Optimization

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Module - 01

Lecture - 25

Numerical optimization - Region elimination techniques

Today, we are considering the numerical methods for solving optimization specific non-linear programming problem. In the last class, I discussed about the classical optimization technique for solving non-linear optimization problem, but there was one restriction for that, the function must be at least twice differentiable within the domain of definition. But, where the function is discontinuous, is not defined within the whole visible space. How to develop the optimization technique for that? The numerical technique or numerical method is most popular in that case, and numerical methods are applicable for constraint as well as unconstraint optimization problems, specifically non-linear programming problems.

A numerical method is mainly based on the searching procedure and search for the local optimal in this feasible region. This is mostly iterating process. We start from a point on the feasible space and we move to the optimal point after certain iterations. Let us consider a hypothetical two variable non-linear programming problem and just I will discuss, how the iterative processes run on this problem to get the optimal solution for the given problem.

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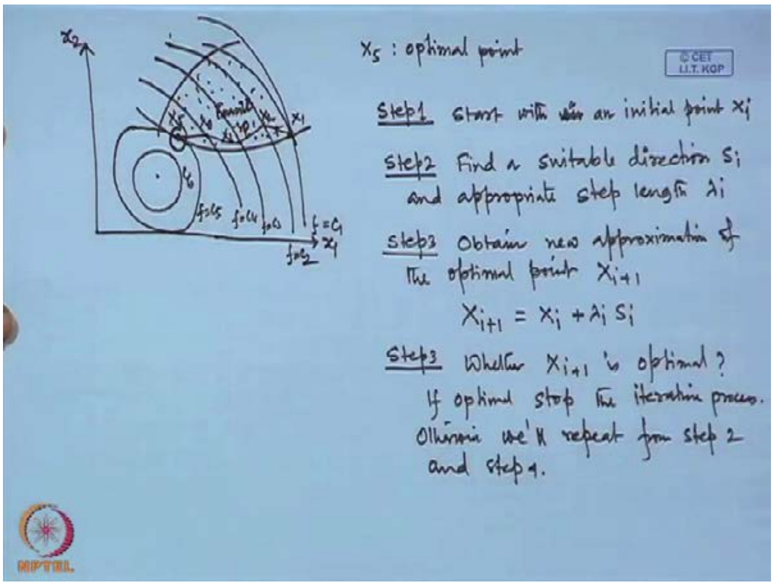

For this let us consider, one problem that is of two variables x 1 and x 2. Let us consider in general the constraint optimization problem, that is why, we are considering there are three constraints; constraint one, constraint two and constraint three. This is the defined feasible space for us. These are all non-linear in nature all the constraints and this is forming the feasible space here.

If you have a function which has to be optimized, let us consider the minimization of the objective function, where objective function is a circle. That is why, for different value of the function that takes different place in the feasible space. If this is one of the places where functional value f is equal to c 1. This is a part of the circle. Since, this is a minimization problem; we will try to minimize the objective function f here. That is why will just find out the space where the objective function is minimum.

This is another position where functional value is a c 2, in that way we can minimize function further and further. How long that dependence on, when the functional value is out of the feasible space. That time, you will just conclude our journey that is why, I can say, this is the space where the functional value is the minimum. This is the corresponding optimal point here. Let us consider this the functional value is c 1, this is c 2, this is c 3, this is c 4, this is functional value is c 5 and there is further, if function moves here, then functional value is c 6 and this is the center for this.

It is clear that if I just move to functional value at c 6 that will be out of the feasible space that is why; we need to stop our iterative process. If we start from f is equal to c 1, it will move to c 2, it will move to c 3; that means, we are getting better and better functional value. Once we are reaching to this point, functional value is coming is equal to c 5 and this is the corresponding optimal point. If this is the process for us, how really we have started to search the feasible space. This is the searching technique I am now discussing with you.

This is first starting point from me, this is the point which is within the feasible space since say this is the point is x 1. We will move to x 2, this is our next optimal point. Our searching mechanism will be such that x 1 will move to x 2, and then it will move to x 3. In the next iteration, it will move to x 4, in the third iteration, it will move to x 5. That is the corresponding optimal point, where x 5 is the optimal point for the given problem.

This is the searching technique for us. How really we did it. We just see the whole mechanism here. Let me write down the whole iterative process as an algorithm. We start our iteration with an initial point x 1. Now x 1 can move in any direction here, x 1 can move in the right, it can go in the north and it can go in the south this way. We need find that direction which gives us the better value for the objective.

