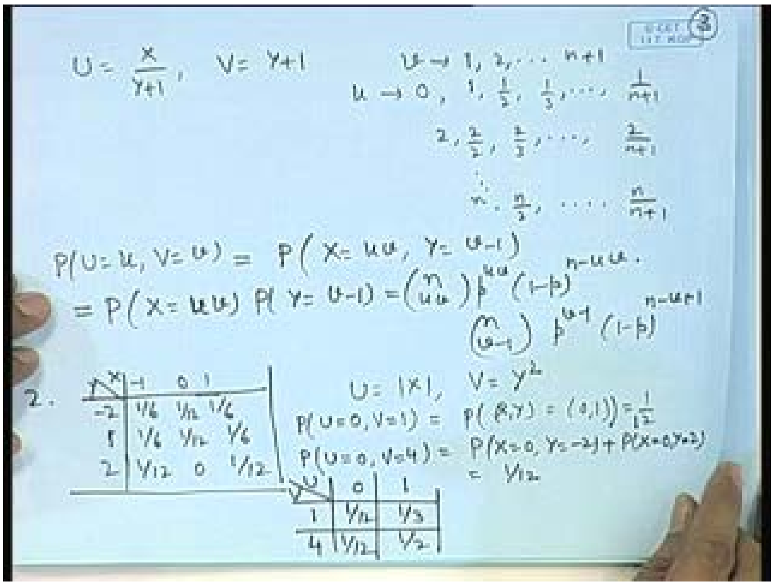

Let us take another case here. Suppose I define say U is equal to X by Y plus 1 and V is equal to say Y plus 1, I want the joint distribution of U and V here, where X and Y follow independent binomials. So, here, you look at the set of values we will follow, since Y is binomial n, p, Y takes values 0, 1 to n, so v will take values 1, 2 up to n plus 1; whereas, the values of u will be, now, here X can take value 0, X can take value 1, in that case, Y plus 1 can take values all these, so, 1, 1 by 2, 1 by 3 and so on, 1 by n plus 1, X can take value say 2, so, this values can be 2, 2 by 2, 2 by 3 and so on up to 2 by n plus 1 and so on, and n by 2 and so on, n by n plus one, so, these are the possible values taken by u.

So, we look at probability of say U is equal to small u, V is equal to small v, where small u and small v take these values then, this can be expressed as probability X is equal to uv and Y is equal to v minus one. So, X and Y are independently distributed, so, this becomes product of, that is equal to ncuv P to the power uv 1 minus p to the power n minus uv then, ncv minus 1 p to the power v minus 1 one minus p to the power n minus v plus one. So, this is a joint distribution of U and V, where u and v take these values.

Let us take another example here say, X and Y have the joint mass function, the probabilities are 1 by 6, 1 by 12, 1 by 6, 1 by 6, 1 by 12, 1 by 6, 1 by 12, 0 and 1 by 12. So, X takes values minus 1, 0 and 1, and Y takes values minus 2, 1 and 2. Suppose I define U is equal to modulus of X and V as Y square then the possible values of U are 0 and 1 and possible values of Y are, V are 1 and 4. So, the joint distribution that is, probability say U is equal to 0, V is equal to 1, that is simply probability of X, Y equal to 0, 1, that is 1 by 12. If we look at what is a probability of U is equal to 0, V is equal to 4, it is the sum of X is equal to 0, Y is equal to minus 2 plus probability X equal to 0, Y is equal to plus 2. So, if we add these probabilities, we get 1 by 12.

In a similar way, we can obtain probability of U is equal to 1, V is equal to 1, U is equal to 1, V is equal to 4 and the joint distribution turns out to be, we can express it as U, V, U can take value 0 and 1, V can take value 1 and 4. So, that distribution is- the 0, 1 is 1 by 12; 0, 4 is 1 by 12; 1, 1 is 1 by 3; and this is half. And from here we can derive the marginal distributions of U and V.

So, in the case of discrete distributions, etcetera, it is possible to derive the distribution of the function of random variables by directly considering the probability mass function. Sometime it is easy to use the direct cumulative distribution function also, I can give an example here.

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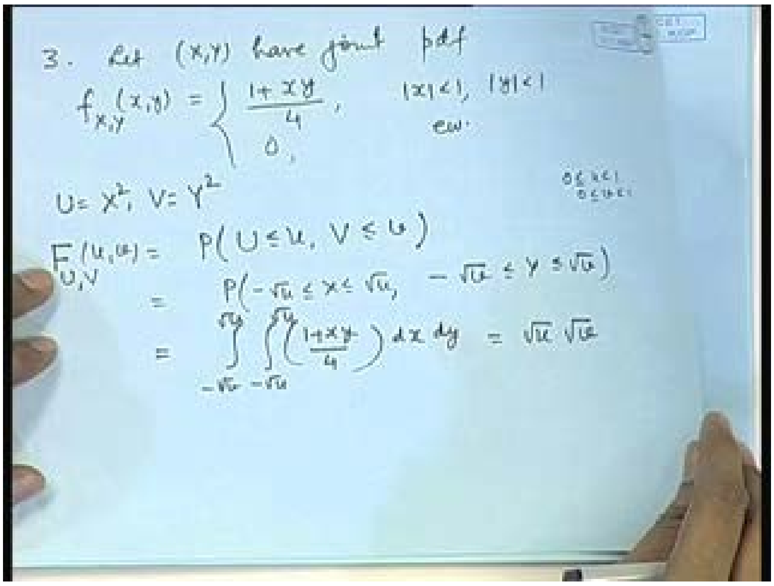

Let us consider say let X and Y have joint probability density function say fXY given by 1 plus XY by 4 where modulus X is less than 1 and modulus Y is less than 1, 0 elsewhere. So, we want to say the distribution of U is equal to X square and V is equal to Y square. Let us consider say cdf of U and V that is, probability of U less than or equal to small u, V less than or equal to small v. Now, notice here that both X and Y lie between minus 1 to 1, so, here the valid region for U and V will be between 0 and 1, so, we consider that 0 less than u less than 1 and 0 less than v less than 1. So, for this case this is nothing but probability of X lying between minus root u to plus root u and Y lying between minus root v to plus root v. So, this is nothing but the integration of the joint density over this region. So, that is integral 1 plus xy by 4 dx dy over minus root u to plus root u, minus root v to plus root v, and we can evaluate it to be root u root v. So, the joint cdf can be obtained.

