Probability and Statistics

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Lecture No. # 01

Algebra of Sets – I

Welcome to this course on probability and statistics. This is an introductory course or you can say first course on the probability and statistics, and it is quite useful for all branches of science and engineering.

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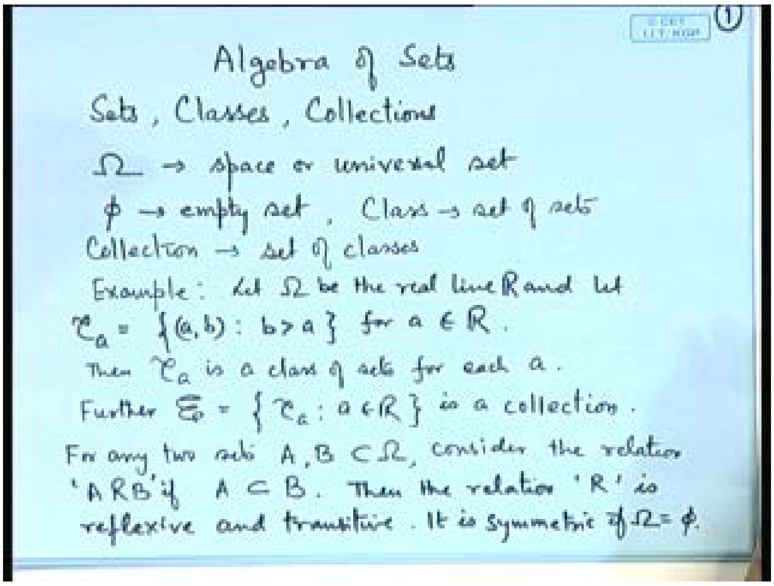

So, to begin with, we introduce algebra of sets; this is required, because the modern theory of probability is based on the set theory. So, here we introduce certain elementary concepts of sets, which will be directly used in our definition and concepts of probability.

So, to begin with, we start the discussion on what is set, classes and collections. So, we have, we start with a universal set; let us call it omega, and then, we also have the usual notation for the empty set as phi. We call a set of sets to be the class and a set of classes to be a collection.

As an example, consider, suppose omega is the real line R and we consider the collection of intervals starting from a to b, where b is greater than a, for every real number a; then, this C a is a class of sets for each a. And then, if we collect all this classes C a into a set called script E, then this is a collection. For any two sets a and b which are subsets of omega, let us consider the relation a related to b; if a is a subset of B, then the relation R is reflective and transitive; it is symmetric, when omega is equal to phi.

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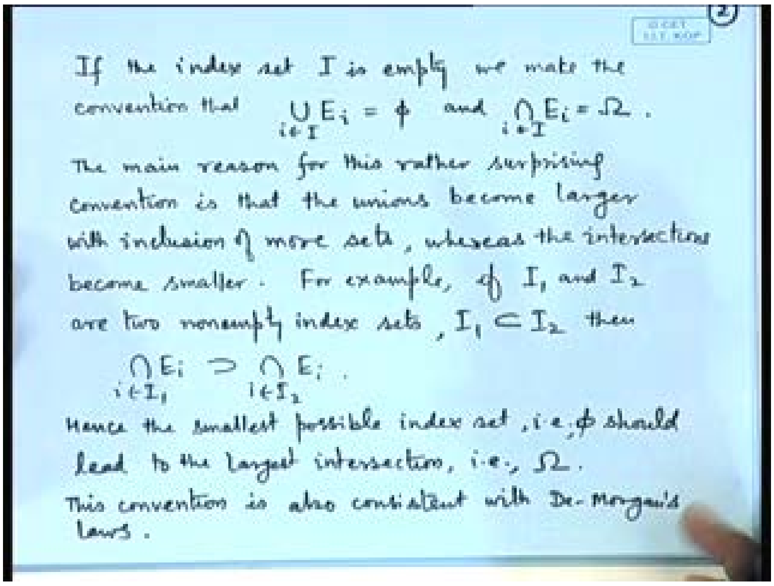

We introduce some further conventions and notations. If the index set i is empty, we make the convention that union of E i is empty and also intersection of E i is the full set omega. The second of this convention looks to be rather surprising. However, it is motivated by the fact, that if we take more and more sets in an intersection, then it becomes smaller; for example, if I have 2 index sets I 1 and I 2, such that I 1 is a subset of I 2, then intersection of E i, where i belongs to I 1 contains intersection of E I, where i belongs to I 2, primarily because the more sets means the intersection becomes smaller.

Hence, the smallest possible index set, that is, phi should lead to the largest intersection, that is omega. This convention is also consistent with De Morgan’s Laws; for example, if I write union of E i, i belongs to I compliment, is equal to intersection E i compliment i belonging to I. Let us take I to be phi, then the left hand side is equal to phi compliment is equal to x; this indicates that, we should take intersection of E i. When i is an a empty index set, then this should be equal to the full space omega

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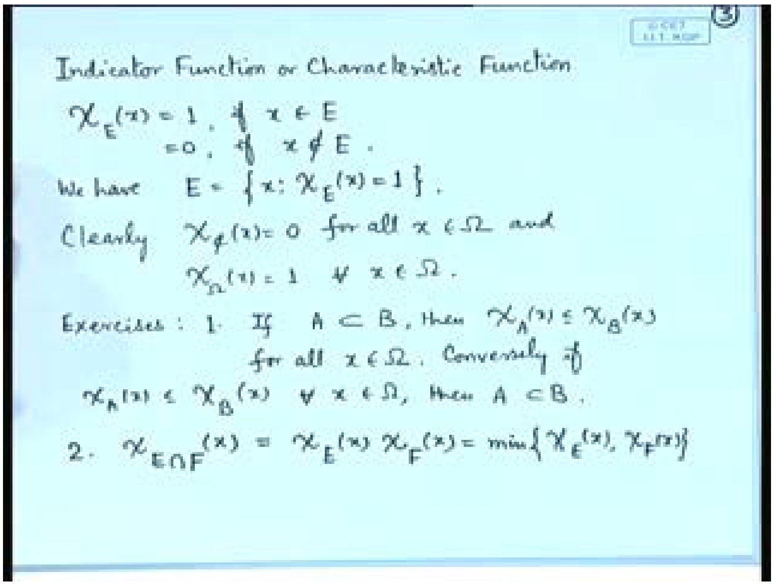

Let us introduce the concept of indicator function or characteristic function. The characteristic function of a set E is defined to be 1, if x belongs to E, and it is 0, if x does not belong to E. Alternatively, we can write the set E as the set of all those points, for which x chi E x is equal to 1; clearly, indicator function of the empty set is always 0 and the indicator function of the whole space is always 1.