Not only that, we will be restricted within the feasible space, that is why, one of the possibility is x 2. If I move from x 1 to x 2; that means, we have decided that I will move through this direction, this is one part and the second part is that the thing will be such that if I move from x 1 to x 2, there is a step length. I will jump from here to here. That is why step two for this iterative process would be finding a suitable direction.

Let us consider the first initial point is x I, instead of x 1 because I will start from 1, 2, 3, 4 like that. I am making the iterative process more general. Find a suitable direction s i and appropriate step length lambda i. Once it is being selected s i and lambda i is being selected we will move to step three. We will go for the new approximation of the optimal point that is x i plus 1.

I am starting from i-th step, I am moving to x i plus 1. What was the formation of x i plus 1, it should be x i plus lambda i and s i, where lambda i and s i are we have chosen the suitable direction and the corresponding step length in the i-th step. Once we are reaching to x i plus 1, our next task could be to find out, whether x i plus 1 is optimal or not. How to check the optimality? There are certain mechanisms for different problem. I will discuss further on that. That is way will check whether x i plus 1 is optimal or not.

If optimal, stop the iterative process; that means, we will stop our movement. If it is not then we will repeat from step 2 to 4, again and again unless, we are reaching to optimal point. Whatever I have written here, the whole algorithm let me explain with this problem again. I am starting from x 1, i equal to 1 with suitable direction s 1 and the appropriate step length lambda 1. I am moving to x 2; that means, x 2 is equal to x 1 plus lambda 1 into s 1 and we have checked s 2 is the optimal, is one of the feasible point, but that is not the optimal because there is a possibility for further improvement.

That is why we are repeating the process, we are moving to step 2 again. We are selecting the x 2 the suitable direction and the appropriate step length lambda 2 and we are getting the next the point that is x three as x 2 plus lambda 2 into x s 2. Again we are moving to x 4, and then we are moving to x 5, x 5 is certainly optimal because if I just proceed for further, I will be out of the feasible space. That is the whole searching mechanism and is based on this appropriate procedure.

Now, there is certain thing to see discussed here, the efficiency of this searching technique is totally dependent on the proper selection for the step length lambda i and the direction s i. Otherwise if the selection for lambda i and s i is such that, it is guiding me to move from the feasible region out of the feasible region. Otherwise this could be another thing I can mention here; if the step length is very small then number of iterations will be more. That is why finding a proper step length will reduce my computational procedure that is why this could be general iterative process. This is the general thing I am discussing. This is numerical technique for the constraint as well as for unconstraint, non liner programming problem also, it is more applicable.

Let us start our discussion with the single variable unconstraint non liner programming problem that is a most simplex case for this. That is why, I am moving to the next part that is the numerical method for solving unconstraint non liner programming where single variable decision variable is involved there. That is why, I am moving to single variable optimization technique.

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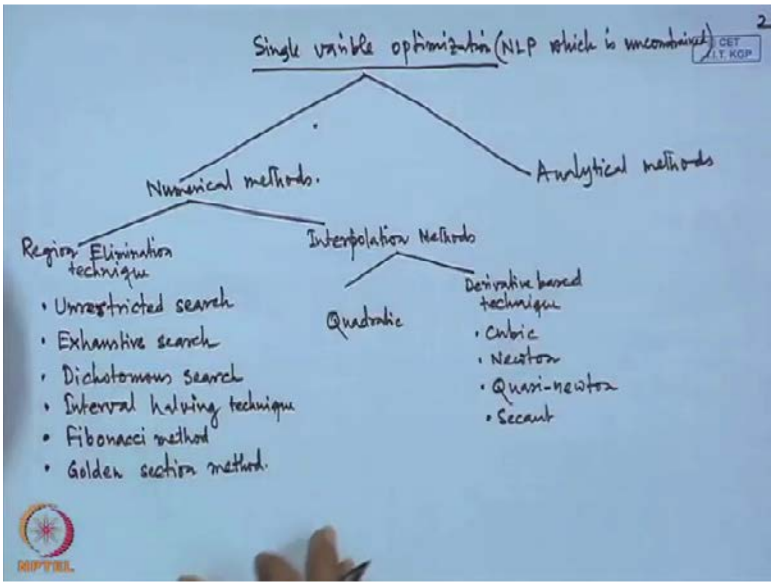

There is certain procedure for solving a single variable non liner programming problem where there are no constraints. That is why; we are dealing with single variable optimization NLP that is non liner programming problem which is unconstraint. There are several techniques for solving it. If I just refer my previous lecture, there is certain analytical procedure that is mostly dependent on the class, that is most specifically the classical optimization techniques, and there is certain other method also that is the numerical methods which I am discussing today.

