Probability and Statistics
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Lecture No. # 02
Algebra of Sets – II

Welcome to this second lecture on algebra of sets. Today, I will introduce some algebraic structures, which are fundamental to our definition of arithmetic definition of probability.

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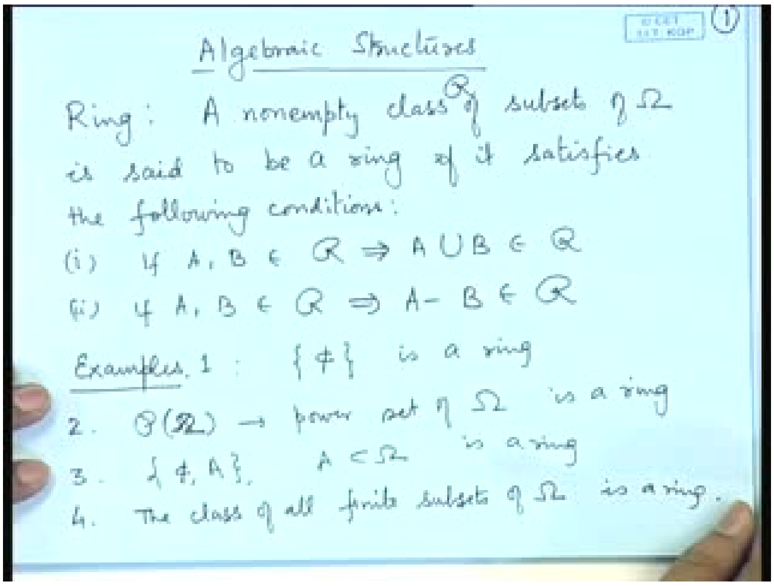

So, let us start with the first structure which I call as a ring; this is a non-empty class of subsets of.., so we continue with our previous definition of the universal set, that is omega; so, whatever sets we are considering, they will be subsets of omega. So, a non-empty class of subsets of omega is said to be a ring, provided it satisfies the following two conditions.

The conditions are that, it should be closed under the operation of unions and differences; that is, if A and B belong to say script R, so let me use a notation the non-empty class to be script R. So, if A and B belongs to R, then A union B belongs to R; and if A and B belong to R, then A minus B also belongs to R; that means, the ring is a structure which is closed under the operation of unions and differences.

Let us consider some simple examples to illustrate what is a ring; so, for example, if I consider a class consisting of simply the null set phi, then this is a ring, because if I consider phi union phi it is phi and phi minus phi is also a phi, so this is a ring. If I consider say the class of the set of all subsets of omega, that is the power set of omega, then this is also a ring, because all the sets under consideration will be subsets of omega only.

If I consider say a class phi and A, where A is a subset of omega; then, this is also a ring, because if I consider phi union A, then it is equal to A. If I consider A minus phi it is A; if I consider phi minus A, then it is phi. So, this is also a ring. So, if I consider say the class of all finite subsets of omega, now suppose I consider two subsets, say, A and B which are finite, then A union B and A minus B, both are finite; therefore, this is also a ring. Thus, we can see that, a ring contains certain subsets of omega with certain property, and therefore, it may be useful to consider such an structure.

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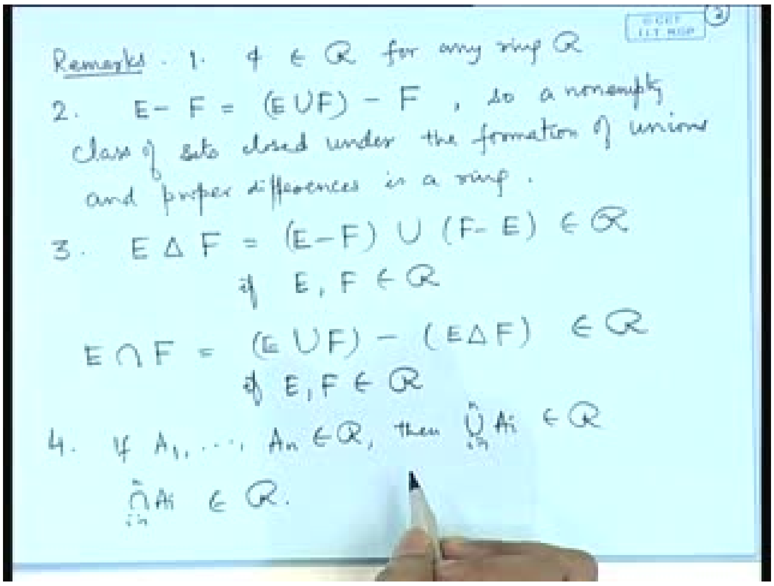

Let me look at some of the properties of ring; for example, if I consider any ring, then phi will belong to that ring; that means, if I consider any set trivially, for any ring R.

Since if I take any set say E belonging to E, since it is a non-empty set; so, at least one set belongs there. And then, if I take E minus E, then it is phi; so, every ring will certainly consist of the null set. If I write say E minus F as E union F minus F, then here you can see that, E union F will be certainly including F; and if E is any non-empty set, then E union F is certainly going to be larger than F; that means, F will be a proper subset of E union F.

So, it follows that a non-empty class of sets closed under the formation of unions and proper differences is a ring. We can also see some other properties; for example, if I look at say symmetric difference, then symmetric difference is defined as, E minus F union F minus E. Now, if E and F are in R, then E minus F and F minus E, both are in R, and therefore, its union is also in R; that means, the symmetric difference of two sets E and F will belong to the ring, if E and F belong to ring; that means, a ring is also closed under the operation of symmetric differences. Similarly, if I consider say E intersection F, then this I can represent as E union F minus E delta F.

Now, just in the previous statement, we have proved that, if E and F belong, then E delta F also belongs to R. E union F is already there and therefore the difference is also in R; therefore, this will also belong to R, if E and F belong to R; that means, a ring is also closed under the operation of symmetric differences and intersections.

