Optimization
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Lecture - 26
Numerical optimization- Region elimination techniques (Contd.)

Today, we are dealing unconstraint optimization problem for single decision variable. We are discussing the searching techniques for solving non-linear programming problem, where there is no constraint and only single decision variables involved in the process.

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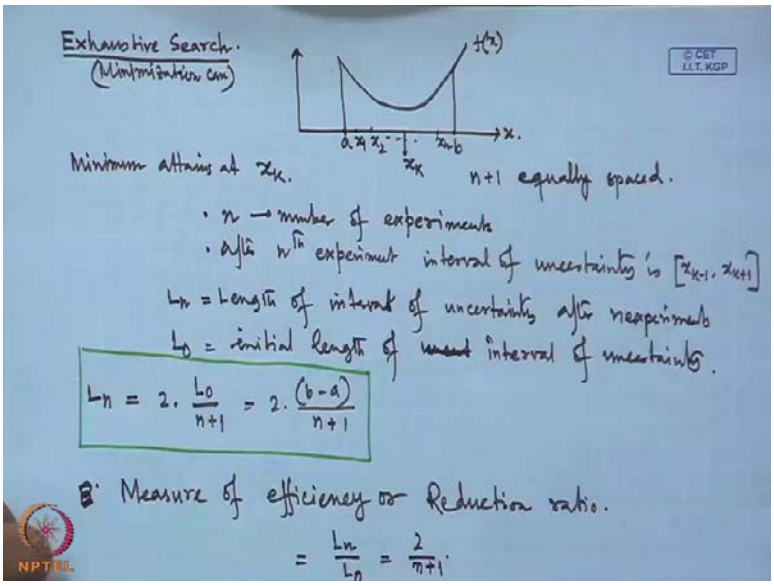

One of those techniques is the exhaustive search technique. I have just introduced this technique in my last lecture. Now, I will just brief it first then I will just explain the whole methodology with some example. The assumption for applying exhaustive search technique is that, the function must be unimodal in the given domain of definition. If this is the axis for the decision variable and this is the objective function within the domain of definition where function is defined a to b, function is unimodal.

Let me discuss the method before the minimization problem, same logic can be just reversed for the maximization problem as well. This is my f x, function is unimodal; that means, there is only one minimum in between. The exhaustive search technique tells us that we have to subdivide the whole interval into n plus 1 equally spaced subinterval and this n value is predetermined. So, we can say x 1 is one of that point, x 2 in this way it is just moving up to x n. We will just find out the functional value at each and every point at x 1 x 2 up to x n and we will see how the functional values are distributed, and from there we will select the minimum value within that.

If minimum attains at x k, then the interval of uncertainty, let me define what is the interval of uncertainty. First, we are assuming the function is unimodal between a and b; that means, the function is having minimum value in between a and b. Thus we can say that the initial interval of uncertainty, where the possible minimum may lie, is from a to b. We are subdividing the whole interval into n plus 1 equally spaced subintervals and we are assuming that minimum is occurring at x k. Then if this is so, there are few things to be observed. First thing is that n is the number of experiments, because we are finding the functional value in n points in the domain.

Then another thing is that, after the n-th experiment, the interval of uncertainty is from x k minus 1 to x k plus 1. Now, let us decide what is the length of the interval of uncertainty? If we consider that L n is the length of interval of uncertainty after n experiments and L 0 is the initial length of interval of uncertainty. Then we can say that L n will be is equal to 2 into L 0 divided by n plus 1 because the value of L 0 is equal to b minus a, because my given interval a to b and there n plus 1 equally spaced points. That is why we can say that the L n will be is equal to 2 into b minus a, n plus 1.

This is the result, we need it for finding out the value of n because if it is not being said that how many points we will divide the whole subinterval, in that case we will just use this formula and then we will use it depending on how much accuracy we want regarding the optimal solution. If I just show you one of the examples, it will be clear to you. In this connection one thing I would just like to mention that, measure of efficiency for this searching technique, measure of efficiency or reduction ratio can be measured equal to L n by L 0 certainly it would be is equal to 2 by n plus 1. Let me solve one of the examples using exhaustive search technique and using all these facts, it will be much clearer to you. Let me take one example that is again we are considering the minimization problem.