Numerical methods can also be divided into two categories, one category is the region elimination technique and another category is the interpolation method. I will discuss each and every method with specific example and corresponding algorithms. Region elimination technique, several techniques are available; most simplex technique is unrestricted search procedure and another procedure is the exhaustive search, dichotomous search, interval halving technique or bi section, Fibonacci method and golden section method.

All these methods are mainly can be categorized as a region elimination technique, what is the meaning of that, I will tell you later on. Now coming to the interpolation method, there are two types of interpolation method; one method is dependent on the derivative of the function that is why, we can name it as derivative based techniques, and the other method which is not dependent on the derivative of the function.

In this category of non-derivative based methods, one of the most popular methods is the quadratic interpolation technique. But for the derivative based techniques one of that the cubic interpolation technique, the Newton method, the Quasi Newton and then is the secant method. We are using these few methods in our numerical problems also. These are the methods I will discuss one by one. I will start by the lecture with the region elimination technique.

If I just move here, we could see there are several methods, but all the methods are having its own advantages and disadvantages. Few methods are very easy for computation process, but we will see that the method is not efficient. So, whenever we are discussing the searching technique for getting the optimal solution, one part is very important for us; we need to check the efficiency of the corresponding searching technique.

In this methods dichotomous interval having Fibonacci golden section method, this methods have been categorized as a sequentially search procedure, because sequentially search procedure utilizes the information generated in the previous iteration in placing the subsequent iterations in the iterative process. We will discuss the region elimination technique and most specifically the unrestricted search method in the next. Region elimination technique is mostly based on one of the most important assumption that is function must be unimodal in the specified domain of definition. That is more important for the region elimination technique.

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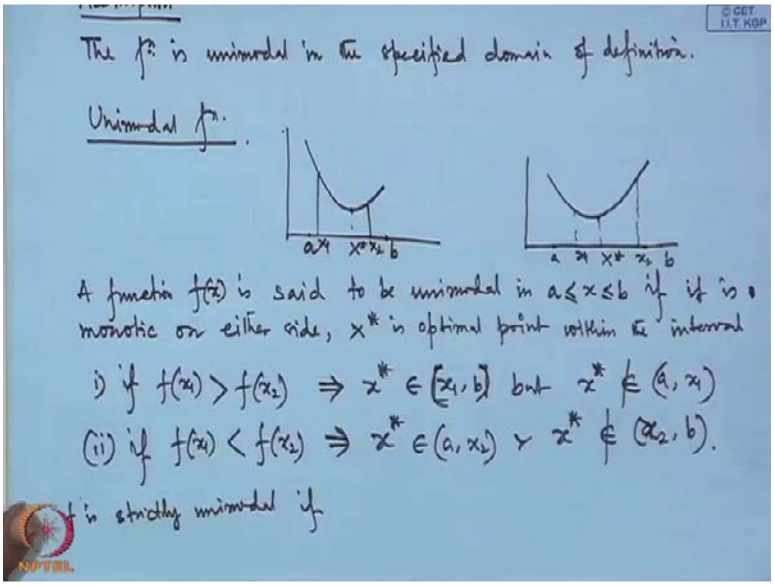

Let me write down the assumption for the region elimination technique is that the function means the objective function, I mean to say because there is no constraint for the corresponding optimization problem and the function, there is a domain of definition and we are assuming that function is unimodal in the specified domain of the definition. What is the definition of the unimodal? That would be our next task to learn within the specified domain of definition.

Domain of definition means, the domain of the corresponding decision variable and we are dealing with a single variable. So, describing the unimodal definition is much easier for us. If the function is multi modal, multi modal means the function is having several peaks or several valleys several maximas and minimas within the specified domain. We will just subdivide the parts of the domain in such a way that, the function is unimodal in the respective parts and we will search for the optimal solution within that specified region, where the function is unimodal.