Since we have taken that, if there are any two given sets A and B, then A union B belongs to R. Then, by mathematical induction, we can prove that, if we have sets A 1 A 2 A n belonging to R, then union of A i is i is equal to 1 to n will also belong to R; that means, the ring is closed under the operation of taking finite unions.

In a similar way, we can also look at intersection A I, i is equal to 1 to n. Since we have already proved that, for given two sets, the intersection is in R; therefore, by induction, we can prove that, intersection A I, i is equal to 1 to n will also belong to R; that means, a ring is closed under the operation of taking finite unions and finite intersections.

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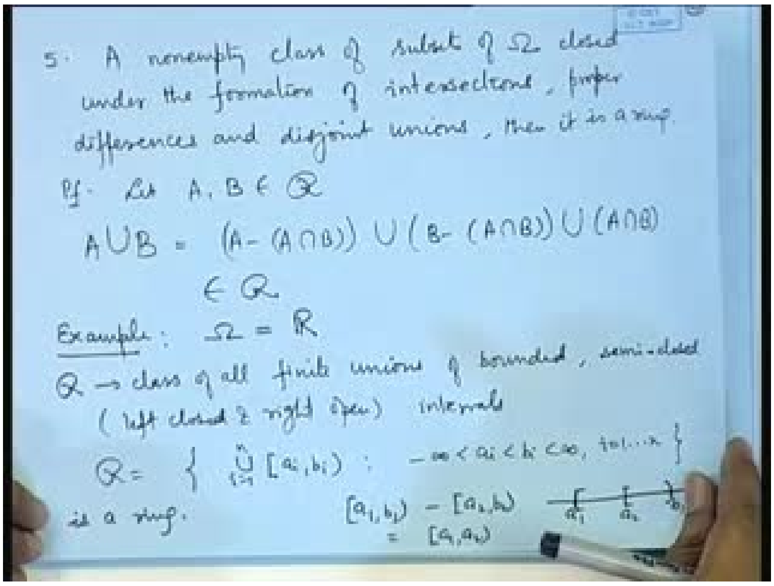

We can consider an alternative definition of the ring in the form, that if we consider a non-empty class of sets - class of subsets - of omega which is closed under the formation of intersections, proper differences, and disjoint unions, then it is a ring. For example, if I consider say two sets A and B in the ring R, and if I look at, since already I have taken the differences, so A minus B is equal to A union B minus b; so, proper differences are already there.

If I take say union, then this union I can represent as A minus A intersection B union B minus A intersection B and A intersection B. Now, here you see these three sets are disjoint; the first two are proper differences and this whole union is a disjoint union; therefore, this will belong to R. So, this can be considered as an alternative definition of the structure ring.

Let us consider some bigger example, in the sense that, it will consist of many more sets. Let me take omega to be the set of real numbers and R is the class of all finite unions of bounded, then semi closed. So, we can consider basically, say, left closed and right open R reverse.

Basically I am saying R is the collection of the sets of the form union i is equal to 1 to n a i b I, there minus infinity less than a i less than b i less than infinity, for i is equal to 1 to n. If we consider such collection, then this is a ring. You can look at the proof of this statement. Suppose I consider two such finite unions, then their union will again be a finite union.

If I take say difference, let me explain the difference, suppose I take only a 1 b 1 minus say a 2 b 2, you draw it on a line. Suppose this is a 1, this is b 1, this is closed, this is open; on this side, you have a 2 b 2. Suppose this a situation, then if I consider a 1 b 1 minus a 2 b 2, then it is simply equal to a 1 a 2 which is again an interval of the same form; therefore, now if I consider unions and then I take their differences, then it will be a unions of the intervals of the same form and therefore this is a ring.

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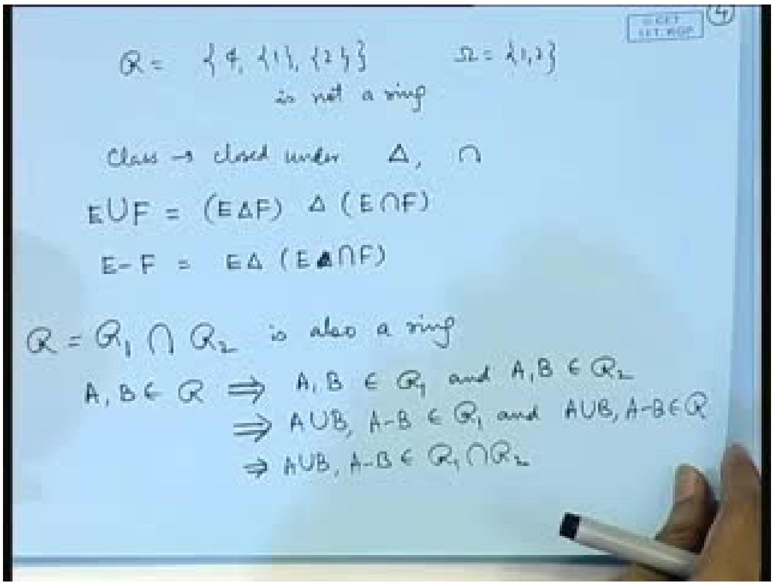

Let us consider say the difference, that here unions are and intersections are treated in the definition of a ring. Here, I say that, if a ring is closed under intersections, then a class of sets closed under the formation of intersection and differences is not necessarily a ring; for example, if I consider R is equal to say phi 1 2, where omega is equal to 1 2, then this is not a ring.

Here, it is closed under the operation of taking intersections, but it is not a ring. So, the definition of ring is not symmetric in the treatment of unions and intersections. If I consider a class, which is closed under the formation of symmetric differences and intersections, then it will be a ring.