From here, we can determine the density of U and V. In general cases, when we have continuous random variables and we make a transformation of that, it may not be so easy to look at the joint cdf, etcetera, in that case like in the case of univariaterandom variables, we have an approach the so called Jacobean approach for determining the distributions of random variables.

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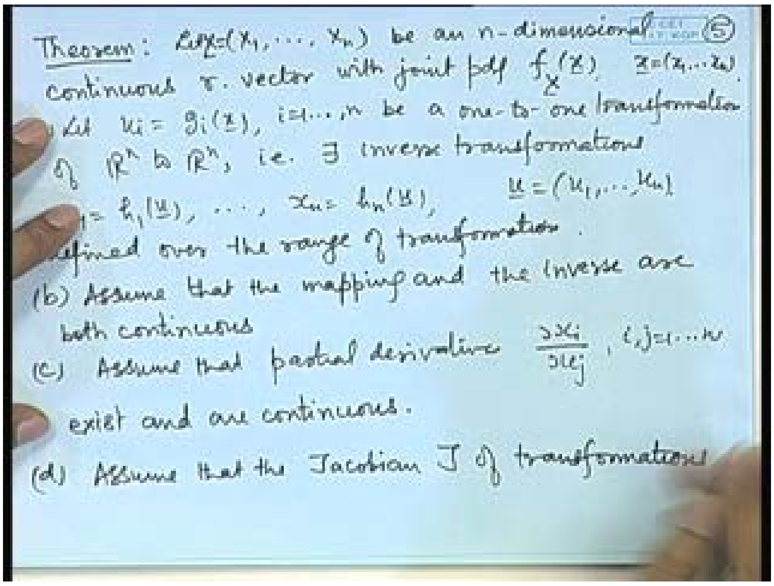

So, we stated it in the form of the following theorem: let X1, X2, Xn be an n dimensional continuous random vector with joint probability density function say, fXX. So, here X is denoting the vector X1, X2, Xn, small x is denoting the vector x1, x2, xn. Let ui is equal to gi of x, i is equal to 1 to n be a one to one transformation of Rn to R n that is, if I am taking one to one, then there exist inverse transformations, let us call it say, x1 is equal to say h1 of u and so on, xn is equal to hn of u, where u is u1, u2, un defined over the range of transformation.

Let us assume that the mapping and the inverse are both continuous. Further assume that the partial derivates delxi over deluj, for i , j is equal to 1 to n that is, all partial derivates, delx1 by delu1, delx1 by delu2, delxn by delu3 and so on, all the partial derivates exist and are continuous. Then, we define, assume that the Jacobean J of transformations.

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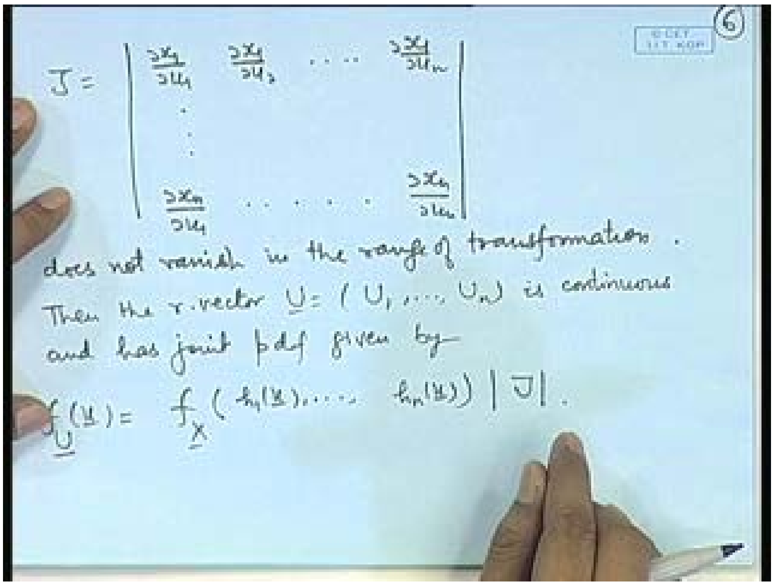

Which is defined by J is equal to delx1 by delu1, delx1 by delu2 and say on, delx1 over delU n and so on, delx n over delu1 and so on, delx n over delun. Assume that this Jacobean does not vanish in the range of transformation then, the random vector U is equal to U1, U2, Un is continuous and has joint pdf given by- so, we write it as fU is equal to fX, now, in place of x1, x2, xn replace it by h1u, h2u, hnu multiplied by the absolute value of the Jacobean over the range of the transformation.

If you see it carefully, it is a forward generalization of the result for one dimensional case. In the one dimensional cases we had considered a one to one transformation and we had looked at the dx by dy term, so, the density of the transformed variable was obtained as the density evaluated at the x equal to g inverse y multiplied by the absolute value of the dx by dy term. So, and we have n dimensional random vector and n dimensional transformation. So, if it is a one to one case, we look at exactly the inverse function and calculate the determinant of the partial derivates called as Jacobean, substitute the values of x1, x2, x n in terms of uis and multiplied by the Jacobean term, absolute value of the Jacobean, that yields the joint density function of the transformed random vector.

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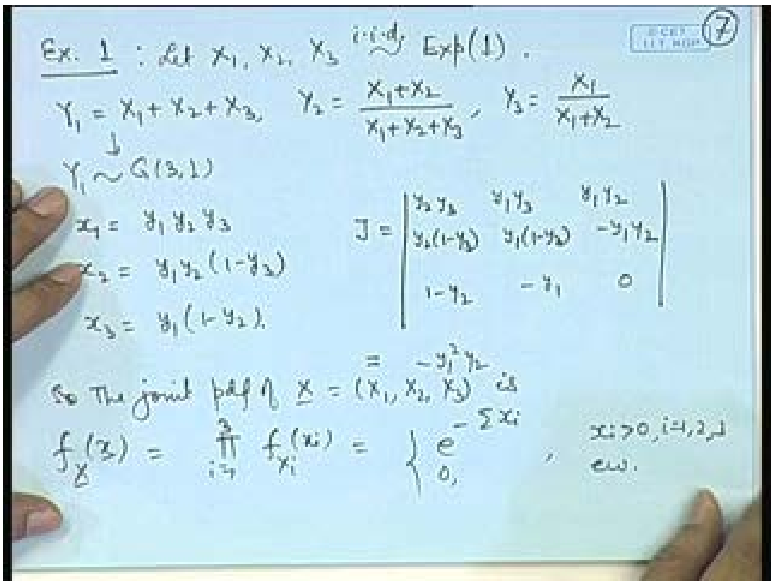

So, let us look few applications here. Let X1, X2, X3 follow exponential with lambda is equal to 1. Suppose they are independent and identically distributed random variables. Let me define Y1 is equal to say X1 plus X2 plus X3, Y2 is equal to say X1 plus X2 divided by X1 plus X2 plus X3 and Y3 is equal to say X1 by X1 plus X2. We are interested in the joint and marginal distributions of Y1, Y2 and Y3.