Unimodality assumption will lead to the conclusion that always we will find out the local optimal in the given range. Let us now define the unimodal function. If I just draw the graph, it will be clear to you. For example, this is the domain of definition, it starting from a, this is our decision variable axis and this is the corresponding objective function f. The function is defined from a to b.

From the figure it is very clear that function is having only one minimum at this point. So,we can say that the function is unimodal in this range and formally let us introduce the definition of the unimodality; a function f x is said to be unimodal on the interval a to b, if it is monotonic in either side, monotonically decreasing in this in the left part and monotonically increasing in the right part, on either side, and x star is the optimal point within the interval.

If it is satisfying following conditions; if f x 1 is greater than f x 2. What is f x 1 and what is f x 2? Let us consider this is f x 1 this is the corresponding x 1 in the decision variable space and this is the corresponding x 2 in the decision variable space that is domain of definition. Then we could see f x 1 is greater than f x 2. Then this implies that optimal solution must exist within x 1 to b, but optimal solution cannot be within a to x 1, alright?

The second condition; if, Let me take another case function is this way, this is the domain of definition from a to b, this is my x 1 and this is my x 2. The optimal point this is x star for us. If f x 1 is lesser than f x 2, this implies that, x star must be within a to x 2 and the other condition is that, the optimal point cannot be within x 2 to b. This is the condition for the unimodality, because the function is having only single optimal point in this region and this is monotic in either side.

If I just move from here to here, this is decreasing, this is increasing. If x 1 is greater than x 2, we have written that, the optimal solution will be in the path from x 1 to b. It cannot be from a to x 1. If the f x 1 in this case is lesser than f x 2, then certainly the optimal point will be from a to x 2. It can’t be from x 2 to b because there is only one minimum point within the definition.

Another definition is that, f is strictly unimodal. If the function is unimodal, this is one of the condition and the second condition is that, there does not exist any sub interval within the range a to b, where the function is having constraint value. Then only we can say the function is strictly unimodal. This is the definition all together regarding the unimodal function.

Now, I am moving to the next. What is the region elimination strategy, we are adopting for region elimination technique? That is the part I am discussing now. We will move to the next, that is the region elimination strategy. We will adopt in general for any region elimination technique, which I have been listed in my previous slide.

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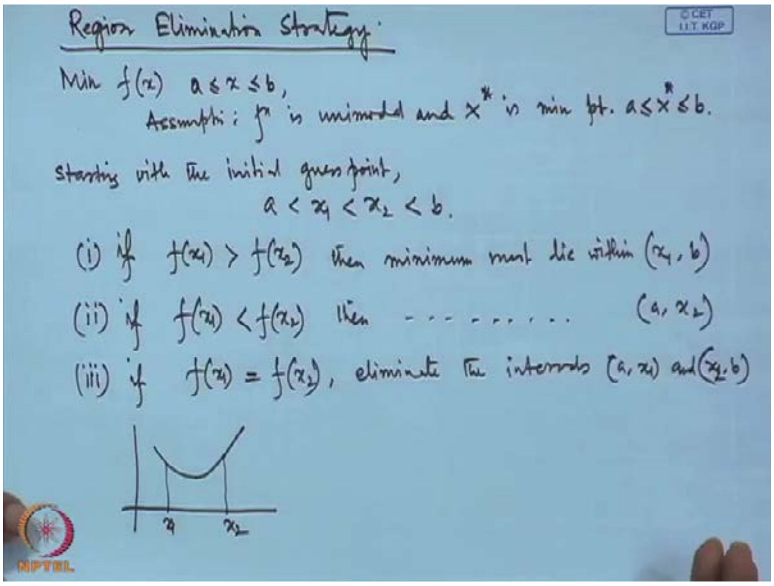

The region elimination strategy is mostly based on the assumption that, the function is unimodal in the given range. Let us start our discussion for the minimization problem and we are assuming that function is unimodal and not only that function is having one minimum point within the domain of definition. That is why we are assuming the function f, we need to minimize f of x where x is in between a to b and we are assuming that function is unimodal, this is the assumption for us and x star is the corresponding optimal point. That is the minimum point where x star is in between a to b.

As I have discussed about the iterative process, it is mainly based on the technique is that we will start from an initial point, from where we will start our journey and we will move to the improved approximation for the corresponding optimal point with some iterative process. That iterative process is based on this region elimination technique. The thing is that the starting with the initial guess point and we will move to a sequence of improvement approximations.