For example, if I consider E union F, then it can be represented as E delta F delta E intersection F. Similarly, if I consider E minus F, then this I can write as E delta of E intersection F. So, you can see that, the class of sets which are closed under the formation of symmetric differences and intersections, they are ring; on other hand, if we consider it is closed under the operation of symmetric differences and unions, then also we get a ring; therefore, the definition of ring becomes symmetric in unions and intersections, if we replace difference by a symmetric difference. Then, further thing is that, suppose I say R 1 is a ring and R 2 is a ring, then if I take the intersection of this, let me call it R, then this is also a ring.

For example, if I take say the sets A B belonging to R, then this implies that A and B both belong to R 1 and both belong to R 2 as well. Now, since R 1 is a ring and R 2 is a ring, that means, that A union B and A minus B will belong to both R 1 and R 2, and this will mean that, A union B and A minus B belong to R 1 intersection R 2; therefore, the intersection of two rings is again a ring.

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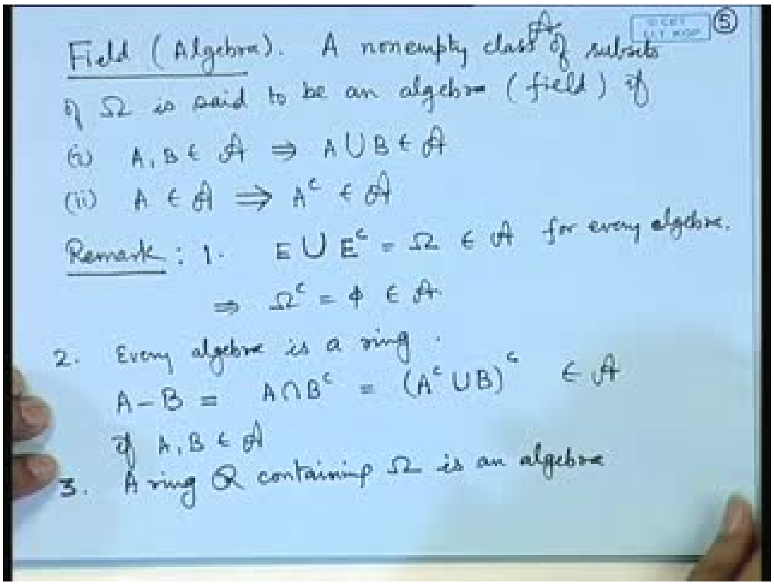

Ring is one of the primary structures in the study of algebraic structures. Now, we proceed to define slightly larger structures; the first of this analization is in the form of a field R and algebra. It is also called an algebra. So, if consider a non-empty class of subsets of omega, it is said to be an algebra are a field, if it satisfies the following two properties.

That is, if A and B belong to the class, so let me denote this class as script A, then this implies that A union B belongs to A; and the second is that A belongs to script a implies that, A complement belongs to script A. Now, if you look at this definition, this definition is a little modification of the definition of the ring. A ring was closed under the operation of forming unions and differences; here, a field or an algebra is closed under the operation of unions, and in place of differences, the property has been replaced by it; it is closed under the operation of taking complementation's.

So, now let us look at the consequences of this definition, how does it behave in the sense or you can say with respect to a ring. So, the first is that, given any set E, its complement will be there, and therefore, its union E union E complement will be there. Now, this is the full set; that means, this will belong to algebra. So, you have seen that, a ring always consisted of the empty set. Now, the full set consist always included in an algebra and now you take omega complement, that is equal to phi, then this also belong to an algebra.

Our second remark is that, every algebra is a ring; now, to prove this statement, we have to prove that the difference between two sets will belong to the algebra. Now, if I take A and B as two sets in algebra, then A minus B I can express as A intersection B complement, which is equal to A complement union B whole complement. Now, by the definition of algebra, if A belongs to algebra, then A complement belongs to an algebra.

Now, A complement union B will belong to algebra, and therefore, its complementation will belong to an algebra; therefore, if A and B belong to the algebra, then A minus B also belongs, and therefore, every algebra is a ring. So, in some sense, now you can say that algebra is an extension of the definition of the ring.

Alternatively, we can look at it, that if I consider a ring and I include the set omega here, then this becomes an algebra, because if I here put omega there, then for any given set, its complementation can be obtained by taking omega minus that given set; therefore, the ring will become automatically an algebra. Next is the property of the intersections; so, since the ring always is closed under the intersections, therefore, algebra is also closed under intersections. Since the ring was closed under symmetric differences, an algebra is also closed under symmetric differences.

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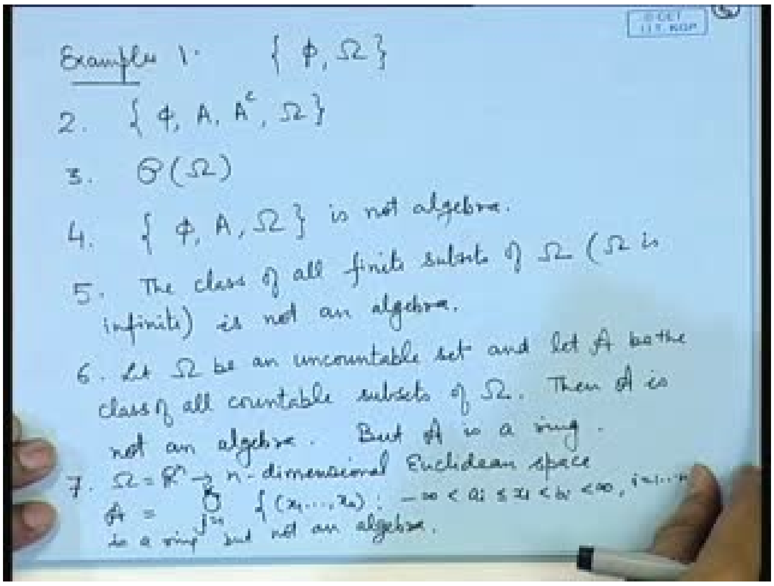

Let us look at certain examples here. So, if I consider say phi omega, then this is you can say an smallest algebra. If I include A and A complement, then you can consider it as smallest algebra containing the set A; if I consider the power set of omega, certainly it is an algebra because all the subsets are already included here.