Of course, here if you are interested only in the distribution of Y1, then that is directly obtained because the sums of independent exponential is a gamma, so Y1 will follow a gamma distribution with parameters 3 and 1, so, that is directly known, however, that does not yield the distribution of Y2, or Y3.

So, we observe here that it is a one to one transformation and inverse functions can be written as x1 is equal to y1, y2, y3; x2 can be written as then, y1, y2 into 1 minus y3; and x3 can be written as y1 into 1 minus y2. So, we can determine the Jacobean of the transformation, doux1 by douy1 is y2y3, doux1 by douy2 is y1y3 and so on, y2 into 1 minus y3, y1 into 1 minus y3, minus y1y2, 1 minus y2, minus y1 and 0. So, if we evaluate this, it out to be minus y1 square y2.

Firstly, we write down the joint density function of X1, X2, X3. So, the joint pdf of X1, X2, X3, so, since X1, X2, X3 are independently distributed the joint density is nothing but the product of the individual density functions of X1, X2, X 3. It is product of fxi, that is equal to e to the power minus sigma xi, each xi is positive. Therefore, the joint density of Y1, Y2, Y3 can be obtained from here by substituting the inverse functions of X1, X2, X3 and the corresponding range and multiply by the Jacobean.

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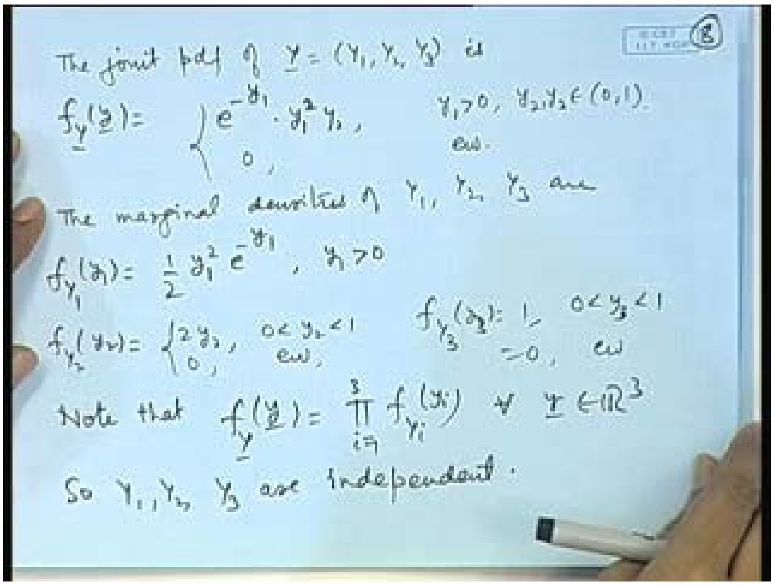

So, the joint pdf of Y is equal to Y1, Y2, Y3 is f, e to the power minus y1 into y1 square y2; the range of the variables we can observe here- that each of the xi is positive random variable, so each of yi is also a positive random variable further, if x2 is positive then y3 will be less than 1 and similarly, y2 will also be less than 1- so, the ranges are y1 greater than 0, y2 and y 3, they belong to the interval 0 to 1.

So, we have been able to determine the joint distribution of Y1, Y2, Y3. In order to get the marginal distributions, we notice here that if we integrate with respect to Y 3 from 0 to 1, we get the same term and therefore, if we integrate this with respect to Y1, Y2, it should give the density of Y3 as 1 on the interval 0 to 1. So, the marginal distributions, the marginal densities of Y1, Y2 and Y3 are obtained as fY1 as half y1 square e to the power minus y1, which is nothing but a gamma distribution with parameters 3 and 1; fY2 is 2 y2 for y2 between 0 and 1; and fY 3 is equal to 1, between 0 and 1. So, this is a uniform distribution.

One interesting feature we can notice here that, if I look at the product of the marginal's, it is equal to the joint. Note that f of y is equal to the product of... so, Y1, Y2, Y3 are independent. So, here we are able to obtain the distribution of a three dimensional function of three random variables here. The important thing to notice here is that apart from substitution in the density function and multiplying by the jacobian we are also judiciously determining the ranges of the variable like, one may simply say that Y1 is positive, Y2 is positive, Y3 is positive without noticing that Y2 and Y3 are less than 1 also, in that case if we will evaluate the integrals of this density, it will not give us 1, so, that will be not determining the density correctly.

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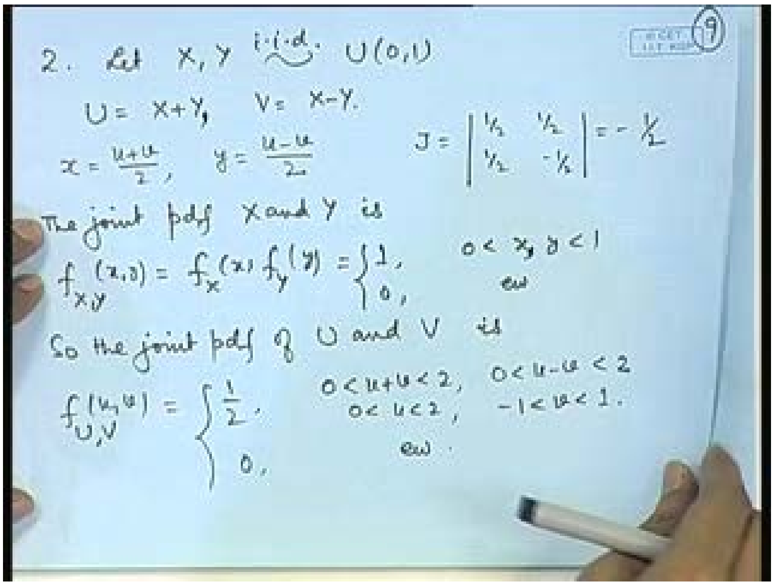

Let us take uniform distributions. Let X and Y be independent and identically distributed uniform random variables. Let us define say U is equal to X plus Y and V is equal to say X minus Y. Now clearly this is a one to one transformation; x is equal to u plus v by 2 and y is equal to u minus v by 2. So, if we look at the Jacobean term douX by douU is half, half, half and minus half, which is equal to minus half.