There are certain exercises here; for example, if a set a is a subset of B, then the indicator function of a is always less than or equal to indicator function of B. Conversely, if the indicator function of the set a is always less than or equal to the indicator function of B, then a is a subset of b.

A simple proof of this is that, if we consider chi A x, then it is equal 1 for all x belonging to a. And since A is a subset of B, this implies that, chi B x is also 1 for these points. Now, for the points where chi A x is equal to 0 includes certain points, where chi B x may be 1 and for other points chi B x will also be 0. Therefore, chi A x is always less than or equal to chi B x.

We have certain elementary properties of the characteristic functions; for example, characteristic function of an intersection is equal to the product of the characteristic functions of the two sets. It is also equal to the minimum of the indicator function values for those two sets. The proof can be simply obtained by definition; for example, chi of E intersection F x is equal to 1, if x belongs to both E and F, and 0 otherwise; that means, it is going to be equal to 1, only if chi E x and chi F x both are 1; and in every other case, it is going to be 0; so, that means, it is equal to the product. In a similar way, the minimum of chi E x and chi F x is going to be 1, only if both chi E and chi F x are 1; that means, chi E intersection F x is equal to 1.

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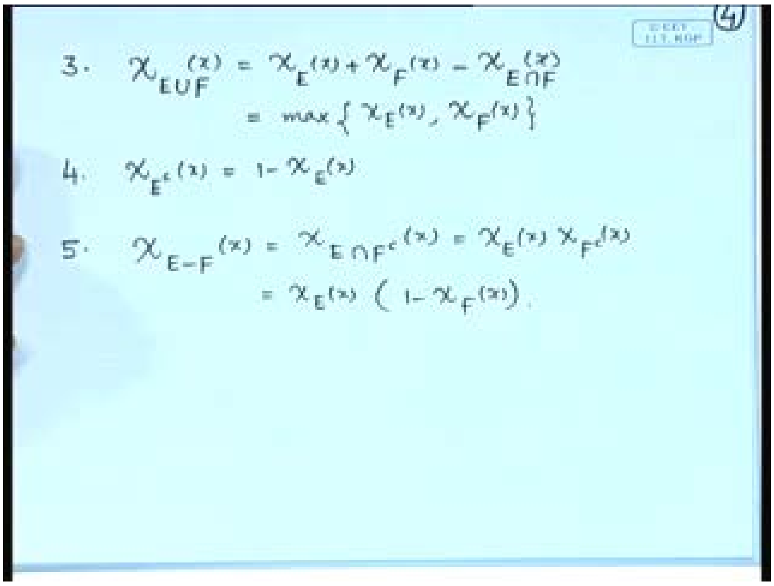

Similarly, the indicator function relations for the unions, complimentations and differences are also there; for example, chi F E union F x is equal to chi E x plus chi F x minus chi F E intersection F x; it is also alternatively equal to the maximum of chi E x and chi F F of x; once again to look at the proof of this.

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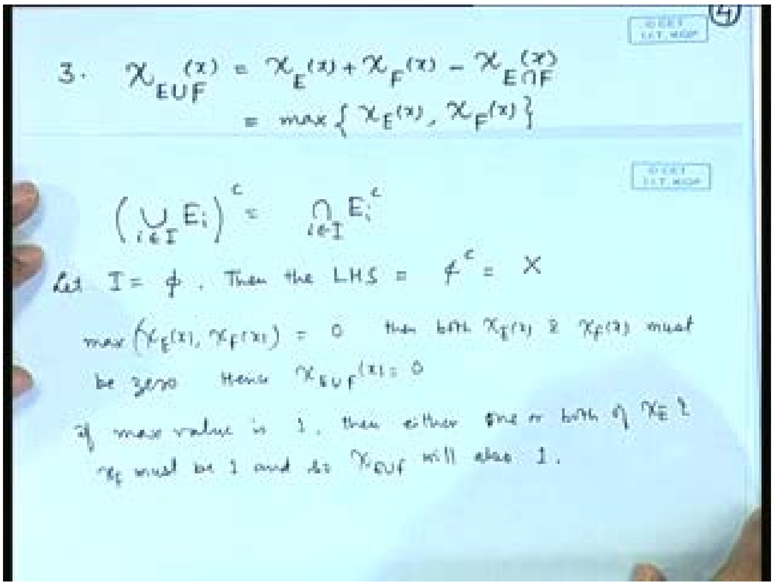

We can see that, if we have, if we write maximum of chi E x and chi F x to be say 0, then both of chi x and chi F x must be must be 0. Hence, chi F E union F x will be equal to 0. If maximum value is 1, then either 1 or both of chi E and chi F must be 1; and so chi F E union F will also be 1.

The other properties of chi function, for example, chi F E compliment is equal to 1 minus chi E; chi F E minus F is equal to chi F E intersection F compliment x, which is chi F E into chi F F compliment from the property 2, and chi F F compliment from the above property is equal to 1 minus chi F x. So, these are some of the useful relationships for the indicator function.

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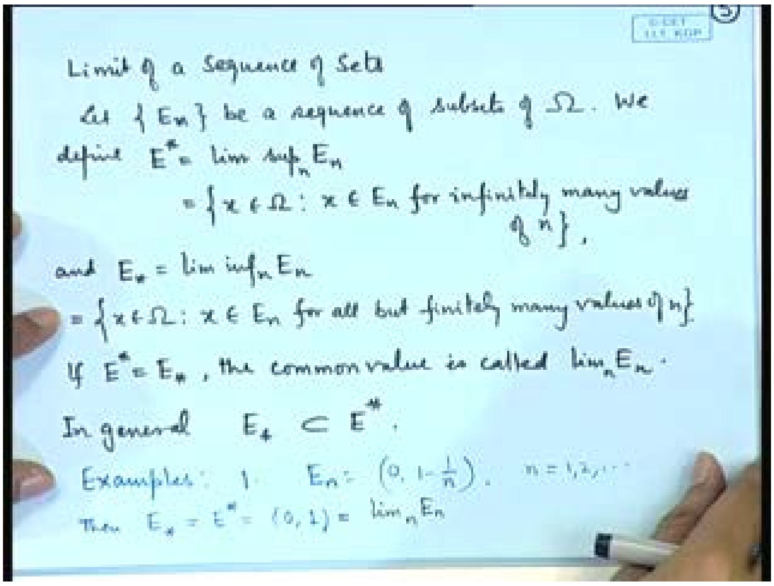

We have all heard about the limit of sequence of numbers; for example, you easily understand that limit of a n is equal to a, if for every epsilon greater than 0; there exist capital N, such that, modulus of a n minus a becomes less than epsilon, whenever n is greater than or equal to capital N.