So, let us consider that x 1 and x 2 there are two points within the domain of definition, where x 1 is lesser than x 2. Once we are moving to the next optimal, what is the strategy we are following is that, if f x 1 is greater than f x 2 then minimum must lie, that is x star must lie within x 1 to b. Just from the definition of the unimodality it comes. In the next iteration we will restrict just ourselves within the region from x 1 to b.

We will start our process, by taking the initial interval that is called, the initial interval of uncertainty, from a to b. There are two points we will take within the region, if we see f x 1 is greater than f x 2, then we are sure that the minimum must lie within x 1 to b. So, we will just discard the interval a to x 1 in the next iteration. But if it is other case, if f x 1 is lesser than f x 2,then minimum must lie within a to x 2. That is why in the next iteration we will just discard the interval x 2 to b. We will eliminate the region.

If we see the other case, if f x 1 is equal to f x 2, then certainly the minimum will lie in between x 1 and x 2 because the function is unimodal. I can just draw graph in this case. For example, this is a graph for me, this is x 1, this is corresponding x 2, and from the figure it is very clear that f x 1 is equal to f x 2 and minimum is in between this and this. In this case we will just eliminate the intervals certainly from a to x 1 and x 2 to b. We will restrict ourselves within x 1 to x 2.

Again we will take two points here. We will just see whether at one point functional value is greater than or less than the other one or not, we will just eliminate the corresponding interval. In that way we will proceed further and further. The beauty of this region elimination strategy is that, every time we are discarding a part of the interval, so that out searching region will be lesser than the previous iteration. That is the technique for us.

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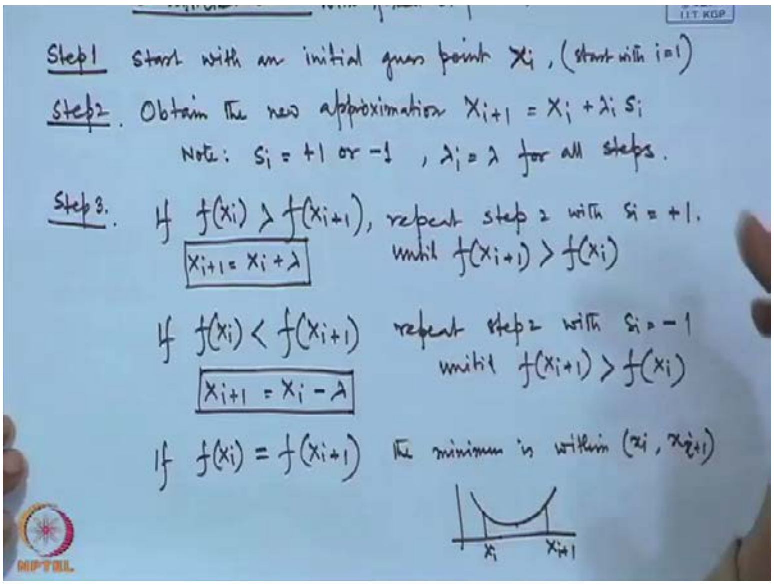

Let us start our discussion with the unrestricted search technique that is, one of the basic region elimination techniques. As you know, for any searching technique, as I have discussed, there are two things; one is the direction and another is the step length. We are discussing the most general unrestricted search technique, simpler unrestricted search technique with fixed step length.

Since we are considering single variable optimization, we are restricted to move in the real line, and if I just start our journey from one of the point in the real line, either I can move to the right or I can move to the left. If I move to the right, then the searching direction would be plus sign and if I just move to left, the searching direction will be minus. Either I will just jump from one point to the other point with the step size plus lambda or we will with the step size minus lambda. This part I will discuss now.

Let me just write down the corresponding algorithm of the unrestricted search technique. Here also the same thing. I will discuss the whole unrestricted search technique and the other things with some example in the next. Search with an initial guess point or the trail point X i, I can start from 1 2 3 etc. That is why start with i is equals to 1. Next step, obtain the new approximation, how with the same technique that X i plus 1 is equal to X i plus lambda i into s i.