So, the joint pdf of say, X and Y that is, fxy, it is the product of the individual distributions of x and y, both are uniform 0, 1, so it is simply 1 for 0 less than x, y less than 1 and 0 elsewhere. So, the joint pdf of U and V is, it will become half for 0 less than u plus v less than 2, 0 less than u minus v less than 2. Now, the ranges of U and V, we can notice further here that since X and Y are between 0 to 1, U will be between 0 and 2 and V will be between minus 1 and 1. This gives the joint density function of U and V. Suppose we are interested in the marginal distributions of U and V. So, in order to get the marginal distribution of U we need to integrate this with respect to the variable V.

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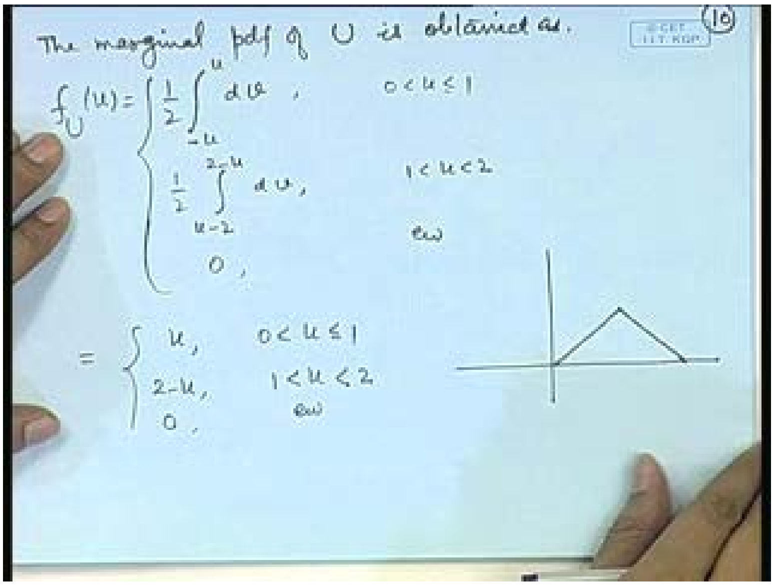

So, the marginal density of U is obtained as, so, fU, integral of this join density, that is half dv, now, notice here the range of V, V's absolute ranges from minus 1 to one, but here V lies between minus u to 2 minus u and V is less than u and V is also greater than u minus 2. So, if we determine the region, it is from minus u to u if u is between 0 to 1; it will be half u minus 2 to 2 minus u dv if v is between 1 and 2 and 0 elsewhere. So, after simplification this turns out to be u for 0 less than u less than or equal to 1, it is 2 minus u for 1 less than u less than 2 and 0 elsewhere.

Notice here, this is a triangular distribution, 0 to 1 and 1 to 2. So, the distribution of the sums of two independent uniform random variables is actually a triangular distribution. In a similar way, we can obtain the marginal density of V also if we integrate with respect to U.

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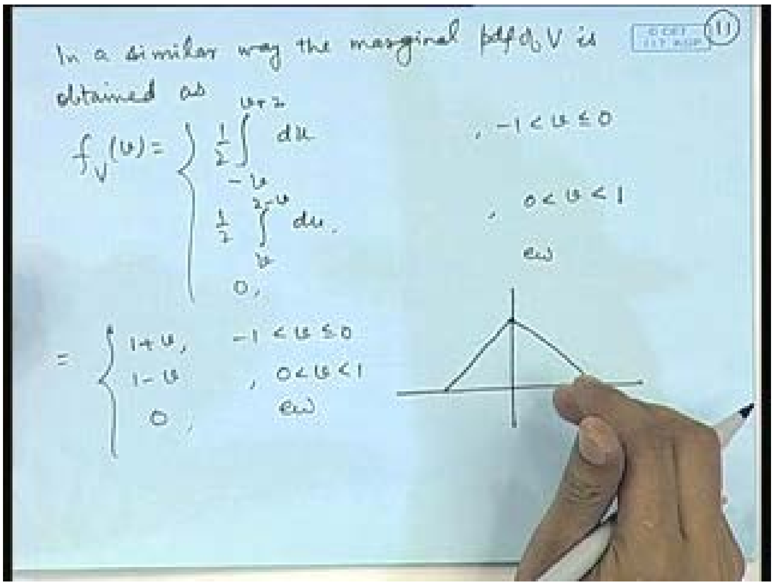

In a similar way, the marginal pdf of V is obtained as fVv, it is integral of half with respect to u from minus v to v plus 2 for minus 1 less than v less than or equal to 0; it is half from v to 2 minus v du for 0 less than v less than 1 and 0 elsewhere. So, after simplification this turns out to be 1 plus v for minus 1 less than v less than or equal to 0 and 1 minus v for 0 less than v less than 1, 0 elsewhere. This is again a triangular distribution on the interval minus 1 to 1. So, minus 1 to 0 the density is 1 plus v and between 0 to 1 the density is 1 minus v.

So, we noticed here that the sums and differences of independent uniform random variables are again, are triangular distributions, and obviously, they are not independently distributed here because the joint distribution of U and V is not equal to the product of marginal distributions of U and V here. Now, in many cases the function from Rn to Rn need not be one to one for example, we have considered the discrete case where U was modulus X and V was Y square, so, it is not a one to one transformation, rather it is a four to one transformation over the range of the variables. So, in that case, we have a result similar to the case of univariate. In the case of univariate when we had many to one transformation we split the domain into disjoint region such that from each region to the range, we have a one to one transformation. We consider the inverse transformation, using that we calculate the density function in each region of the domain, disjoint regions, and we add all of this, that gives the joint distribution.