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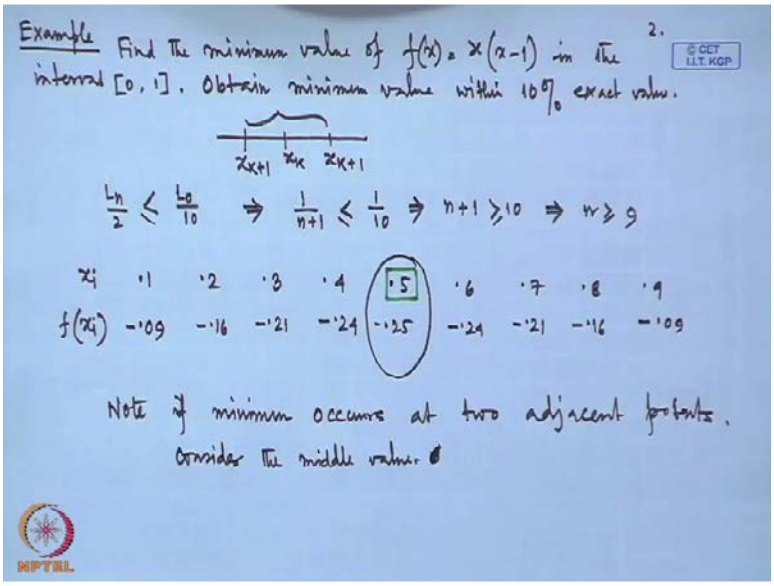

The same logic can be extended for the maximization problem as well. Find the minimum value of f x is equal to x into x minus 1 in the interval 0 to 1 and obtain minimum value within 10 percent of exact value. That means, if I consider the case that there is the optimal value is x k then this side it should be 10 percent of the exact value and this side also the 10 percent of the exact value. That is why the whole interval would be altogether 20 percent. These are all equally spaced that is why the we can say that the length of the interval of uncertainty after the n-th experiment L n by 2 must be lesser than L naught by 10.

In the given example there is no mention that how many equally spaced points we will consider, but there is one information that the minimum value must be within 10 percent of exact value. From there we just develop this result that this side 10 and this side 10 L n by 2 that is a half of the whole interval of uncertainty is lesser than 10 percent of the original interval, that is the initial interval.

And here the initial interval is 0 to 1 that is why L 0 is 1. From here we are getting one relation that, 1 by n plus 1 must be lesser than 1 by 10. In otherwise we can say n plus 1 must be greater than 10. In otherwise we can say that n must be greater than equal to 9. Let me consider the equality case because within the 10 percent of exact, that is why 10 percent is acceptable to us that is why number of points, which I am just deciding that how many equally spaced points we will consider within the interval 0 1 that must be greater than equal to 9. If we want that the minimum value is within the 10 percent of the exact value that is the case.

Now, let me subdivide the whole interval 0 1 into 9 plus 1 points rather 10 points. That is why we consider x i is starting from 0.1, 0.2, 0.3, 0.4, 0.5, 6, 7, 8 and 9. If I consider the subinterval size as 0.1, then it will be sufficient because n is greater than equal to 9 and one thing to be remembered that we have considered only the minimum value of n. If we say increase the number of experiments for example, instead of taking 10 numbers of such subintervals, if we consider 20 number of subintervals, we will get better result. If I just increase number of n much higher than the corresponding optimal value, we are getting through the exhaustive searching technique that will be much better, than the previous result.

If this is so, let me consider the functional value at each point, 0.1 f x is equal to x into x minus 1. At 0.1 the value is 0.09. 0.2 The value is minus 1.16, minus 0.21, minus 0.24. These are the objective functional value at individual points, minus 0.25, minus 0.24, minus 0.21, this function is very much symmetric in nature because looking at the values we can conclude in that way. We can say that minimum is occurring at 0.5, because that is the only minimum function value for us and we can declare the corresponding functional value as minus 0.25. That is the minimum value for us. This is the minimum and we are sure that if we consider the interval of uncertainty in such a way 0.4 to 0.6, then certainly we will get the 10 percent of the exact value that is the case.

One thing one note I just want to make here, if we see that at two points the minimum occurs at two adjacent points, then we need to consider the minimum as the middle value of these two. In this way, through the exhaustive search technique, we can find out the minimum value for any given non-linear function within the given domain. Another very important technique is there, that is dichotomous search technique, and this technique is little bit complicated than the previous techniques which I discussed before. But dichotomous search technique is better technique to achieve optimal value for any non-linear optimization problem.