Let us note in this case that s i is equal to plus 1 or minus 1 and lambda i is equal to lambda for all steps because we are considering the fixed step size lambda here. We are moving to the next, that is why x 1 is going to x 2. Either it is a x 1 plus lambda or x 1 minus lambda and we will check. Step 3 the optimality, how to check the optimality? Checking optimality means, we will just check whether there is any possibility for the improved objective functional value or not; that means, we will just check whether we are getting the better objective functional value within by considering the corresponding domain of definition that is the corresponding range of the decision variable.

If I just move from one initial guess point to the other point, I will just check whether f x i is greater than f x i plus 1 or not. If f x i is greater than f x i plus 1; that means, we are getting better functional value. Since, we are considering the minimization problem; we will prefer the lesser functional value. That is why, x i plus 1 is most preferable for me. We will just move to the next step with s i equal to plus 1. If I just write down the algorithm, I can write it down in this way. Repeat step two with s i is equal to plus 1, until f x i plus 1 is greater than f x i; that means, we are not getting improved approximation in that case.

If it is the other case, that is f x i is lesser than f x i plus 1, then we will repeat the process. Repeat step 2 with s i is equal to minus 1. How long the same condition? Until f x i plus 1 is greater than f x i. In the previous case, I can say x i plus 1 is equal to x i plus lambda. In the second case, we will consider x i plus 1 is equal to x i minus lambda that is the consideration in the next case; that means, we are moving to the left, because if I move to the right we will get the unimproved approximations.

If I just see, we are mostly dependent on the region elimination strategy, I have just discussed in the previous case. One thing I must again mention here; we are considering that function only in the unrestricted search, where the function is unimodal in the given domain of definition. If the function is multimodal in the similar case, we will just consider, we will subdivide the interval into the different parts, so that the function is unimodal in the respective parts, and we will proceed in that way.

There can be another possibility that f x i is equal to f x i plus 1; that means, that function is the third condition for the region elimination strategy, that function is having minimum within x i and x i plus 1. That is why we will start our procedure from step two again. That means, minimum is within x i to x i plus 1 and we will start our journey in that way. That means, if this is the case, this is my x i, this is my x i plus 1 and I know the minimum is here.

So, we will discard this region and that region, again we will take a guess point here. We will move either in the right or in the left according to the condition, whether we are getting better functional value or not. With this two condition separately, we will move to the next and will repeat the process, until and otherwise we are not getting any better functional value. Let me explain this procedure with the simple example in the next.

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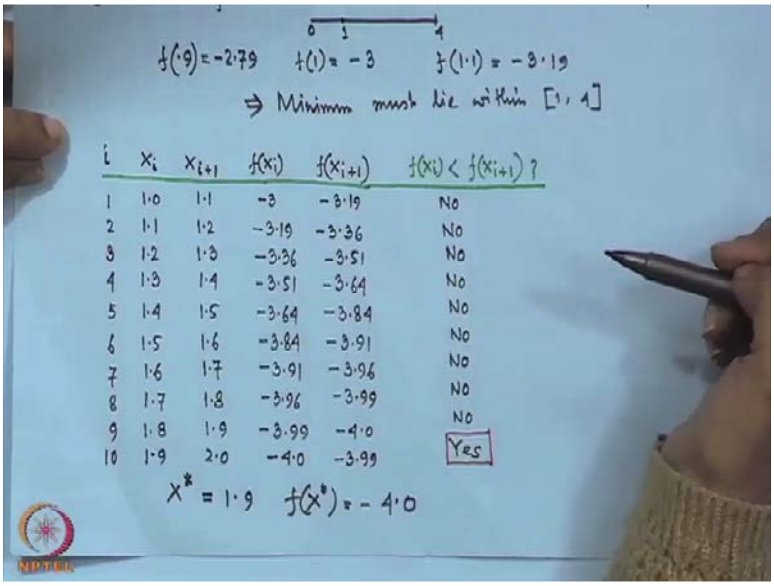

We are considering a simple example here. Consider for problem of minimization again. We are trying to minimize f x, that is x into x minus 4 that is a non-linear function of a single variable x. Here it is given that x is in between 0 to 4 that is the domain of definition. It is also given that the function is unimodal. If I draw the graph, I will see the function is unimodal within the range 0 to 4. Otherwise it is given to us and it is also given that starts with the initial guess point or the trail point as one. Start with x 1 is equal to 1 and the step size lambda is equal to 0.1. If this is the case we will just do the entire thing.