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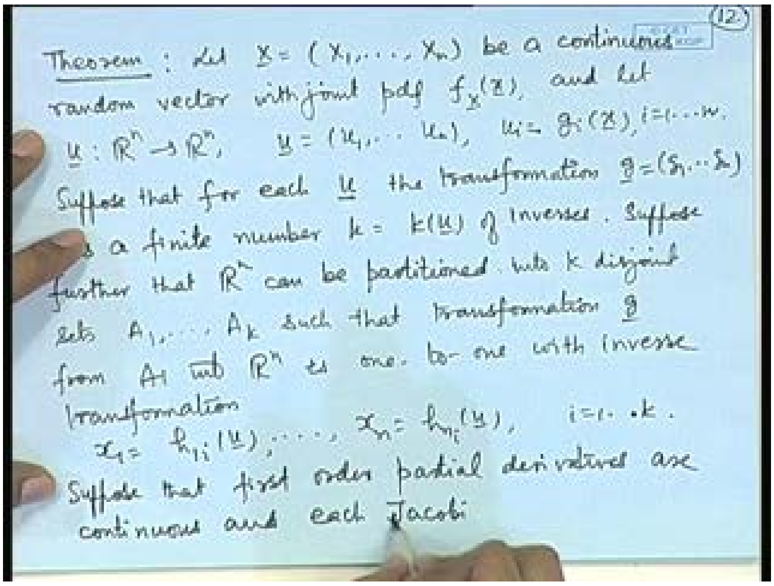

So, generalization of this result is available for the n dimensional case also and we state in the form of the following theorem: let X is equal to X1, X2, Xn be a continuous random vector with joint pdf fX; and let U be a mapping from Rn into Rn, where U is equal to u1, u2, un, ui is equal to some gi of x for i is equal to 1 to n. So, we are not assuming that it is a 1,1 on 2 functions. Suppose that for each U the transformation g that is, g1, g2, g n has a finite number say, k of inverses. Suppose further that Rn can be partitioned into k disjoint sets say, a1, a2, ak such that transformation g from ai into Rn is one to one with inverse transformation say, x1 is equal to h1i u and so on xn is equal to hni u for i is equal to 1 to k. As in the case of pervious theorem, you have to assume that the mapping and the inverses are continuous, and these first partial derivatives are continuous. Suppose that first order partial derivatives are continuous and each Jacobean.

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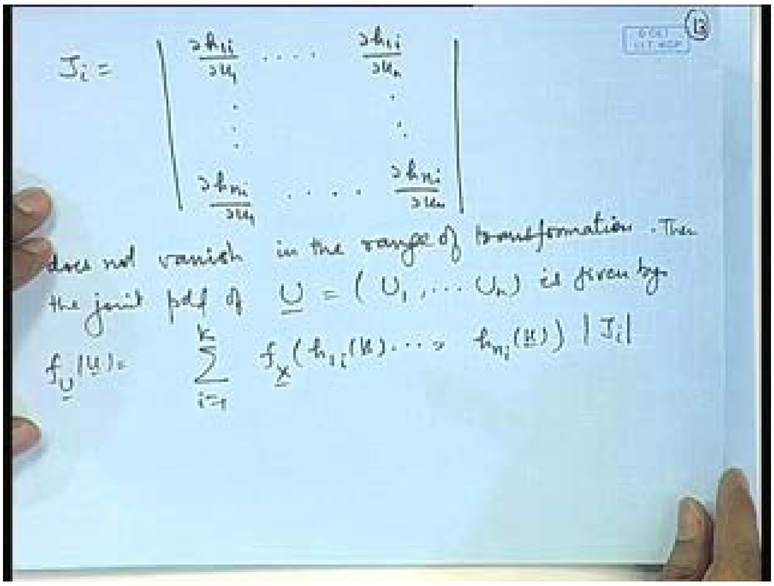

That is, Ji that is, delh1i by delu1 and so on, del h1i over delun and so on, delhni over delu1 and so on, delhni over delun, does not vanish in the range of transformations. Then, the joint pdf of U is equal to U1, U2 to Un is given by fU is equal to sigma f of X h1iu and so on, hniu multiplied by absolute value of the Jacobean, i is equal to 1 to k.

So, note here, if we consider this term, it is the density determined by the one to one transformation from ai into Rn. So, we calculate this density for each region, a1, a2, a k and add, this gives the joint distribution of U1, U2, U n.

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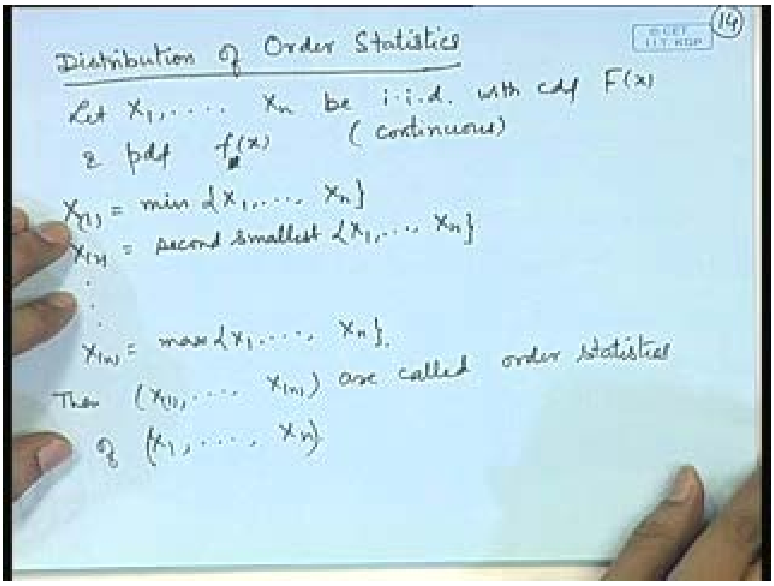

We consider an example. For example, we can consider distribution of order statistics, what are order statistics? Let X1, X2, Xn be iid with some cdf say, FX and pdf say f X; and let me assume it to be continuous. We define X1 to be the minimum of X1, X2, Xn; X2 to be second smallest of X1, X2, Xn; so, like that X3 will be third smallest and so on; Xn will be the maximum of X1, X2,Xn. Then, X1, X2, Xn these are called order statistics of X1, X2, Xn. In many practical aspects, the order statistics are quite important for example, the raw observations may be something and suppose the raw observations denote the marks by the students, but we may be interested in the order observations that who is getting the highest marks, who is the second highest, etcetera, if we are selecting certain candidates on the basis of certain scores, so, we will be interested in selecting the best ten, so, we will be interested in say, Xn, Xn minus 1 up to Xn minus 9 say- so, the top ten students. So, in general we are interested in order statistics and therefore, the distributions of the ordered statistics.