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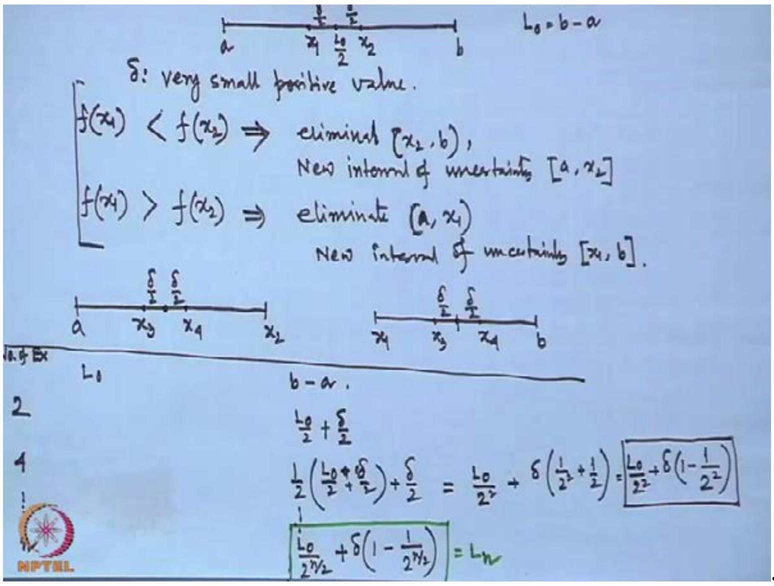

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Let me discuss the dichotomous search technique. Here also the given information is similar, the function is unimodal in the given range, function is having only one minimum or maximum in the given domain of definition and we need to find out the optimal value. Here also the same thing, the number of experiments is given beforehand. That is why how many numbers of points, we need to evaluate the functional values. These are all given, otherwise an information may given to us that how much accuracy we want regarding the minimum or maximum value of the given objective function.

Let me tell you first, how the whole strategy is being done. Then I will write down the corresponding algorithm. Say a to b, that is on the real line certainly because we are considering the single variable case. The function is defined. One thing to be mentioned here, whatever we are discussing, these searching techniques are applicable not only for the continuous functions and these are also applicable for the discontinuous functions as well. That is why, we need not to depend on the differential coefficients. We need not to differentiate the function etcetera. That is why only the calculation of functional value will be sufficient for finding out the optimal value.

The dichotomous search technique tells us that if L naught is the initial length of interval of uncertainty, so, L 0 will be b minus a. Then we will just take the middle value of that. Then certainly it will be L 0 by 2, and we will consider two points in the left and right side of this middle point. These are my first two approximations for the optimal solutions. One of that approximation is x 1 another is x 2. Both are equally distant from the middle value and this distant distance is being considered with a specified parameter delta that is given to us, this side is delta by 2 and this side is delta by 2.

If this is so, only information we need to have what is the value of a and what is the value of b and what the value of delta. Then we will get two approximations of the optimal solution one is x 1 another one is x 2. In this connection we need to say that delta is very small positive value and the value as much as small it will be, we will get accurate result. But the number of iterations will be more, that is in other case.

But if delta is a very small positive value we will we will get two approximations x 1 and a y 1. After that we will find out the value of function f at x 1, value of function y at x 2, and if we see that f x 1 is lesser than f x 2, certainly we will eliminate x 2 b because we are considering again the minimization problem, f x 2 is giving us the better value higher value, that is why minimum cannot occur from x 2, b certainly the new interval of uncertainty will be a to x 2.

If this is the other case, f x 1 is greater than f x 2, certainly we have to eliminate a to x 1 because minimum cannot lie within this region, I have already explained regarding the region elimination technique, the strategy we are adopting for each and every searching technique, then we have to eliminate a to x 1 and new interval of uncertainty will be x 1 to b. This is the first step for us. The next step, what we will do for the first case when the new or the interval of our uncertainty is a to x 1 again we will consider a to x 1. We will again take the middle value and again we will consider another two points x 3 and x 4 delta by 2 distance apart from the middle value. Again we will find out the functional value at x 3 and x 4.

According to this strategy again we will just apply the same and accordingly by the using the assumption of unimodality, we will discard a portion of the interval and we will just consider the rest portion as the next level of interval of uncertainty in the next. For the other case as well, after first step interval of uncertainty is from x 1 to b again we will take the middle point and we will consider two points x 3 and x 4 delta by 2 distance apart from the middle value. And we will find out the functional values at x 3 and x 4 accordingly by using the unimodality assumption, we will just consider the part of the interval and we will just eliminate another part of the interval.

In this way we will proceed further and further and in the each step of the iteration, we will discard a portion of the interval and we will consider another portion of the interval as the new interval of uncertainty. In this way if I proceed further and further in the iteration and we will reach to a smaller interval of uncertainty and after certain stage, according to the value of n or according to our desired accuracy we will stop our iteration. If I do number of experiments more, then I will get better result.