See, if I consider a set like phi a and omega, then this is not an algebra, because A complement is not there. We have considered the class of all finite subsets of omega, where omega may be infinite; then, this is not an algebra, this was a ring, but this cannot be an algebra, because the complementation of a finite set will become infinite set and that is not there in this class; so, this cannot be an algebra.

Let us take some nontrivial example here. Let omega be an uncountable set; let us define the set a to be the class of all countable subsets of omega, then once again if I consider a countable subset of omega, its complement will become uncountable; and therefore, this A cannot be an algebra.

However, you can see that it will be a ring, because if I take any two countable sets its union, it is again a countable set; and if I take difference of the two countable sets, it can be a finite or a countable set. Therefore, it will be a ring. From the examples, it is clear that, an algebra are a field is an extension of the definition of a ring, in the sense that, it contains some more sets.

Let us consider omega to be n-dimensional Euclidean space, and let us consider A to be the class of all finite unions of the intervals of the form, say, x 1, x 2, x n, where a i is less than or equal to x i less than b i less than infinity; for and let me put this as, say j is equal to 1 to k, and here i is equal to 1 to n. So, basically you can consider them as unions of certain cells.

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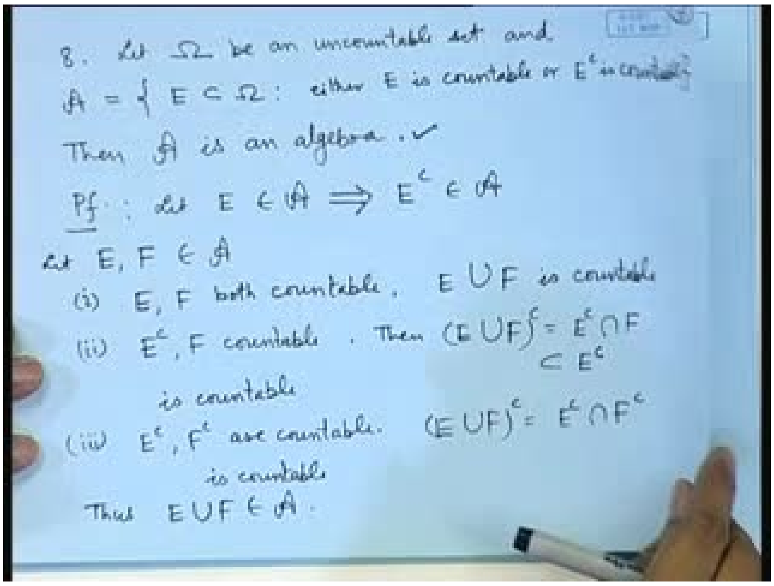

Then, this is a ring but not an algebra. The reason is that why it is an not an algebra, because if I take the complementation of such a set, then that will be say minus infinity to a i and union b i to infinity, which is not a set of this form. Let us construct some sets which some classes which may be algebra. So, let us consider, say, x to be an uncountable set, and A is the class of all subsets of omega, such that, either E is countable or E complement is countable.

So, you can see that, this is an extension of the example 6, where we considered only countable subsets. Now, here I am considering those sets which maybe themselves countable or their complements maybe uncountable, then the claim is that, A is an algebra. To look at a proof of this statement, let us see; suppose I consider E to be in A, then by the definition of the class E, either E is countable, then if I look at E complement, then the complement of that is countable. If E is uncountable, that is E complement is countable, then E will satisfy the same property; therefore, this implies that E complement belongs to A.

If I consider two sets E and F belonging to A, then there are different cases; see, E and F both countable; if both of them are countable, then E union F is also countable. If I have say E complement and F countable, then we can write E union F complement as E complement intersection F, which is a subset of E complement; so, this is countable. Suppose both E complement and F complement are countable, then we can express E union F complement as E complement intersection F complement. And since both E complement and F complement are countable, the intersection is also countable.

The case where E and F complement are countable is similar to this, because in that case, E union F complement will become a subset of F complement. So, in all the cases, E union F will belong to A. So, A is an algebra, in fact, given a ring we can always construct a bigger class, which will be an algebra; we can do it in the following way.

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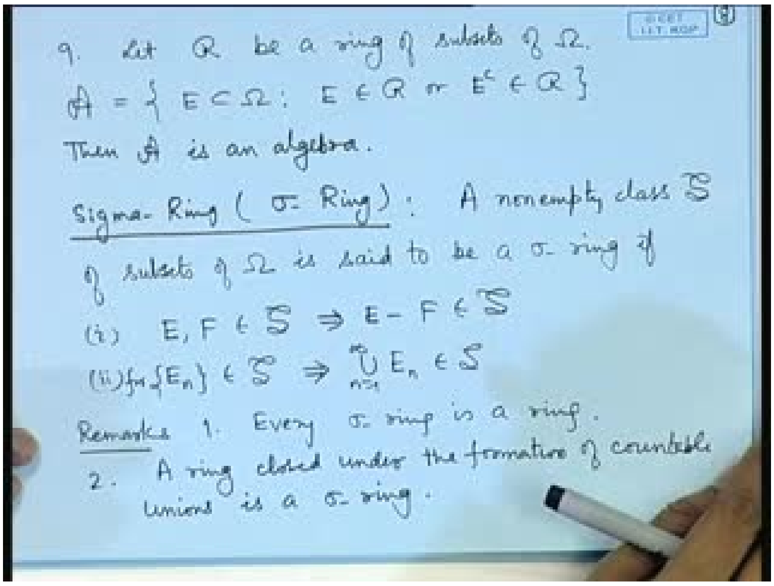

Let us consider R to be a ring and we define the class A to be the class of all subsets of omega, such that, either E belongs to R or E complement belongs to R. Then, once again A is an algebra. The proof of this statement is almost the same as the previous exercise, because in the previous exercise, if you replace the ring to be the class of all countable subsets of omega and you implement this definition, then the proof will be same.