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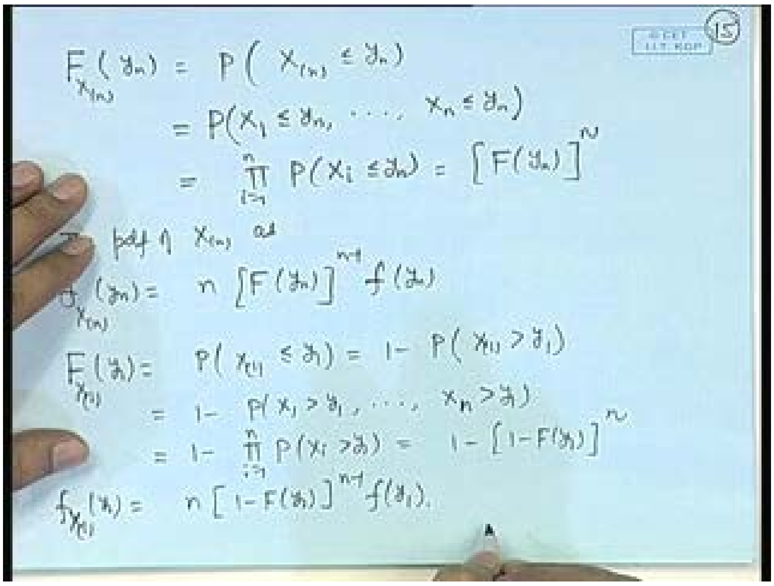

Suppose, you want to find out the distribution of X n. So, we may use a direct approach. For example, we look at F of Xn. Let me use a notation here say, Yi is equal to Xi, where i is equal to 1 to n. So, the distribution of the largest that is, probability of Xn less than or equal to yn, now, notice here this is maximum being less than or equal to yn, this event is equivalent to that each of the Xis is less than or equal to yn- now, we make use of the fact that the random variables are independent and identically distributed, all of them have the cdf Fx- so, each of these values is f of yn and therefore, we get this as f of yn to the power n.

If we are assuming the distributions to be continuous, then we can differentiate it and get the pdf of Xn as n times Fyn to the power n minus 1 f of y n. So, using a direct a cdf approach the distributions of the largest can be determined. In a similar way, we can obtain the distribution of the minimum also. That is, probability of X1 less than or equal to say y1, this we can note as 1 minus probability of X1 greater than y1. Once again, if you are saying that the minimum is greater than y1, it means that each of the observations is greater than y1. At this stage you can use the independence, 1 minus product Xi greater than y1- and since each of this is identically same- so, it is 1 minus 1 minus F of y1 to the power n. So, the cdf of the smallest can be determined. And the pdf of the smallest can be determined by differentiating this with respect to y1, that gives n times 1 minus Fy1 to the power n minus 1 f of y1.

So, the distribution of the largest and the smallest order statistics can be determined using the cdf approach. However, this approach will be complicated suppose we want to determined the distribution of third order statistics, or say the joint distribution of fourth and seventh order statistics, so, in that case, if you are assuming the continuous random variables, we make use of the Jacobean method.

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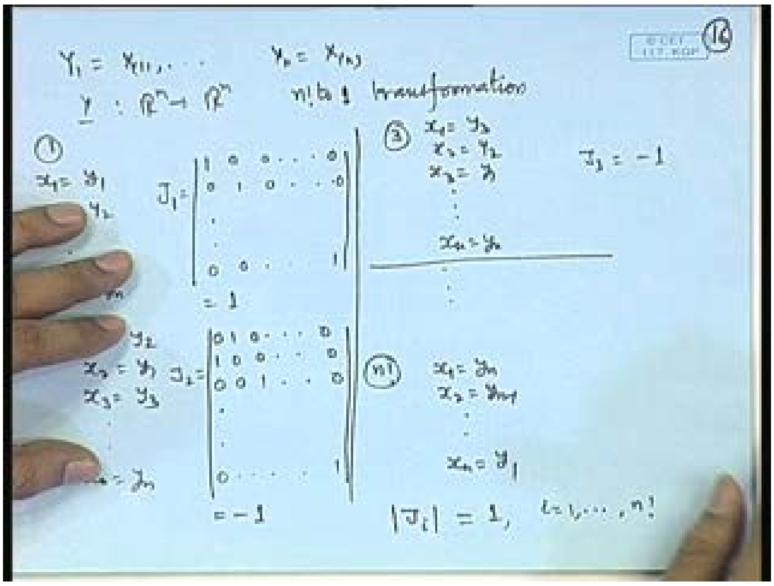

So, let us consider the joint distribution of Y1, Y2, Yn. So, here the function Y1 is equal to X1, Yn is equal to Xn. Now, this is a transformation from Rn to Rn, but it is a many one transformation in fact, n factorial combinations of X1, X2, Xn give the same values of Y1, Y2, Y n.

Let us consider say n is equal to 2. Suppose I say X1 is equal 1, X2 is equal to 1.5 then Y1 will be 1, Y2 will be 1.5, we can take X1 is equal to 1.5 and X2 is equal 1 once again, Y1 and Y2 will remain the same that means, two sets of X1 and X2 give the same value of minimum and maximum. In a similar way, suppose we have three values X1, X2, X3 then six different combinations of X1, X2, X3 will give the same values of Y1, Y2 and Y3. So, in general Y is Rn to Rn- this is n to, n factorial to one transformation. So, we can partition Rn into n factorial different regions, so that in each region it is a one to one transformation. In the region one for example, you may have x1 is equal to y1, x2 is equal to y2 and so on, xn is equal to yn; in the region two, we may have x1 is equal to say y2, x2 is equal to y1, x3 is equal to y3 and so on, xn is equal to yn; in the region three, it may be x1 is equal to say y3, x2 is equal to y2, x3 is equal to y1 and so on, xn is equal to yn and so on; in the n factorialth region, we may have x1 is equal to say, yn, x2 is equal yn minus 1 and so on, xn is equal y1.

Let us look at the Jacobean in each case. The Jacobean here is the determinant of the identity matrix 1, 0, 0, 0, 0, 1, 0 and so on, 0, 0, 1, which is simply equal to 1; if we look at the Jacobean in the second case, this is 0, 1, 0 and so on, 1, 0, 0 and so on, 0, 0, 1 and so on- note here that this is obtained from J1 by interchanging the first and second row or first and second column- so, the value of this will be minus 1. So, likewise the Jacobean in each case will be either plus 1 or minus 1 because it is obtained by permuting rows of identity matrix. So, Jacobean absolute values for each of them will be 1 only.