Whatever I have said, there are a few things to be noted here that, after the first stage what will be the length of interval of uncertainty. Initial length of interval of uncertainty is L 0 is equal to b minus a. After the first step as I have explained to you after considerations of the two approximations of the optimal value as x 1 and x 2. We are eliminating a portion of the initial interval a b and we are accepting the other part and we are proceeding further and further with the same logic that is why this is the process of iteration.

If we consider that f x 1 is lesser than f x 2, then the interval of uncertainty is from a to x 2. So, we will next consider the middle value of these two points and we will consider another two points x 3 and x 4, these are the new approximations for the optimal value, delta by 2 distant apart from the middle value. Then we will say that x 3 and x 4 may be the possible approximations for the optimal value, but the main thing is that again here we need to find out the functional values at x 3 and x 4 again.

If we see that f x 3 is lesser than f x 4, then again the same thing we will discard the region x 4 to x 2 and we will consider a to x 4. If this is the other case, then we will just proceed similarly. Similarly for the other case, when we are eliminating the region a to x 1 that is why my new interval of uncertainty is from x 1 to b. Here also we will take the middle value and in the middle value, we will take two points x 3 and x 4 delta by 2 distant distance apart from the middle value.

And this x 3 and 4 could be the corresponding possible approximations for the optimal value and we will find out the value for function at x 3 and x 4. And we will see which one is greater and which one is lesser using the unimodality assumption, and we will discard we will eliminate a portion of the interval. We will consider another portion of the interval as the next level or interval of uncertainty for the next step.

In this we will proceed further and further and if raise up n experiments we will see that we will have a smaller interval size. And that interval size will be declared as the final interval of uncertainty, if the n is given to us, how many number of experiments have to be performed is given to us or otherwise, it is given that, how much level of accuracy I want regarding the optimal value. Depending on that, we will just select the number of experiments.

There are few things to be noted for this dichotomous searching technique is that, at each stage we are generating two approximations and we are evaluating functional value at two approximations. That is why initial level of uncertainty was b minus a, after doing two experiments. If I consider another column as number of experiments, after doing two experiments my level of uncertainty will be L by 2 plus delta by 2. Why it is so? Because look at these figure, initially it was from a to b. Either I am accepting from a to x 2 or we are accepting from x 1 to b; that means, we are considering the half of the whole interval plus delta by 2 portion, either this side or that side. That is why the whole interval of uncertainty will be L naught by 2 plus delta by 2.

In the next stage, when number of experiments are 4 that time x 3 and x 4, this is the size that is L naught by 2 plus delta by 2. Again we are considering the same and the whole part is L naught by 2 plus delta by 2 half of that plus delta by 2. If I just proceed further and further after n experiments we will get, if I just modify this one, then we are getting L naught by 2 square plus delta 1 by 2 square plus 1 by 2. This can be written as L naught by 2 plus delta 1 minus 1 by 2 to the power 2. That is why I will use this result here.

In general we can declare that after n experiments, the length of the interval of uncertainty would be L 2 by 2 to the power n by 2 plus this was 2 square 2 to the power n by 2 plus delta into 1 minus 1 by 2 to the power n by 2. This figure is my L n that is the length of the final uncertainty, in the dichotomous search technique. Here also we can just measure the efficiency of this searching technique, as the reduction ratio L n by L 0 that will be this value divided by L 0 that will be the corresponding efficiency of the dichotomous search technique. The same technique, let me just explain with a example in the next.

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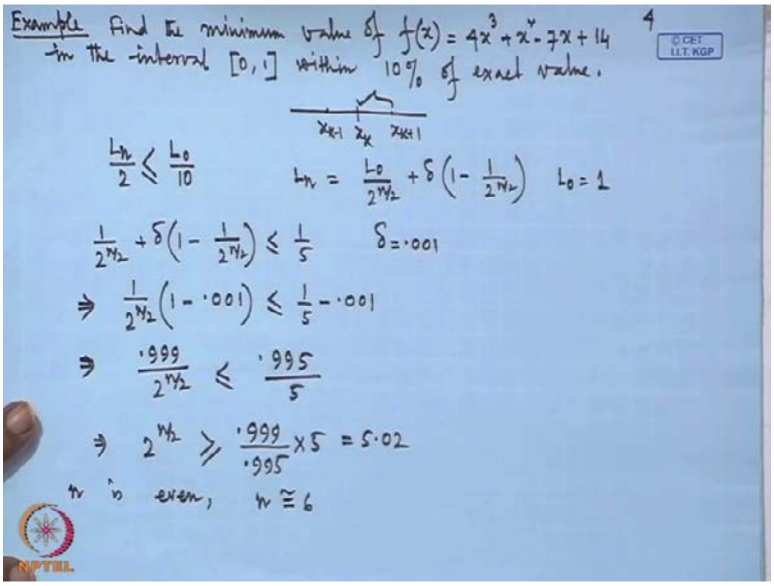

Consider one minimization problem again here. Find the minimum value of f x, where f x is non-linear function of single decision variable 4 x cube plus x square minus 7 x plus 14, in the interval 0 to 1 within 10 percent of exact value. If this is the case, one thing is that it has assumed the function is unimodal in the given range from 0 to 1. And if we are reaching to the final interval of uncertainty, if this is the optimal value as x k, then x k minus 1 and x k plus 1, then this ratio must be L naught by 10 because 10 percent of the exact value is allowed for us.