Now, let me define a another extension of the definition of ring and we call this structure as a sigma-ring. We can also write it in this fashion sigma-ring; so, this is an extension of the definition of ring, in the sense, that a ring was closed under the operation of taking finite unions and differences. If we change the finite unions to countably many unions, then it becomes the definition of a sigma-ring.

So a formal definition is, a non-empty class of subsets of omega, so let me denote this class by say script S, is said to be a sigma-ring, if it satisfies the following two properties; that given any two sets their difference must be in the class, and secondly, if I consider any sequence, then its union must belong to… so, easily you can see that, the second one is a generalization from the definition of a ring, because there we assumed only finite union. So, naturally the consequences that, every sigma-ring is a ring; secondly, if I have a ring and it is closed under the formation of countable unions, then it is a sigma-ring.

As you had seen in that in the definition of a ring, we assumed only closeness under the formation of unions and the differences. However, we could show that, it is closed under the formation of symmetric differences, the formation of intersections, etcetera. In a similar way, if I am considering a sigma-ring, then it will be also closed under the formation of intersections. So, let us write down that statement.

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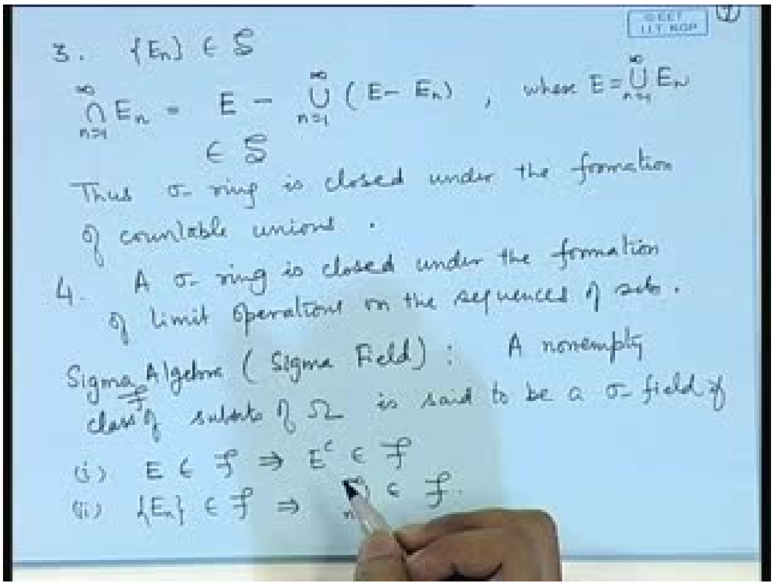

If I have a sequence say E n in S, then we can write intersection E n n is equal to 1 to infinity, as E minus union E minus E n n is equal to 1 to infinity, where E denotes the set union n is equal to 1 to infinity. Now, if script S is a sigma-ring and if I am considering a sequence E n there, then naturally the countable union of the sets belongs to the class S.

Now, E minus E n also belongs, because this is the differences of that sets; and therefore, if I again take the countable union, that is again belonging to the class S, and therefore, E minus this is again in the S. So, this belongs to S. Now, since this is an E, this is A alternative representation of the intersection E n, this means that, every sigma-ring is closed under the formation of countable unions.

In the previous class, we introduced the concept of the limit of a sequence of sets; it was actually defined as a limit superior of a sequence of sets, limit inferior of a sequence of sets, and if the two are equal, then the limit exist. Now, we had alternative representation of the limit superior and limit inferior in the form of countable unions of countable intersections or countable intersections of countable unions. Now, if my initial structure is a sigma-ring, and it is closed under the formation of countable unions and countable intersections, therefore it is also closed under the operations of limit superior and limit inferior; and therefore, if the limit exists, then under the operation of taking limits.

So, we can mention that a sigma-ring is closed under the formation of limit operations on the sequence of sets. One more extension of the definition of ring are algebra or sigma-ring is the so called structure called sigma-algebra or a sigma-field. So, let us define it in the following fashion: sigma-algebra or a sigma-field. So, a non-empty class of subsets of omega, so let me define this class as a script F, is said to be a sigma-field, if it satisfies that, for any given set, its complementation will also be in the given class. Secondly, for any sequence of sets, the countable union must also be in the class.

So, you can see that, it is a generalization of the definition of a field as well as it is a generalization of the definition of sigma-ring; it is a generalization of the definition of field, in the sense, that in a field, we had closed under the operation of taking complementation’s and unions, but unions were taken to be finite. Here, we have taken countable unions; it is an extension from the definition of a sigma-ring, in the sense that, countable unions are there, and the differences have been replaced by complementation.