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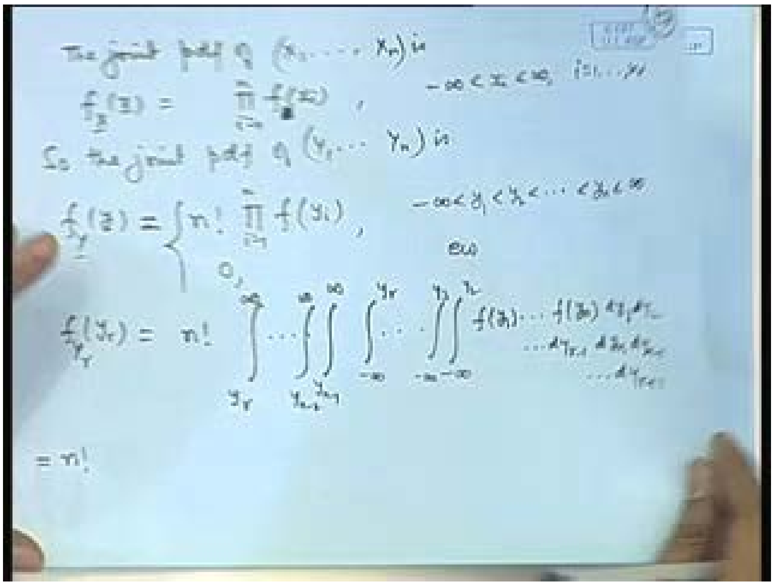

Now the joint distribution of X1, X2, Xn is product of fxi, i is equal to 1 to n. So, in general the regions are... So, the joint pdf of Y1, Y2, Yn that is, order statistics fy, now, in the first region, we will substitute xi is equal yi multiplied by 1, in the second region, we will multiply by 1 and substitute x1 is equal to y2, x2 is equal to y n and etcetera; notice here that in each of the cases it is only a permutation of y1, y2, y n, since it is the product of all these terms it will always give total product of f of y1, f of y2, f of yn.

So, this becomes product of f of yi, i is equal 1 to n and we have n factorial regions, so it is n factorial times and the region is y1 is less than y2 less than yn less than... We may have the cases where y1 is equal y2, or say ys is equal to ys plus 1 for certain s, since we are assuming the continuous distributions the probability of two variables being equal is 0 and we can ignore this. So, you can see here that the joint distribution of Y1, Y2, Yn can be written. Now, suppose we are interested in the distribution of say Rth order statistics say, what is fYr? Then, we need to integrate y1, y2, yr minus 1 and yr plus 1, yr plus 2 up to yn. So, we can devise a scheme for integration, it will be integral f of y1, y2, f of yn.

So, we can integrate with respect to y1 from minus infinity to y2, with respect to y2 from minus infinity to y3 and so on, up to yr minus 1, minus infinity to yr. Then, we can integrate yn from yn minus 1 to infinity, yn minus 1 from yn minus 2 to infinity and so on; dyr plus 1 will be from yr to infinity. So, let us look at the evaluation here, if we integrate with respect to y1, f of small f of y1 gives capital F of y1, and when we substitute the limits here, at minus infinity, it is 0 and at y2, it is capital F of y2.

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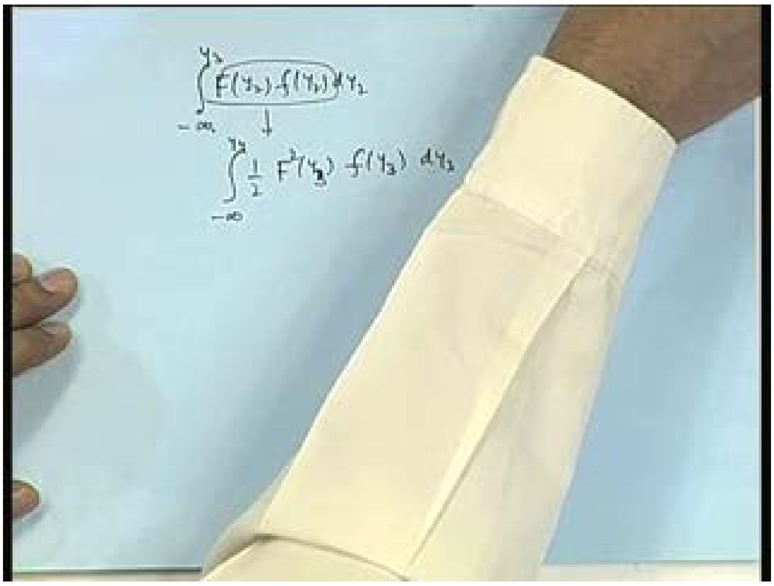

So, this gives us capital F of y2 into small f of y2 in the next step. This will be integrated from minus infinity to y3, now, notice here is that if we integrate this we will get half F square y2, and again, if we substitute the limits, I will get it as half F square y3 and at minus infinity this will be 0, so at the next stage the integrant will be this term from minus infinity to y4. So, next stage it will give F cube of y4 and here it will be 1 by 2 into 3.

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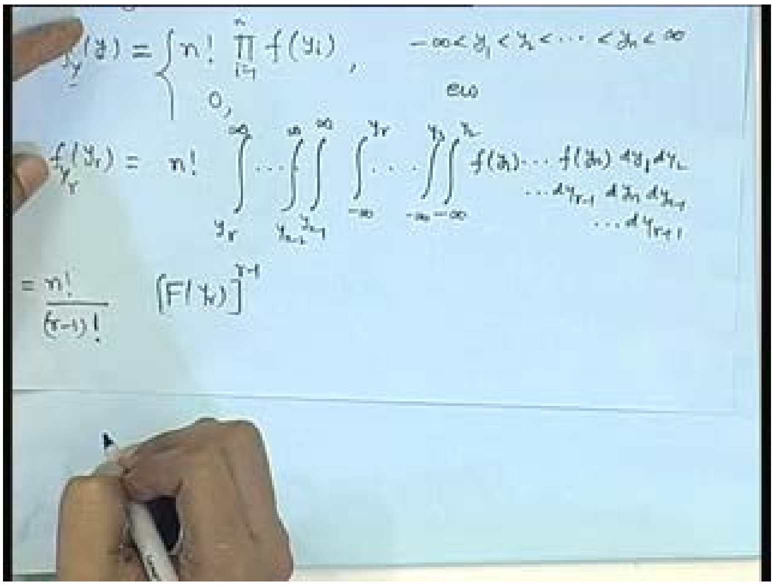

Now, this step will continue up to F of y r minus on, so, the last step will give us up to r minus 1 factorial F of y r to the power r minus 1 because in the first stage, it is capital F of y2, in the second stage, it is F square by 2, in the next stage it is F cube y4 by 3 factorial, so, at the r minus 1th stage, this will give F of yr to the power r minus 1 by r minus 1 factorial.