This shows that, the sequence a n is closer to the number a, the values of the sequence a n become closer to the number a, as a becomes large. However, the similar concept is not available for the sequence of sets. Here, we define it in a slightly different fashion; so, in order to introduce the concept, we introduce the of concept of lim sup of E n and lim inf of E n are called limit superior of E n or limit inferior of the sequence E n.

So, consider a sequence of subsets of omega called E n. We define E upper star is equal to lim sup of E n, which is equal to the set of all those points, which belong to infinitely many values of n. What does it mean? It means that, from the whole space, we take up those points which belong to E n for infinitely many values of n; the collection of such elements will be called limit superior of E n.

In a similar way, we define limit inferior of E n called E lower star; it is the collection of those elements of omega, which belong to all, except may be a finite number of values of n; that means, they belong to almost all the values of E n, except may be for a few of them. If E upper star and E lower star are the same sets, then the common values called the limit of E n, and we say that the limit of the sequence of sets exists.

By the definition, it is clear that, in general, we will have E lower star as a subset of E upper star, primarily because, if x belongs to E lower star, then already we are committing that it is belonging to infinitely many values of n, because it is belonging to all except a finite number; that means, definitely it is belonging to infinitely many values of n. Therefore, in general, E lower star is always a subset of E upper star; when it is equal, we say that the limit exists.

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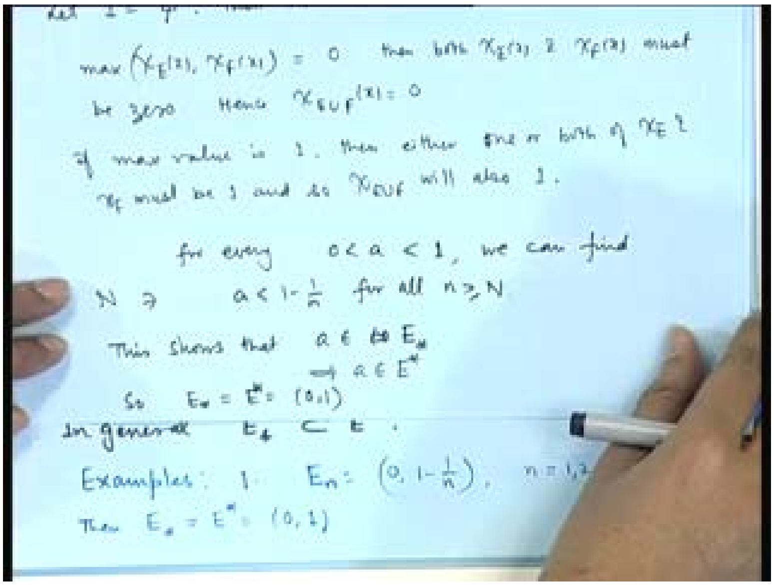

Let us consider some simple examples of application of the concept of the limit of sequence of sets. Consider for example, E n set defined to be an open interval 0 to 1 minus 1 by n for n is equal to 1 2 and so on.

Here, if we see, if we calculate the limit inferior, then if you consider any point in the interval 0 to 1, then since 1 by n goes to 0 as intense to infinity, for every number a between 0 to 1, we can find for every 0 less than 1, we can find capital N, such that, a is less than 1 minus 1 by n, for all N greater than or equal to n. This shows that, a will belong to the interval; a will belong to E lower star. This also shows that a will belong to E upper star.

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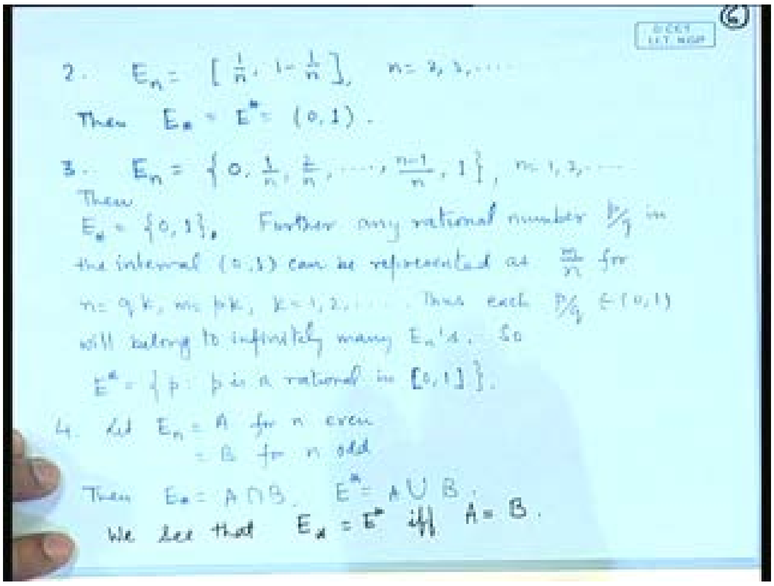

So, E lower star is equal to E upper star is equal to the interval 0; so, this is a case where the limit exists. Let us take another example; the sequence of closed intervals from 1 by n to 1 minus 1 by n, for n is equal to 2, 3 and so on. Once again if you utilize the argument as given in the example 1, if we consider any element a between 0 to 1, then after a certain stage, it will belong to the set E n, for all n greater than or equal to sum capital n.

Therefore, once again, the E lower star, that is a limit infimum of this sequence will be the interval 0 to 1. And naturally, E upper star will also be 0 to 1, because it is a superset of E star in general, and it cannot go beyond the interval 0 to 1. Notice that, here the point 0 itself and the point 1 itself do not belong to E lower star or E upper star, because for no value of n, these points belong to the set. Now, I will give one example, where the set E n is defined has a finite number of points; and here, E lower star and E upper star are different.