In general we can say if we consider the middle value as the final of the final interval of uncertainty as the optimal value. Then we can say that L n, that is the nth length of the final interval of uncertainty divided by 2 because that is the middle value. This must be lesser that equal to 10 percent of the original that is the initial interval of uncertainty. Let me repeat once again for you, I wanted to say that if the final interval of uncertainty is having length L n.

Then if we consider the middle value as the optimal value of the final interval of uncertainty. Then we can say that this L n by 2 that is the one side of that must be lesser than equal to L naught by 10 because we are allowing the value of in such n in such a way, that this side we are we can allow the 10 percent error. And this side we can allow 10 percent deviation from the original value.

That why this is the case, now we have achieved that for dichotomous searching technique L n is equal to L naught by 2 to the power n by 2 plus delta 1 minus 2 to the power n by 2. If we just substitute this value here, then we can say and here L naught is equal to 1 because 0 to 1 is my interval. Then in this case, we can say that 1 by 2 to the power n by 2 plus delta 1 minus 2 to the power n by 2 must be less than is equal to 1 by 5 by considering to this side. This is same as, now we need to supply the value for delta here then only we can proceed further.

Let us consider delta is equal to 0.001 as I have said that as small as possible we will consider the value of delta and since we are doing it manually let me consider a reasonable the smaller value that is delta is equal to 0.001. If we consider these value here, then we are getting the value of n in this way. Just let me adjust it that is 1 minus delta; that means 1 minus 0.001 and less than is equal to 1 by 5 and delta this side that would be 0.001.

That means, we are getting here 0.999 by 2 to the power n by 2 lesser that is equal to 0.995 divided by 5. Which implies 2 to the power n by 2 must be greater than is equal to 0.999 divided by 0.995 into 5 and this value is coming as 5.02. One thing is clear from the dichotomous technique is that, n is always even because at each iteration we are considering two experiments at a time, that is why n is even here. If we consider minimum value of n is equal to 6 that is approximately sufficient for us. This is the way we will proceed in the next. We will consider 6 experiments for solving this problem then we can conclude that we will get 10 percent accuracy of the exact value. Let us solve the next problem.

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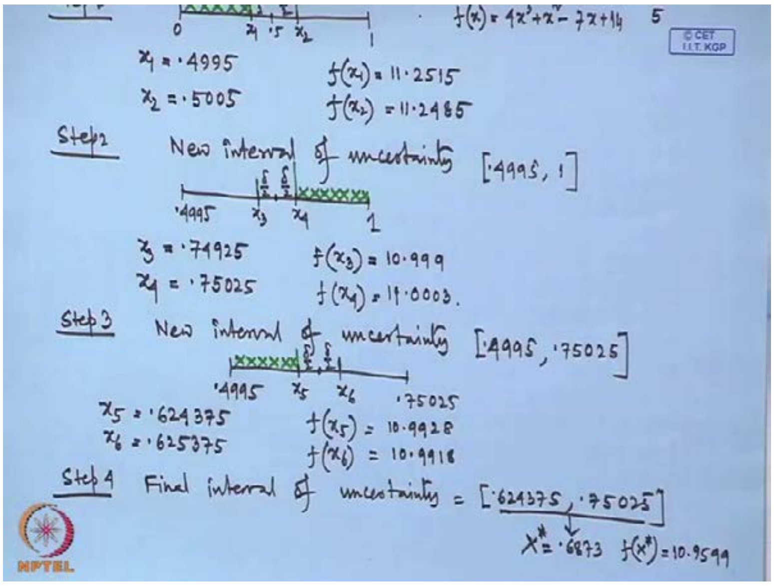

Now our first step would be step 1. There are two points 0 and 1, the middle value is 0.5 and we will consider x 1 here and x 2 here in such a way that, this is delta by 2 that is 0.001 by 2 and this is delta by 2. That is why we are getting x 1 is equal to 0.4995 and x 2 is equal to 0.5005. Let us see the corresponding functional value at these two points, f x 1 will be is equal to 11.2515 and x 2 will be is equal to 11.2485.