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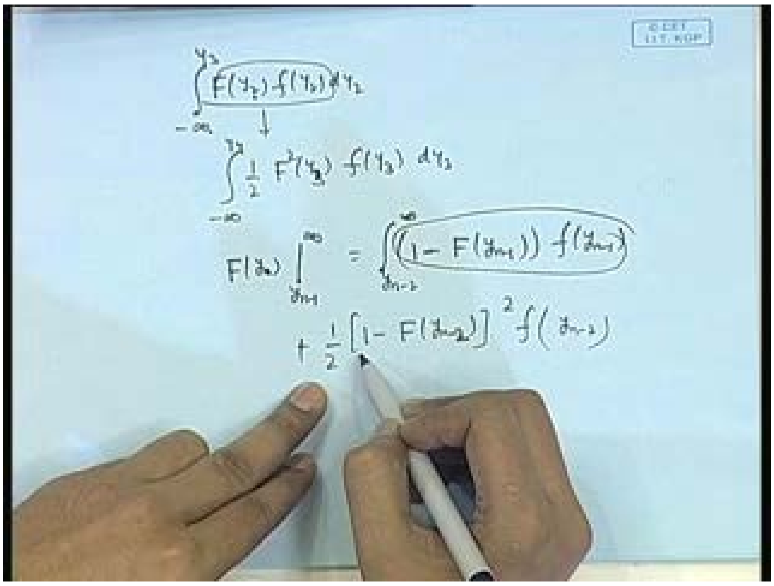

Now, let us look at the other set of variables to be integrated. Then we integrate f of yn then, that will gives us capital F of yn, now, the region of integration is from y n minus 1 to infinity, now, at infinity this is 1, so it is 1 minus F of y n minus 1. So, at the next stage, the integrant is F of, 1 minus F of y n minus 1 into f of y n minus 1. This we are integrating from y n minus 2 to infinity. Again, we notice here is that the integral will be 1 minus F of y n minus 1 square by 2 with a minus sign. So, at infinity this will become 0 and at yn minus 2 this will become this term. So, at the next stage again, and this will give us cube divided by 3 into 2.

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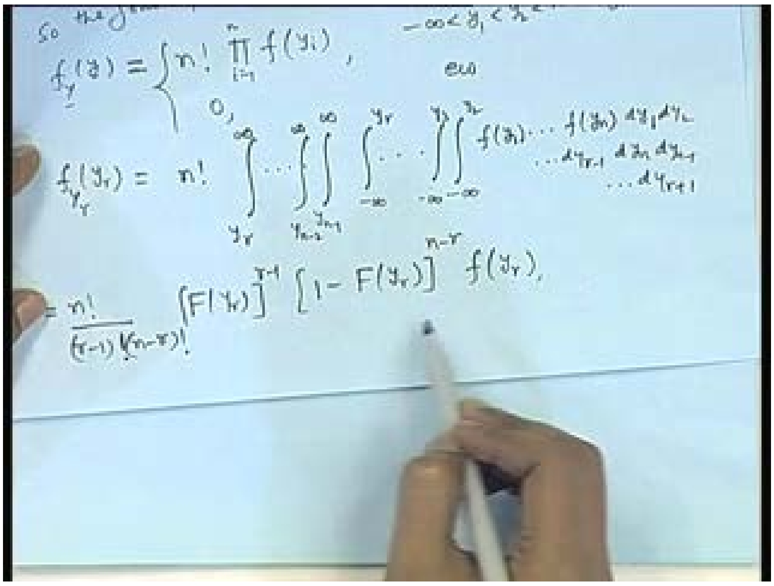

Like that, we have to continue up to dy r plus 1, so this will give us n minus r factorial 1 minus F of yr to the power n minus r multiplied by f of yr. So, we are able to determine the distribution of the rth order statistics here.

The particular case we can see, suppose the random variables are uniformly distributed on the interval 0, 1 then this will become yr, this will become 1 minus yr to the power n minus r, which is nothing but a beta distribution. So, that is one of the origins of, or you can say applications of beta distribution.

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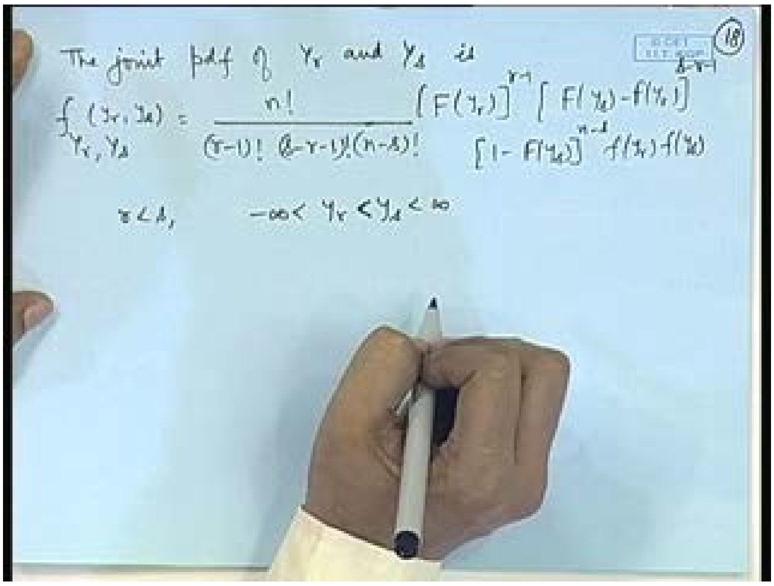

Likewise, if we want to integrate leaving variables say yr and ys, so, the joint pfd of say, two order statistics say, Yr and Ys, so, that is determined as f of Yr, Ys, yr, ys, that is equal to n factorial divided by r minus 1 factorial s minus r minus 1 factorial n minus s factorial F of yr to the power r minus 1 F of ys minus f of yr to the power s minus r minus 1 1 minus F of ys to the power n minus s f or yr f of ys; here I am taking r to be less than s, so, yr will be less than ys. In particular, we may write the joint distribution of the smallest and the largest, from there we can determine the distribution of the range that is Yn minus Y1 etcetera. So, in the next lecture, we will be considering various applications of the transformations here. Thank you.