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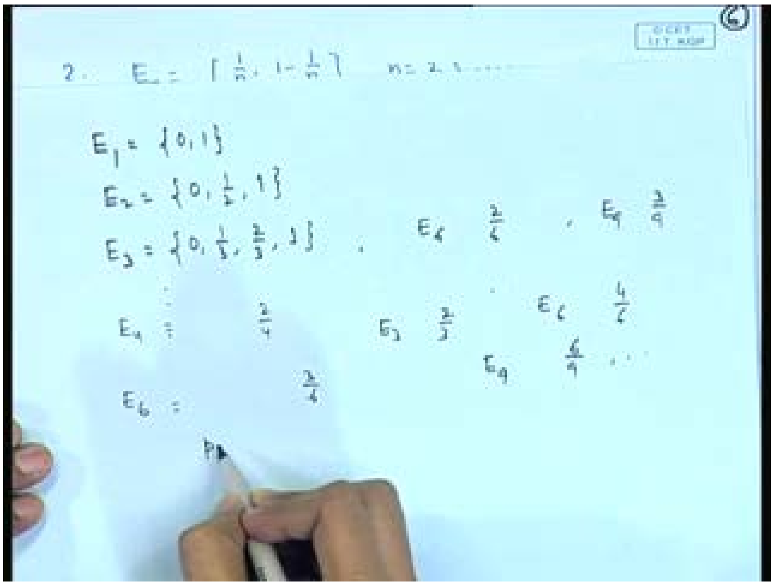

Consider the set E n given by 0, 1 by n, 2 by n, n minus 1 by n to 1, for n is equal to 1, 2 and so on. Then, this particular sequence can be explained like, that if we consider E 1, then it is the set 0 to 1. If we consider E 2, then it is the set 0 1 by 2 1. If we consider the set E 3, then it is equal to 0, 1 by 3, 2 by 3, 1. Clearly, the points 0 and 1 belong to all the sets; therefore, E lower star, that is the limit infi mum must contain the 0. 01. Now, we say that, no other element of the firm, say, p by q can belong to all the sets E n's, even after a certain stage.

For example, the point half, it will belong to once again E 4, because 2 by 4 will come; it will belong to E 6, because three by 6 will come. So, it will belong to infinitely many sets, but it will not belong to all but finitely many. Similarly, if i consider, say 1 by 3, then it will be there in E 3, it will be there in E 6 as 2 by 6, it will be there in E 9 as 3 by 9, etcetera.

So it will once again belong to infinitely many sets, but it will not belong to all but finitely many. Suppose we consider a point, say, 2 by 3, then it is belonging to E 3, then it will again belong to E 6 as 4 by 6, it will again belong to say 9 as 6 by 9, etcetera. Therefore, we conclude that, all the rational numbers in the interval 0 to 1, all the rational numbers of the form p by q in the interval 0 to 1, will belong to infinitely many of the E n's.

Therefore, E upper star will be the set of all those rationals in the interval 0 to 1, whereas the limit inferior is equal to the interval, the set consisting of the points 0 and 1. Another example is, we consider the sequence as split into two parts, that for even ordered sets E n is one particular set, say, A and it is equal to B for n odd. Then, E lower star is the set of points, which are common to both A and B, because if we consider a point which belongs to both of them, then for all but finitely many sets of E n, x will belong to them.

However, if a point is not in one of A and B, then for infinitely values of n, it will not belong. Therefore, the only points which can belong to E lower star are, which are common to both A and B. On the other hand, any point which is belonging to either A or B will be there infinitely many sets, and therefore, the E upper star will be a union B. Naturally, you can see that, E lower star is equal to E upper star, if and only if a is equal to b.

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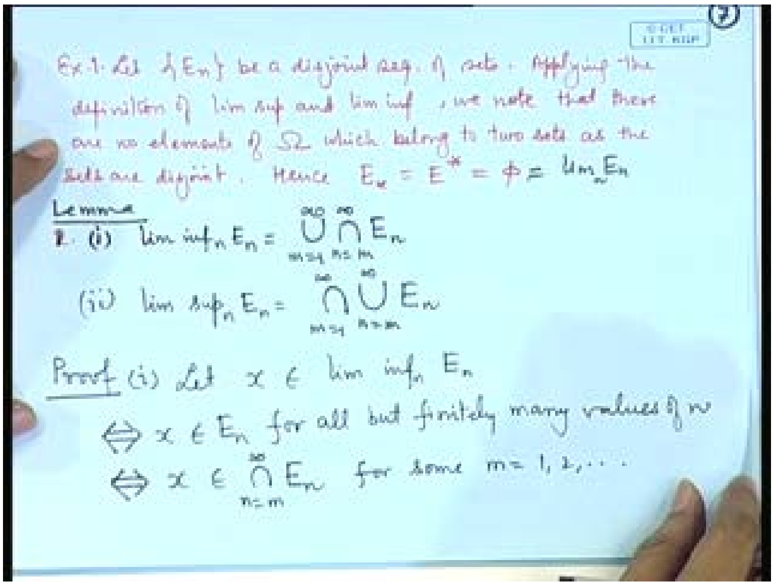

Let us consider a adjoint sequence of sets. If we apply the definition of limit superior and limit inferior, then we note that, there are no elements of omega which belong to two sets, because the sets are disjoint. Therefore, no point in the set omega can satisfy the property of belonging to either infinitely many E n's or all, but finitely many points of E n.

Therefore, limit inferior as well as the limit superior, both are the empty set and it is equal to limit of the sequence of the sets. In order to derive the limit inferior and limit superior using mathematical arguments, one look for the alternative representations for these sets; for example, limit inferior can be written as, union intersection E n, n is equal to m to infinity and n is equal to 1 to infinity.

In a similar way, limit superior can be written as, intersection union of E n, n is equal to m to infinity, n is equal to 1 to infinity. To prove these statements, let us take the first case, consider a point x belonging to limit inferior of E n; by definition of the limit inferior, it implies that, x belongs to E n for all, but finitely many values of n. This implies that, from a given m onwards, x will belong to all the sets E n; that means, x will belong to the intersection of E n, n is equal to m to infinity, for sum n is equal to 1, 2, etcetera.