We will start from x 1. This is my 0 to 4 and this is my 1. If I start from x 1, let us see what is the functional value at 1. The functional value at 1 is certainly minus 3. I know that I have to jump to the next point with the step size 0.1, but I do not know in which direction I will move. I can move in the left direction, I can move in the right direction as well. If I move in the right direction, I will see whether we are getting better functional value or if I move in the left direction, we are getting better functional value. So, let me take two points in either side of one.

That is we are considering 0.9, x equal to 0.9 and we could see that the functional value is coming minus 2.79. If I just move to 1.1, we could see that we are getting the value. Then what is the conclusion for us? The conclusion is that, if I move to the left, then we are getting the functional value, which is higher than the functional value at one and the function is unimodal. The minimum cannot lie in between 0 to 0.9, I am sure for that.

So, I will move to the right hand side because in the right hand side, we could see that, if I just move to a 1.1, the functional value at 1.1 is minus 3.19. That means, this is better value than the one value. That is why if I move further from 1.1, there is a possibility that we will get the better value for the function f x. Better in the sense that function is minimization, better means we will get lesser functional value, that is more acceptable in this case. We can conclude that minimum must lie within 1 to 4 that is the conclusion for us. We are discarding the region, we are eliminating the region from 0 to 1 in the next case and we are moving further.

For that thing, the whole information can be tabulated in this way. Just see, the first step we are considering x 1 as 1, x 2 as 1.1. This is the corresponding functional value minus 3 and minus 3.19. Certainly, f x 2 is lesser then f x 1, that is why the condition f x i is lesser than f x i plus 1 is not fulfilled and we have written no. So, we will move to the next, where to move? From 1.1, with the step size 0.1, we will move to the right that is 1.2.

Just see the next. The second step, from 1.1 we are moving to 1.2. For this the functional value is changing from minus 3.19 to minus 3.36. That means, we are getting better functional value. So, there is a possibility to proceed further. Let us see what is happening in the next? We are again moving from 1.1 to 1.3 and just see we are getting better functional value. That means the lesser functional value in this case.

Again there is a possibility to improve further and further. Let us see what is happening? We are moving from 1.3 to 1.4, better functional value. 1.4 to 1.5, again the better functional value. We started our journey from minus 3 and we see, we are getting better and better functional value and reaching to the functional value as minus 3.84. Again we are moving to the next, from 1.5 to 1.6, better functional value. For the next 1.6 to 1.7 better functional value, 8 better functional value.

In this way if I just move to 1.8 to 1.9, we could see the functional value is coming again better minus 4.0. But if I just move from 1.9 to 2.0, we see that the functional value is now the higher value. That is why, we will stop our journey at 1.9 and we will declare that the optimal point, the minimum point must be 1.9 and the corresponding functional value is minus 4.0. This is the case. We will stop our iterative process.

So, we could say that x star is equal to 1.9 and the corresponding functional value is f x star is equal to minus 4.0 and this is my conclusion. Whatever logic we have adopted for the minimization problem, if I just consider the maximization problem we will just adopt the reverse logic. We will conclude that we are getting the better functional value, if the functional value is increasing instead of decreasing. That is the case for us.

Coming to the next, we will move to the next that is the unrestricted search technique with accelerated step size. Now, just look at this table how many iterations we have to perform, here we have to perform ten iterations. One advantage is that this process is easy to implement. Either we can to do it manually or we can write a simple program to implement the iterative process, so that within ten steps will reach to the optimal solution. That is why, as an advantage of this method, I can mention that it is easy to implement.

But there is a disadvantage. Disadvantage is that selection of the initial guess point is more important for us. If the initial guess point is given to us that is fine, if it is not given; if I select randomly within the region 0 to 4, there is a chance that I need to do more number of iterations. So, selection of initial guess point is much more important thing for us. Not only that, selection of step length, though we are considering the fixed step length, still if the step length is not proper one, there is a chance that I will be out of the optimal point within fewer steps.

Now, that is why it may happen that we will start our journey from the initial guess point and we will move with the corresponding step length, but it may happen we will far from optimal point if the step length is very high. But if the step length is very small, number of iterations will be more. If you have some mechanisms, so that if I see the functional value is increasing and increasing further and further with the step length 0.1, if there is a mechanism that I will change the step length in between from 0.1, I will move to 0.2. Then in that case instead of 1.1 to 1.2, I will just directly move from 1.1 to 1.3. In that case if I see the functional value is again better one, I will just cross it with the same step length or with accelerated step length, so that we could reduce the number of iterations. Then the searching procedure will be much more efficient. That is why, we are having the next stage of searching technique that is called the unrestricted search with the accelerated step length.