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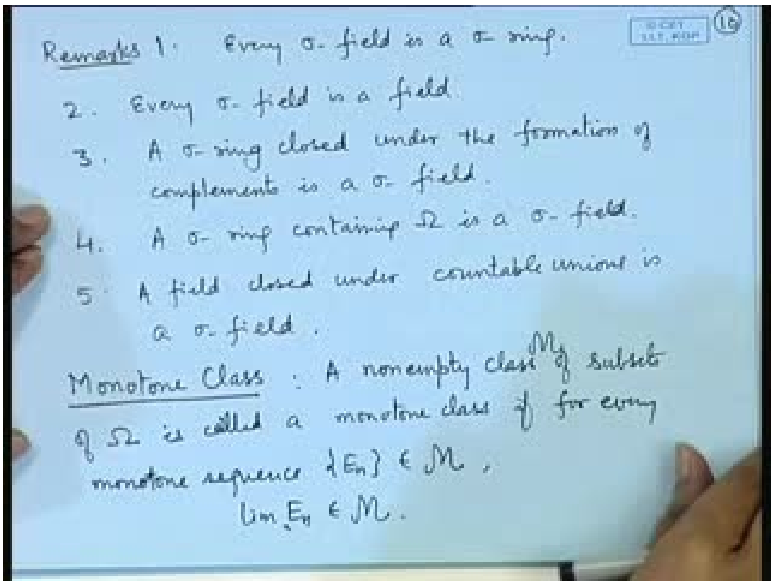

So, in some sense, it is one of the largest structures among the four structures that we have defined. So, we can write the comments that, every sigma-field is a sigma-ring, every sigma-field is a field, and therefore, every sigma-field is also a ring. We can also say that, a sigma-ring closed under the formation of complements is a sigma-field, a sigma-ring containing omega is a sigma-field. A field closed under countable unions is a sigma-field; so, among the four structures, this sigma-field or sigma-ring is the most general structure. A related structure is that, of a monotone class, a non-empty class of subsets of omega is called a monotone class, if for every monotone sequence E n in this class limit of E n belongs.

We proved that, a monotone ring is a sigma-ring and a sigma-ring is a monotone class. See, we already said that, in a sigma-ring, the limit operations are valid, and therefore, in a sigma-field also, the limit operations are valid. So, monotone class is a particular structure, which is something like in between a ring and a sigma-ring or between a field and a sigma-field. However, it is useful in the sense that, given a class, if I just look at whether the limits are there, then monotone class is confirmed; and in many operations, that is what we finally need.

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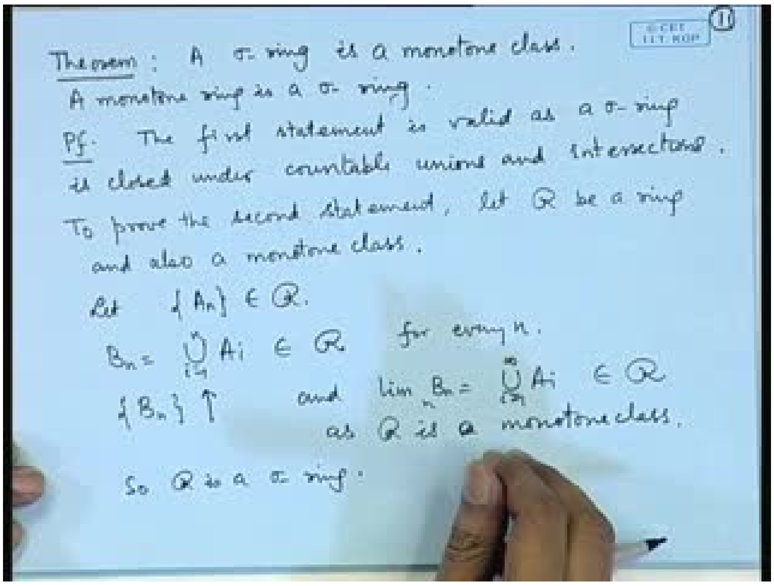

So, let me prove the following theorem; a sigma-ring is a monotone class. A monotone ring is a sigma-ring; to prove that a sigma-ring is a monotone class, we notice that a sigma-ring was closed under the operations of taking infinite unions and infinite intersections; and therefore, the limit operations were valid; therefore, a sigma-ring is naturally a monotone class, because if a monotone sequence is taken its union or intersection is the limit, depending upon whether you have a monotonically increasing sequence or a monotonically decreasing sequence.

So, the first statement is valid as a sigma-ring is closed under countable unions and intersections. To prove the second statement, let us see in the following fashion, let R be a ring and also a monotone class; so, since it is already a ring, we have only show that countable unions will belong to the given class, to prove that it is a sigma-ring. So, let us consider a sequence A n in R. Now, if I take say B n is equal to union of A I, i is equal to 1 to n, then it is a finite union of the sets in R; and therefore, it will belong to R, for every n.

Now, the nature of the set B n is that, if I take B n plus 1, then one more set will be coming. So, B n is naturally a monotonically increasing sequence of sets, and limit of B n will become equal to union A I, i is equal to 1 to infinity. Since R is a monotone class, limit of B n will belong to R as R is a monotone class; so, R is a sigma-ring.

This result is useful in the sense, that if we want to create a sigma-ring from a given class of sets, then taking all the unions etcetera may be quite complicated, whereas if it already a ring, if we ensure that the limits of the sequence of the sets is present in the given class, then it will become a sigma-ring.

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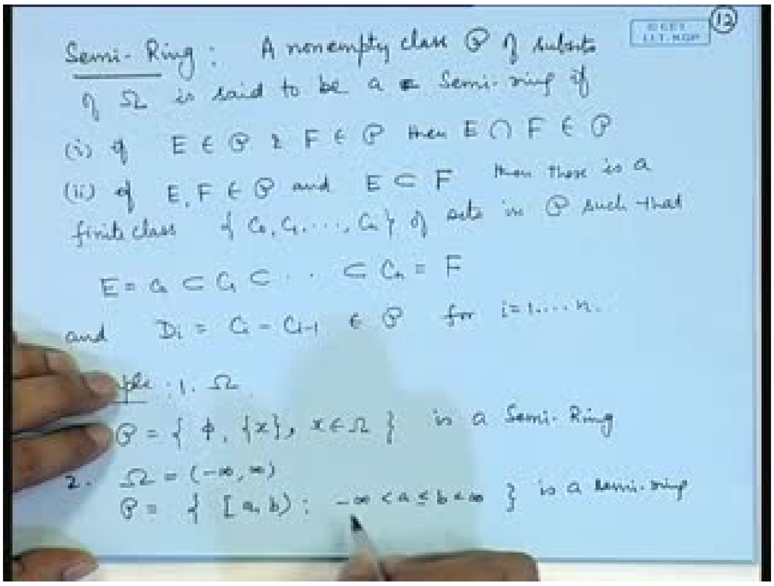