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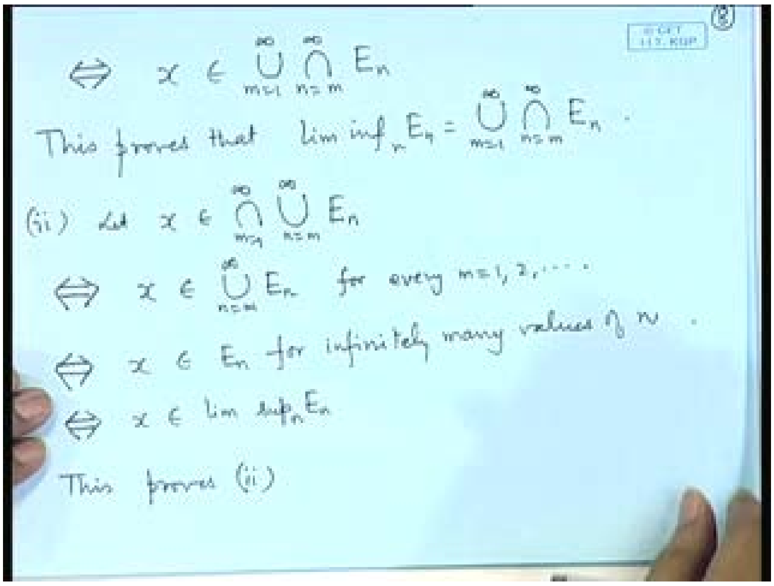

Clearly, this implies that, x belongs to union intersection E n, n is equal to m to infinity and m is equal to 1 to infinity. On the other hand, if we take x belonging to union m is equal to 1 to infinity intersection n is equal to m infinity E n, then this statement implies that, x belongs to intersection of E n for some m is equal to 1, 2, etcetera.

Now, this statement once again implies that, x belongs to E n for all but finitely many values of n, which was actually the definition of limit inferior. Therefore, we have both the implication for all the statements, and this proves that, limit inferior of E n is equal to union intersection E n, n is equal to m to infinity, m is equal to 1 to infinity.

If we apply the same argument for the limit superior, then we get that, if x belongs to intersection union E n, n is equal to m to infinity, m is equal to 1 to infinity; so, I am taking the hand side first. Now, this implies that, x belongs to union E n, n is equal to m to infinity, for every m is equal to 1, 2, etcetera. Now, if we say that x belongs to the union starting from every m, how so ever m large may be, this definitely implies that x belongs to infinitely many values of n, because if it was belonging to a finitely many values only, then after a certain stage, x will not belong to the union of E n's. However, it is belonging for every value of m, this means that, x belongs to E n for infinitely many values of n, which is actually the definition of limit superior. Now, do look at the argument in the reverse way; if x belongs to the limit superior, then this implies that x belongs to E n, for n infinitely many values of n.

Now, once again if x belongs to infinitely many values of n, then no matter for which value of m you start with, x will belong to certain sets after that, and therefore this statement implies this. Now, since this is true for every m, this statement implies that, x belongs to intersection union of E n's; this proves the statement 2.

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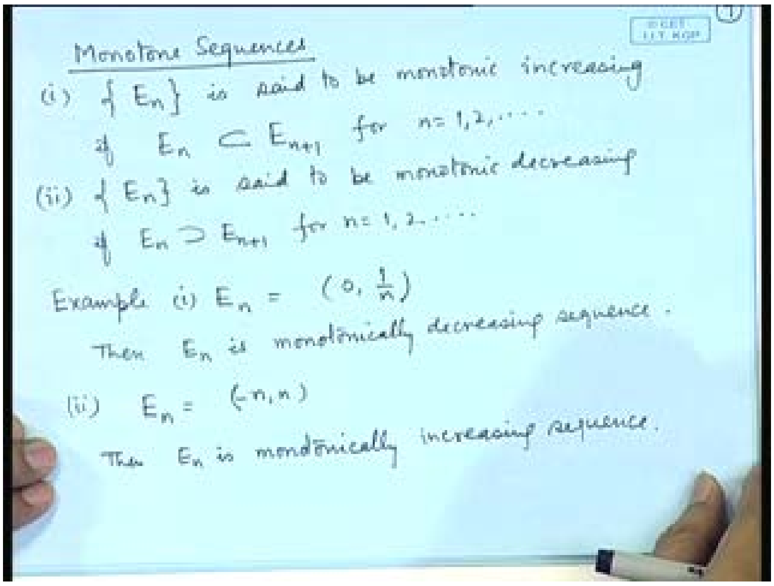

Further, we consider certain particular kind of sets, certain particular type of sequences, where the limits will always exist; they are called monotones sequences. So, we say that, a sequence E n is said to be monotonic increasing, if E n is a subset of E n plus 1, for n is equal to 1, 2, and so on. In a similar way, we define E n to be monotonic decreasing, if E n is a containing E n plus 1, for n is equal to 1, 2 and so on.

Let us consider some example here; let E n be the interval, say, 0 to 1 by n, then E n is monotonically decreasing sequence. Further, we can consider, say, sequence E n is equal to interval say minus n to n, then E n is monotonically increasing. One important result which is true for the monotonic sequences is, that the limit always exist; I will state it in the form of a theorem here.

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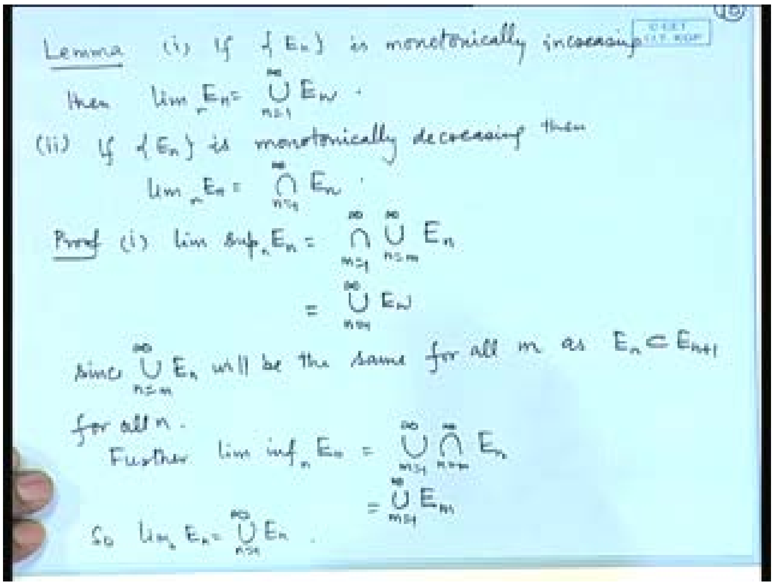

If E n is monotonically increasing, then limit of E n is equal to union of E n, n is equal to 1 to infinity. In a similar way, if E n is monotonically decreasing, then the limit of E n is the intersection of the sequence of sets E n. To look at the proof, let me prove the statement 1 first, if we take the E n to be monotonically increasing sequence of sets, then the union E n will be same for all n, and therefore, limit superior will be same.