If this is the case, then let me draw the figure here. We are getting higher value for x 1 and we are getting lower value for x 2 and we are considering that we need to find out the minimum value. So, if this is the case, then certainly the minimum cannot occur in this region. That is why we discarding this region and the new interval of uncertainty we are considering in step 2 as from x 1 to 1. What is my x 1? That is 0.4995 to 1. Again the same process we will take the interval 0.4995 to 1, we will take the middle value of this and we will consider x 3 and x 4 in such a way this distances is 0.0005 because delta value is given as 0.001 and this is delta by 2.

That is why we are getting the x 3 value as 0.74925 and x 4 is equal to 0.75025. Let us see what is the corresponding functional value? We are getting the functional values f x 3 is equal to 10.999. If we just substitute the value in the given f x, our given f x is f x is equal to 4 x cube plus x square minus 7 x plus 14, just substitute these values there and we will get corresponding functional value as this one and f x 4 is equal to 11.0003.

What we see here again? We see that f x 3 is having lesser value and f x 4 is having higher value. That is why we are sure that minimum must lie within this point to this point minimum cannot be in this side. That is why we are eliminating this interval again. We are reducing the size of the interval of uncertainty. That is why in next step in step three our new interval of uncertainty will be 0.4995 to x 4 that is 0.75025.

Again, we will take the middle value of these two points, let me consider the new interval of uncertainty 4995 to 0.75025 take the middle value. Again consider this side x 5 this side x 6 and this is again delta by 2, again delta by 2, in this way we are getting x 5 is equal to 0.624375, x 6 is equal to 0.625375 and corresponding x 5, at x 5 the functional value is 10 0.9928. And here just see that difference in functional values is now reducing these are much more closer now, but still we are getting at x 6 the lower functional value. That means, the here the functional value is high here the functional value is low.

But I am not considering further because we have already decided that for getting 10 percent accuracy, I have to do minimum six numbers of experiments, that is why I can stop here the iteration process and we will declare the new interval of uncertainty is from x 5 in the step 4, which part we are discarding here? This is higher value, this is lower value, that why I am discarding this interval, and we are declaring the last final interval of uncertainty is equal to from x 5. That means, from 624375 to 0.75025 and consider the middle value of these two, if I consider the midpoint of this interval midpoint will come as 0.6873 and we can declare this as my optimal value, that is the minimum value.

If we consider this as a minimum value that is sure that we are accepting the fact that this value is within 10 percent of exact value and one thing to be mentioned again. The corresponding functional value is 10.9599. If I do more number of experiments, certainly we will get better result for this case. Let me go to another very well known technique, this is another searching technique for solving non-linear unconstraint optimization problem.

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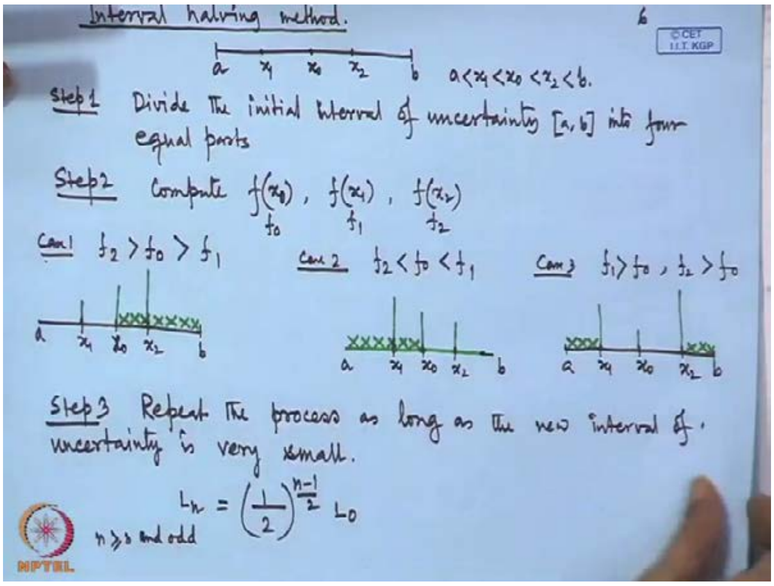

Name of this technique is either interval halving technique or the bisection method. Bisection method is popular in numerical analyses, here also the same method follows. Let me tell you the process for solving an unconstraint optimization problem using interval halving technique. In the dichotomous technique we are considering the initial interval of uncertainty a to b. In the interval halving technique, what we do, we will consider equally spaced three points, rather four equal parts here and we will see how functional values are distributed in these? That is why we will consider x 1, x 0 is the middle value x 2 is the right hand side value.