So, it is something like a you can say a method of obtaining generated sigma-rings from a given class, which is already a ring. Somewhat simpler structure in this direction is a semi-ring: a non-empty class say P of subsets of omega is said to be a semi-ring, if it satisfies the following properties; that is, if a set is there, then E intersection F belongs to P; that means, it is closed under the formation of the taking intersections. However, another property which looks somewhat different than the ones which we have given till now, is that if E and F are any two subsets of omega in P, and say one of them is a subset of the other one, then there is a finite class c naught c 1 up to c n of sets in p, such that, e is equal to C naught subset of C 1 subset of C n, which is equal to F, and the successive differences of C i is must be in P.

I mentioned that it is a structure which is simpler compare to the structures that we defined just now, in the sense that, I have not assumed that, union will be there or complementation’s will be there.

You can see here through an example, if I consider omega to be any set and P is the class consisting of the phi set, and the set of consisting of singleton sets for all x belonging to omega, then this is a semi-ring. Let us see how; if I take intersection of any two sets, then that will be empty set; if I take any two sets, since no set is a subset of another one, the second property is trivially satisfied. So, this is a semi-ring.

Now, why I said that it is a structure which is more simpler in nature compared to ring sigma-ring field or sigma-field? Because now you see this set P, it is not satisfying properties of none of the previous structures.

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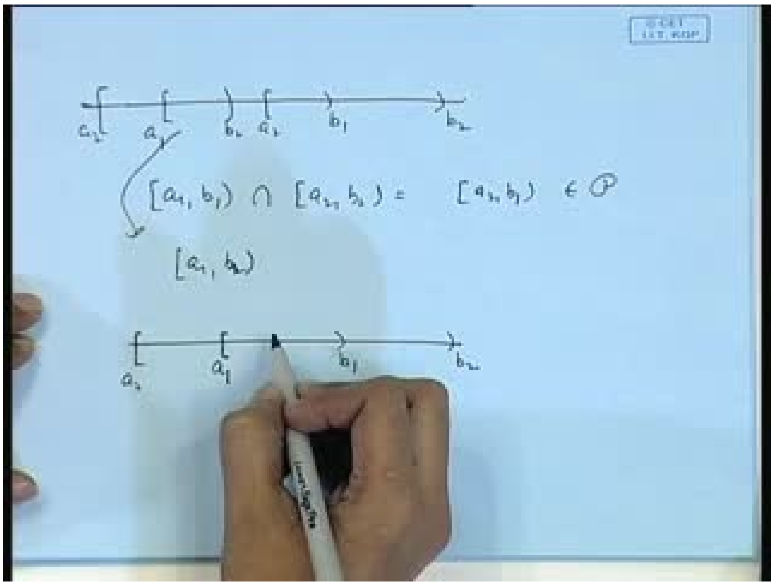

For example, unions of two sets are not there; so, it cannot be a ring, sigma-ring field or sigma-field, whereas it is a semi-ring. Let me take another example; suppose I consider omega to be the set of real numbers, and I consider P to be the class of intervals of the form a to b, that means, semi closed intervals, I am taking left closed and right open. Then, this is a semi-ring.

To see the structure of this, suppose I am considering any two intervals of the form a to b, now if I take say a 1 b 1 and say a 2 b 2, then if I take the intersection of this, then if they are of this form, then the intersection of a 1 b 1 with a 2 b 2 is of the form a 2 b 1, which is again an interval of the same form. Suppose I take this to be disjoint, then the intersection will be phi, which is corresponding to the choice a is equal to b here. Similarly, I may take this on this side, say, a 2 b 2; in that case, the intersection will be a 1 b 2, which is again an interval of the same form.

So, in all the cases, the intersections will be existing in the class P. Now, suppose I consider two intervals, such that, one of them is contained into another one; so, let me take a 1 b 1 contained in the interval a 2 to b 2.

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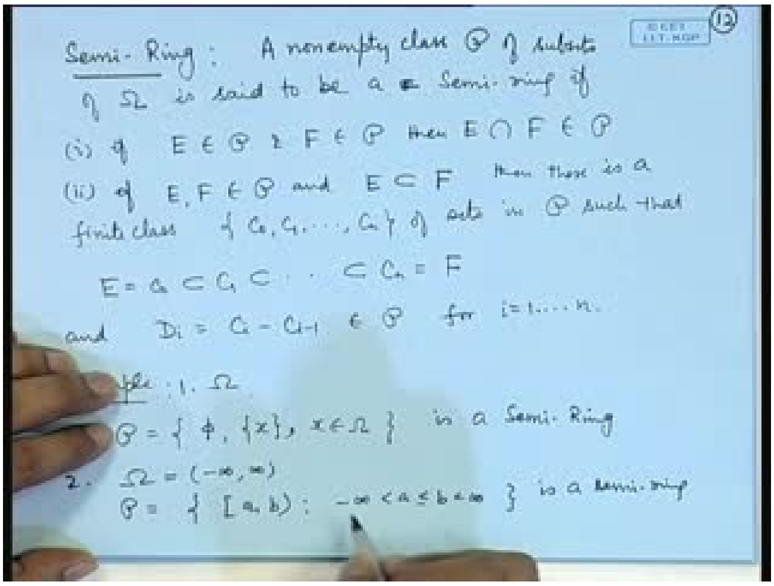

Now, if I look at this, then I can consider n classes which are n sets. Here, if you look at the second property of the definition of the semi-ring, then there is a finite class of sets in P c naught c 1 c n, such that, the smallest of this is same as E and the largest of this is same as F, such that their differences belong to P.