Let me explain it further; if we consider limit superior of E n, then it is equal to intersection union E n, n is equal to m to infinity, m is equal to 1 to infinity; this will be equal to simply union n is equal to 1 to infinity.

Since union E n n is equal to m to infinity will be the same for all n, as E n is subset of E n plus 1 for all n. Further if we look at limit inferior of the sequence, then this is union intersection E n, n is equal to m to infinity. Now, in the intersection E n, this is starting from E m E m plus 1 intersection E m plus 2 etcetera.

Since the sequence is monotonically increasing, the first set in the sequence is the smallest set and it is contained in all the sets, which are coming after this. Therefore, the intersection will be equal to the first set itself, and therefore, it is equal to E m; that means, we are getting limit infimum as also union of E m's.

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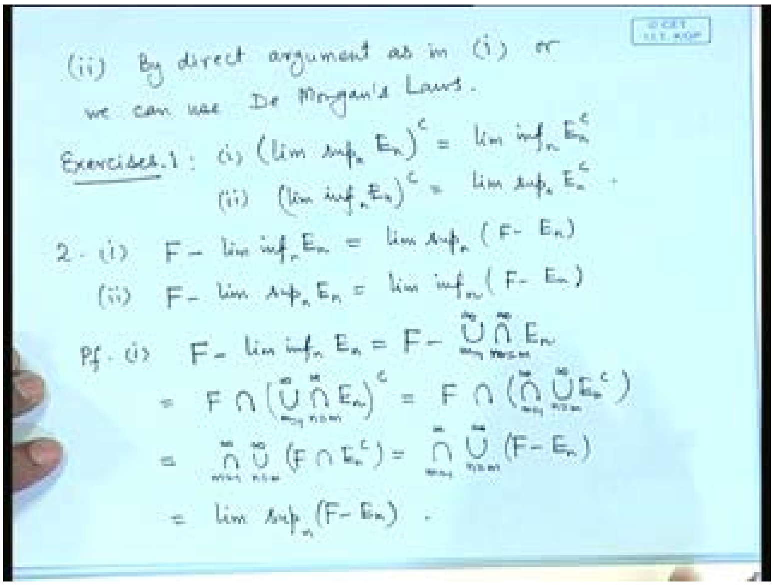

If we combine these two results, the limit exists and it is equal to the union of the sets. The proof of the second statement can be given, by direct argument as in 1 or we can use De Morgan's Laws. Let me state as exercises a few more results, which are related to the limit of the sequence of the sets; for example, if I consider limit superior of a sequence of sets and if I take complementation of that, then it is equal to limit inferior of the compliments of the sets.

In a similar way, if I consider limit inferior of the sequence of sets and I take its compliment, then it will be equal to the limit superior of the compliments of the sets. The proofs will be almost clearly feel make use of the lima 1, which gave the representation of the limit superior and limit inferior, and make use of the De Morgan's Laws. An extension of this exercise would be, if I consider F minus limit infimum of the sequence E n, then it is equal to limit superior of F minus E n.

In a similar way, if I consider F minus limit superior of E n, that is equal… To prove, say, the first part of this, we can consider F minus limit inferior of E n and let us write down the representation of the limit inferior in terms of unions and intersections, n is equal to m to infinity, m is equal to 1 to infinity. And at this, we just use a set theoretic notation, where a minus B is equal to a intersection B compliment; so, this becomes F intersection union intersection E n, n is equal to m to infinity, n is equal to 1 to infinity compliment; that is equal to F intersection intersection union E n compliment, if we apply De Morgan's Laws.

Now, at this stage, we can apply the distributive properties of the unions and intersections. And this will give us intersection m is equal to 1 to infinity union n is equal to m to infinity F intersection E n compliment, which is equal to intersection union F minus E n, where n is equal to m to infinity, m is equal to 1 to infinity; and this is nothing but the limit superior of the sequence F minus E n.

In a similar way, we can prove the statement 2 of this; there are certain relations, which include the characteristic functions of the limit superior and limit inferior, and I will state it as a statement in the following exercise.

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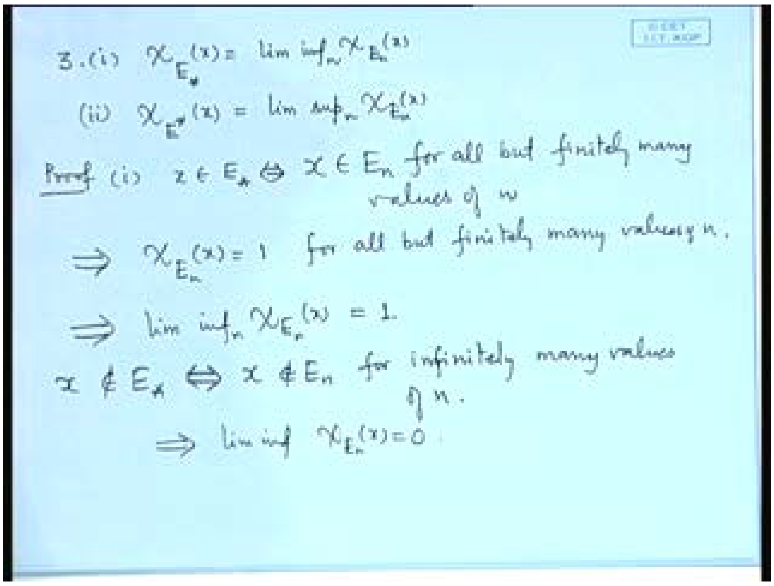

Indicator function of the limit inferior of a sequence of sets is the limit infimum of the indicator functions of the sequence of the sets. In a similar way, if we consider indicator function of the limit superior, then it is equal to limit superior of the sequence of the sets.