So we can say, is lesser than x 1, less than x naught, lesser than x 2, lesser than b. If this is the case, in the interval halving technique, in step one, what we do? Let me write down the algorithm; divide the initial interval of uncertainty a, b, into four equal parts and we will say that x 1, x naught and x 2 these are the possible approximations for the optimal value. Step two, compute f x 1, let me consider f x 0 first, we can say this as f 0 as well, f x 1 we can consider shortly as f 1 and f x 2, f 2 and we are adopting some strategy here. We are considering three cases just see if we are having f 2 greater than f naught greater than f 1.

That means if you are having this is the case, this is my a, this is my b, x naught, this is x 1 this is x 2; that means, f 2 is having higher value than f naught and f 1 is having lesser value. Then certainly the minimum cannot lie within this region that is why we will just discard this region in this case. In the next level we will consider the interval of uncertainty as from a to x 0. This is the first case 1. Go for the next case.

Case 2, if this is the case, f 2 is lesser than f naught lesser than f 1. In the figure, this is my a, this is my b, x naught, x 1, x 2.That means we are having higher value in x 1, then the next value, then this value. Then certainly we have to discard another portion which portion shall we discard for this, then we can say the minimum cannot lie in this interval. We will discard this interval and we will consider in the next level as interval of uncertainty as from x naught to b.

Let us consider the other case, case 3, f 1 is greater than f naught and f 2 is greater than f naught; that means, if we consider this thing again the same figure x naught middle x 1, x 2 we are considering f 1 is higher than x naught, not only that f 2 is again higher than x naught. Then certainly the figure tells us that minimum cannot lay either this region with assumption of unimodality.

Let me just point it out once again, we are considering all functions are unimodal function within the given domain, then only it is possible. Then the minimum cannot lie within this region. What exactly we are getting that, for each case we are discarding half of the interval that is the beauty of this interval halving technique and each iteration we will discard, half of the interval and we will go to the next. Again we will repeat the same process, we will consider the portion and again we will take three points in between and we will consider. So, let me write down the next step as repeat the process as long as the new interval of uncertainty is very small.

One thing to be mentioned here, in each time we are discarding half of the interval. That is why after nth experiments, n must be, certainly one thing to be noted here and at each level we are considering three points, in the first in the next level one point is given another two points were considering. That is why always n is greater than equal to 3 and it is odd numbered. L n will be is equal to 1 by 2 to the power n minus 1 by 2 L naught. This value we need it, when we will just the efficiency of the searching techniques and there we use this value and we will go further. This is the whole algorithm for solving unconstraint optimization problem using interval halving technique.

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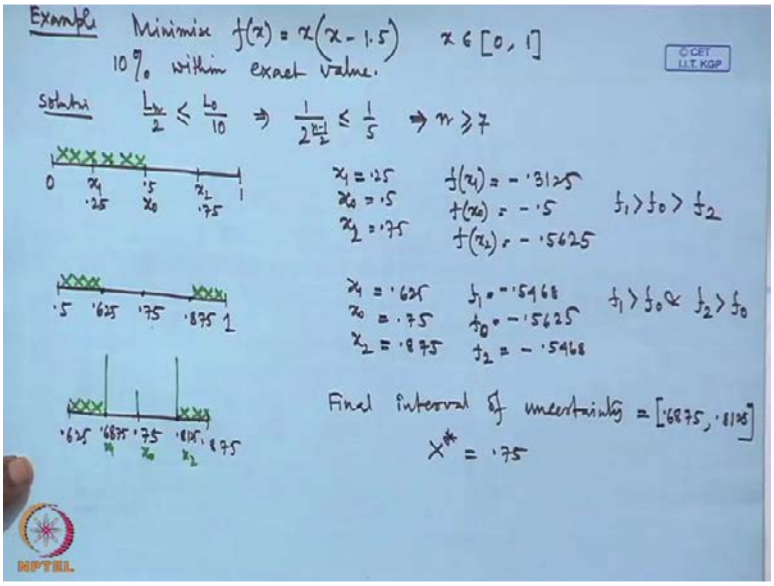

Let me give you one small example on this, and let us see what is there in the next example. We have to minimise f x is equal to x into x minus 1.5 within the interval 0 to 1. We have to have 10 percent within exact value. Here also the same thing, the same logic we will apply like previous one for solution that, half of the length of the final interval of uncertainty must be lesser than is equal to L naught by 10. That means, for this method L n by 2 is equal to, if we see that that must be 2 to the power n minus 1 by 2 must be lesser than is equal to 1 by 5. From here we can conclude that n must be minimum value of n must be 7. Because as I have said that n is greater than equal to 3 and n is odd. That is why we can calculate the value of n from here and we will see the value of n is greater than equals to 7.