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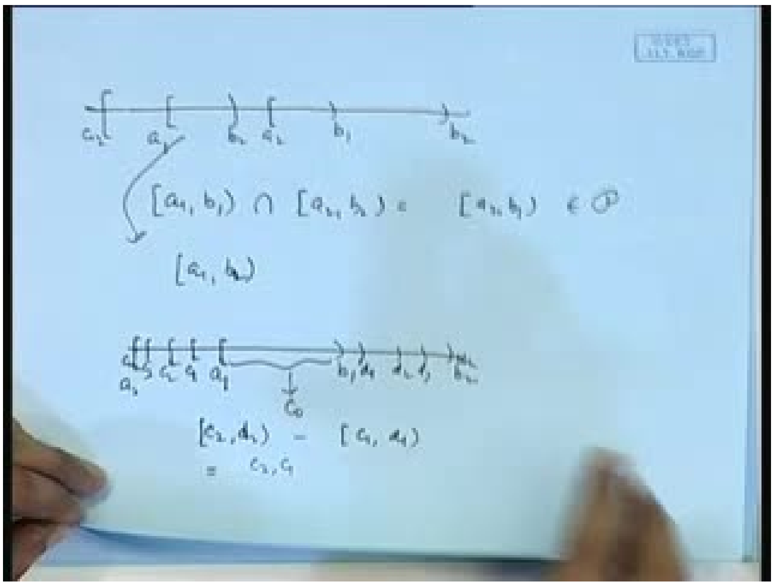

So, if we look at this one, I can always construct sets like this; so, this set itself can be my c naught, this can be some set, let me call it say c 1 to d 1, say, c 2 to d 2, c 3 to d 3 and this is say c n to d n. Then, each of them is the superset of the previous one; the largest is equal to the bigger interval a 2 to b 2, the smallest is equal to the interval a 1 b 1. All of the sets are of the same form, and therefore, they are satisfying the property that they are in P.

If I consider the difference of any two successive sets, so for example, if I take c 2 d 2 minus c 1 d 1, then it is equal to, now in this particular structure, the difference is equal to c 2 d 2. So, I am removing c 1 d 1 from here, then it is becoming equal to c 2 c 1. So, if I look at the definition of the semi-ring, then it is belonging to the class P.

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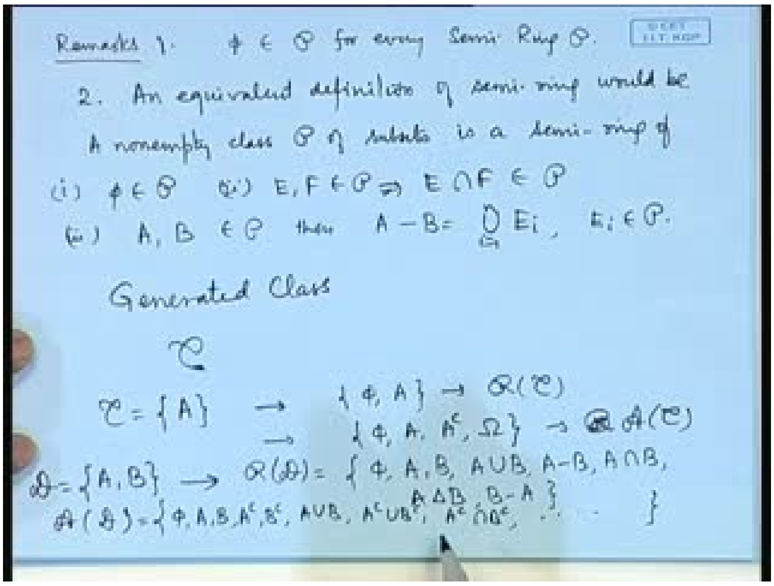

So, we have the following remarks: phi belongs to P for every semi-ring, because I can take two sets to be equal. And if I take their differences, then it will be there. An equivalent definition of semi-ring would be a non-empty class P of subsets is a semi-ring, if empty set is there for any two sets, intersection must be there. And thirdly, if A and B belong to P, then I should be able to represent A minus B as the union of a finite unions of sets in P. So, this is an alternative representation of the definition of semi-ring.

A useful concept in this context is that of a generated class. What is a generated class? So, given a class of sets, if I consider the smallest ring containing C, the smallest sigma-ring containing C, the smallest algebra containing C or the smallest sigma-algebra containing C or the smallest monotone class containing C, then it is called a generated ring, a generated sigma-ring, a generated algebra or a generated sigma-algebra. Now, to see this, suppose I look at a set consisting of singleton set A, then phi A, suppose I say this is my C, then this the generated ring of C. Suppose I take this same thing, and I take phi A A complement and omega, then this is a generated algebra from C.

Since this is a finite class, therefore, this is also a generated sigma-ring or a generated sigma-algebra. To see that how this process becomes more complicated, if I consider larger classes is that, suppose I consider say class say A B, now you can see, in order to generate a ring out of this, I have to consider the empty set, the two sets, their union, their differences, as we have said its intersection will also be there, its symmetric difference will also be there.

So, now you can see that, variety of sets B minus A must also be there, because we can take the difference of these two. So, a generated ring is of this nature, which is consisting of many sets. Now, you see how the process will become, even more complicated if I try to generate a field out of this.

If I consider a field generated from D, then the number of sets is much more. I will have to take phi, A, B, then I have to take A complement, B complement A union B, then A complement union B complement, then I have to take A complement intersection B complement, and so on so forth. So, the number of sets will be much more. And this shows that, this kind of structures will be useful in the definition of probability, because when we want to introduce the probability of different events, then their simultaneous occurrence, their differences, their unions, etcetera, also will be needed in defining the probabilities.

So, this type of definition will be extremely useful for our formal definition of probability. So, we plan to take these things in our next lectures, when I introduce the concept of probabilities.

Thank you.