We may look at the proof of say one of them. Consider say belonging to E star, then this implies that, x belongs to E n for all but finitely many values of n. This implies that, the indicator function of the set E n is 1 for all but finitely many values of n, which is equivalent to the statement that limit infimum of the indicator function chi E n x is equal to 1, and these statements are both if and only if.

Further if I consider x not belonging to E lower star, then this will imply that x does not belong to E n for infinitely many values; and this implies that limit infimum of chi E n x is 0. In a similar way, if we use a definition of the limit superior of a sequence of functions, then we will be able to prove the second statement; useful relation which is used for in set theory is that of symmetric differences.

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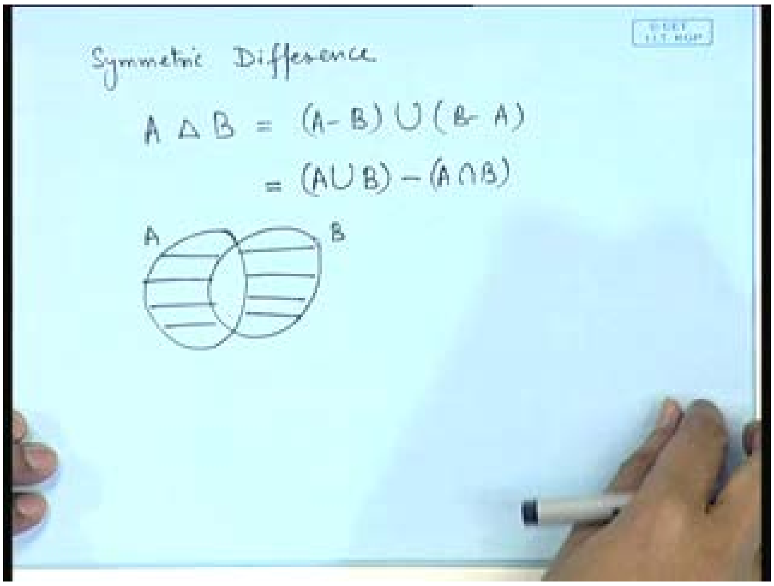

The concept of symmetric difference is defined by, A delta B is equal to A minus B union B minus A. So, you can see here that, it is A minus B and B minus A, both are combined together and that is why it is called a symmetric difference. And an equivalent interpretation for this is A union B minus A intersection B. From the Venn diagrams, you can see that, if I have two sets A and B, then the symmetric difference is the shaded portion; certain relationships which are true for set theoretic operations are given in the form of exercises below.

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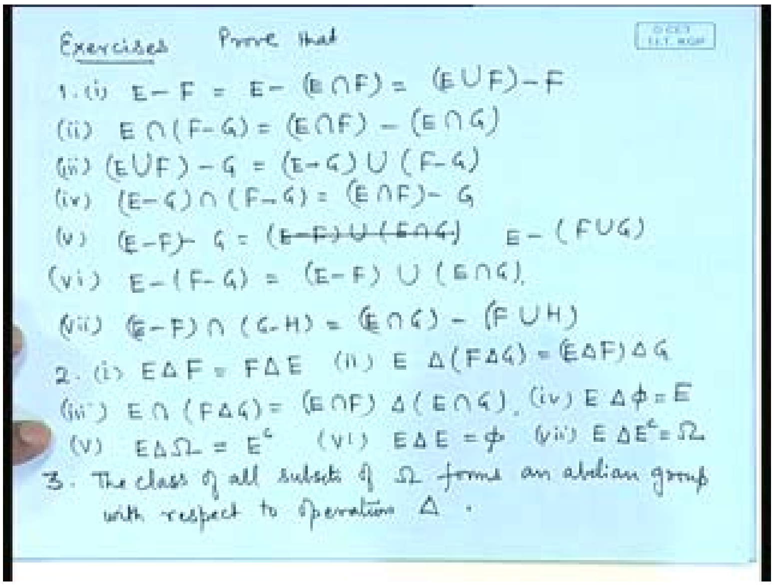

Prove that, E minus F is equal to E minus E intersection F, it is equal to E union F minus F. E intersection F minus G is equal to E intersection F minus E intersection G. E union F minus G is equal to E minus G union F minus G. E minus G intersection F minus G is equal to E intersection F minus G. E minus F minus G is equal to E minus F union E intersection G. E minus F minus G is equal to E minus F union E intersection G. E minus F intersection G minus H is equal to E intersection G minus F union H.

Certain relationships which are true for the symmetric differences are given in the next exercise; for example, symmetric difference of E with F is same as symmetric difference of F with E. Symmetric difference satisfies associative property, that is, E delta F delta G is same as E delta F delta G. E intersection F delta G, it is equal to E intersection F delta E intersection G; that means, intersection and symmetric differences are distributive.

If we consider the symmetric difference of a set with the empty set, then we get the same set; that means, as a group theoretic operation empty set acts as an identity operator. If we consider with the full space, then I get the complementation. If we consider with itself, then we get empty set; that means, with respect to group theoretic operation, E is its own inverse. And if we consider E delta E compliment, then I get the full space. So, one can ask that, the class of all subsets of omega forms an billion group with respect to symmetric difference operation.

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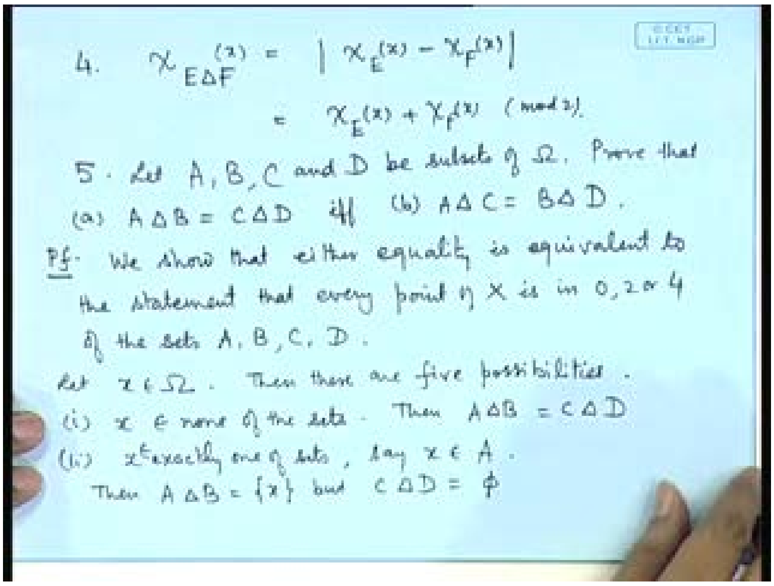

Further if we consider the indicator function of the symmetric difference, then it is equal to the absolute difference between the indicator functions of the two sets. An alternative way of telling it is that, it is equal to chi E plus chi F, where the sum is taken modular 2. And additional exercise in this direction can be that, if we consider A, B, C and D, then a delta B is equal to C delta D, if and only if a delta C is equal t to B delta D.