Let us start our process from here step 1, we are having two points 0 and 1. We are taking the middle value 0.5, this is as x 0. Let me consider x 1 and x 2 equally spaced that is why this is 0.25 and this is 0.75. Let me consider the functional values at each point x 1, 0.25, x naught 0.5, x 2, 0.75. Here we will get f x 1 is equal to minus 0.3125, f x naught is equal to minus 0.5 and f x 2 is equal to minus 0.5625 if this is so, what we see here that f 1 is greater than f naught is greater than f 2. That is why we will discard the region a to x naught. This region we will discard because we are having higher value for this, f 1 is the higher value then f naught then this. That is why minimum must occur within this region, we will just discard this region.

We will delete this interval and in the next level we will consider the interval of uncertainty from 0.5 to 1 again. Here also we will consider the same, we will take the middle value as 0.75 and x 1, x naught, x 2 and this value as 0.625, this value as 0.875 let me write it down 625.75 and 0.875 and corresponding functional values f 1 is equal to minus 0.5468, f 0 is equal to minus 0.5625 and f 2 is equal to minus 0.5468.

Here, we will see that f 1 is greater than f naught and f 2 is greater than f naught. That is why we will discard two portions from here, this part as I explained before and this part. My new level of uncertainty again will be from 0.625 to 0.875. The middle value is 0.75 and this value is 0.6875 and this value is 0.8125. Again we will calculate the functional values and we will see that the functional value f 1, if I consider this is as x 1 this is as x naught and this is as x 2 we will see that f 1 is greater than f naught. Just calculate the functional value, you will get the same and f 2 is again greater than f naught. Again the same logic, we will apply and we will discard this region.

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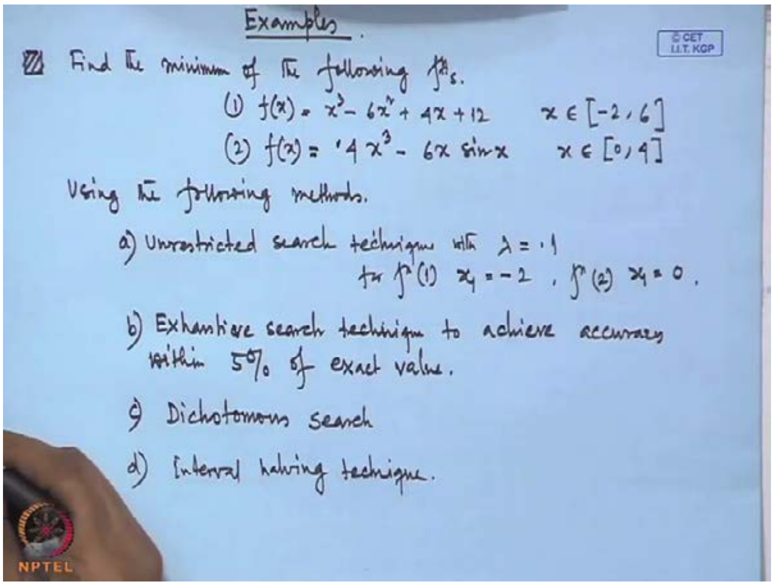

We will declare, final interval of uncertainty is equal to, up to this we have evaluated 7 functional values. That is why we can conclude here for getting 10 percent accuracy, we will declare this as the final interval of uncertainty 0.6875 to 0.8125 and middle value of that is a optimal value, that is 0.75 for us. In this way with an interval having technique we can solve the unconstraint optimization problem. Now for exercises, let me give you few problems to be solved. Whatever we have covered on that part some examples I am just writing for your practise. Find the minimum value of the following functions.

One function is given as f x is equal to x cube 6 x square plus 4 x plus 12 x is within the interval from minus 2 to 6 that is the initial interval of uncertainty. Another function could be considered as f x is equal to 0.4 x cube minus 6 x sin x and consider x within the interval 0 to 4. If this is the case, then using the following methods, solve the problem.

One is that unrestricted search technique with fixed step size lambda is equal to 0.1 and for function one, take the initial point as minus 2, and for function two take the initial point x 1 as 0. You can extend this method for accelerated step size as well, with less number of iteration; we will get the same result. Use the exhaustive search technique to achieve accuracy within 5 percent of exact value. You can use the dichotomous search technique. Again get the 5 percent accuracy and do the same problem with the interval halving techniques and have a feel which method is giving you better result rather which method is efficient and in the next lecture I will discuss one the efficiency of the searching technique.

Thank you for today.