In order to prove this statement 5, we proceed as follows. We show that either equality is equivalent to the statement, that every point of x is in 0, 2 or 4 of the sets A, B, C, D. So, let us consider x to be any point in the space omega, then there are five possibilities. Let us consider these possibilities; one possibility is that, x belongs to none of the sets; if x does not belong to any of the sets, then with respect to this point, A delta B and C delta D, they must be equal, because x is none of them.

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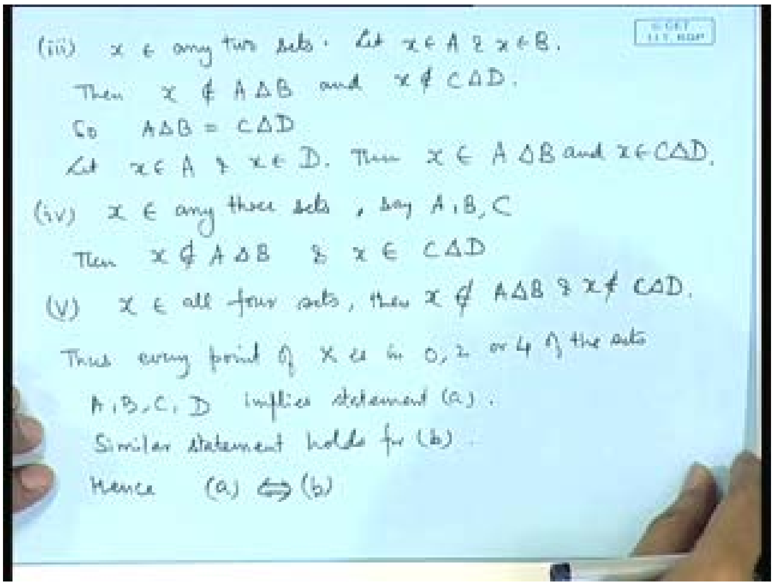

If we consider x belongs to exactly one of sets, say, x belongs to A; in that case, A delta B will consist of the point x and C delta D will not include this point. If we consider x belonging to any two sets, let us take x belongs to A and x belongs to B, then clearly x does not belong to A delta B, because the points which are common to both are excluded from the symmetric difference, and x does not belong to C delta D. So, A delta B will be equal to C delta D. Let x belongs to A and x belongs to D, then x will belong to A delta B and x will belong to C delta D. Let us consider the possibilities that, x belong to any three sets, say, A, B and C, then x will not belong to A delta B and x will belong to C delta D.

If we consider x belongs to all four sets, then clearly x does not belong to A delta B and x does not belong to C delta D. Thus, we have proved that, every point of x is in 0, 2 or 4 of the sets A, B, C, D, implies statement 1; similar statement holds for B, hence A and B must be equivalent.

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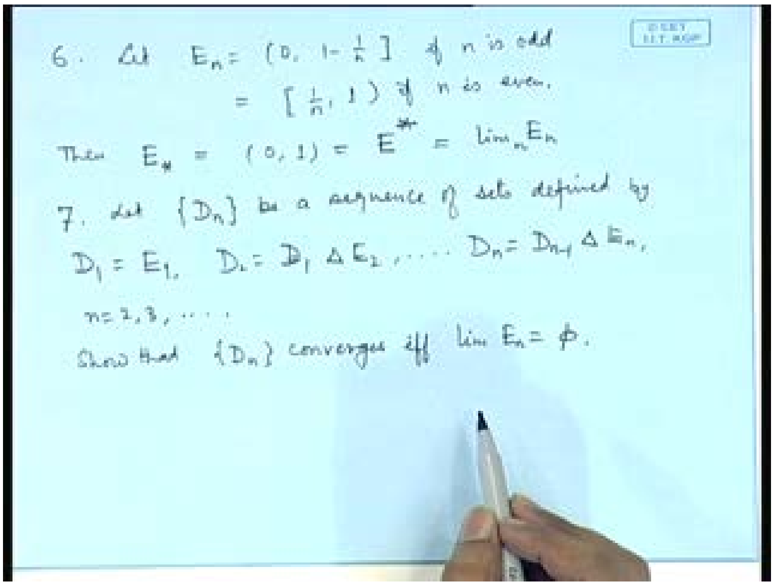

To end this class, we look at 1 or 2 more examples of the limit infimum and limit supremum, and let E n B equal to 0 to 1 minus 1 by n semi-open interval, if n is odd, and it is equal to 1 by n to 1, if n is even. Then, if we look at limit inferior, it is the open interval 0 to 1. If we consider any real number between 0 to 1, then we can always find capital n, such that, the point a will belong to both 0 to 1 minus 1 by n and 1 by n to 1, for all n greater than or equal to capital N.

Therefore, the point will certainly belong to the limit inferior set. Since it will belong to limit inferior, it will also belong to the limit superior, and limit superior cannot be bigger than the interval 0, 1. Therefore, it is also equal to limit superior, and therefore, the limit of the sequence exists and it is the open interval 0 to 1.

Let me complete today's lecture by giving the final exercise. Let us consider a sequence of defined by that, D 1 is equal to E 1 D 2 is equal to D 1 delta E 2, in general, D n is equal to D n minus 1 delta E n, for n is equal to 2, 3 and so on, so that the limit of the sequence D n exists, if and only if limit of the sequence E n is equal to phi.

In next class, we will introduce the concepts of certain algebraic structures such as rings, sigma rings, fields and sigma-fields, which are eventually going to be used for the definition of probability function.

Thank you.