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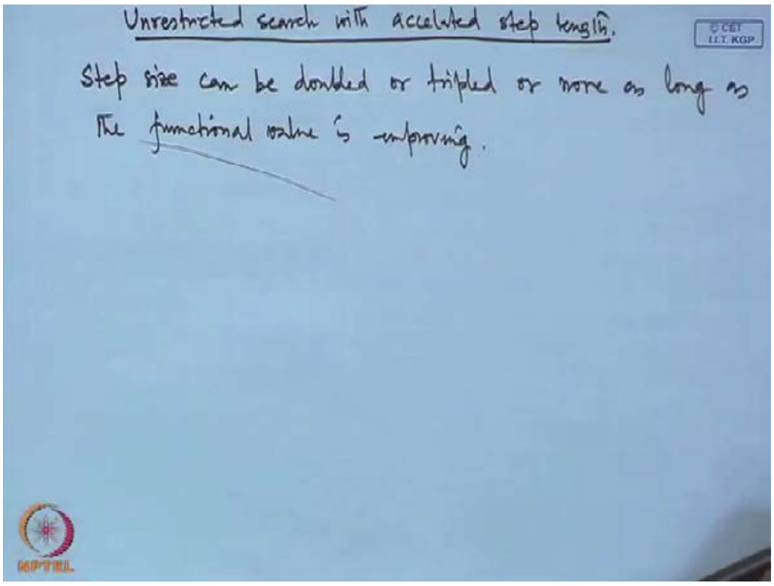

What is the thing is that, in the process we can increase the step size, it can be doubled, it can be tripled or more as long as the functional value is improving. Let us consider the same example here and let us see, whether with accelerated step size, we are getting better optimal or not.

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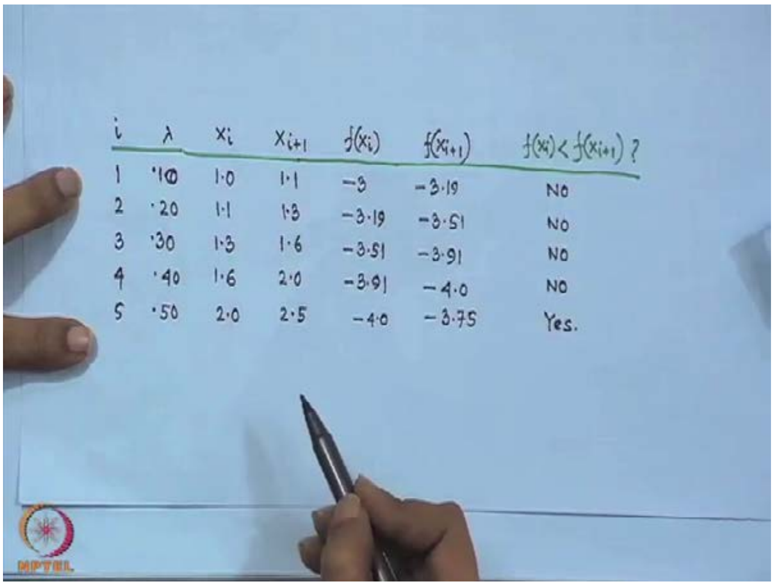

And this is the case for this we are starting with the step length lambda is equal to 0.1. Now, we are moving from 1.0 to 1.1. This is my staring point 1, it was given there. Now functional value is better. So, there is a chance for improvement. Now, I am increasing the step length, instead of 0.1 I am making it 0.2. Then 1.1 to 1.3, we are getting better functional value.

Let me do a try; we are changing the step length again from 0.2 to 0.3, so 1.3 will move to 1.6. Again we are getting better functional value. Again we are changing the step length 0.4. We could see that we are getting better functional value. I can move to the next with 0.5 and we could see that from 2.0, it is just jumping to 2.5 and we are getting not better functional value. We are getting the higher value of the objective function. So, we have to stop here. If I stop our searching procedure, we will declare in the present situation is that 2.0 is the corresponding functional value for it.

But one thing, it is clear that, we have increased the step length from 0.4 to 0.5 in between the optimal value we are missing. So, we can revise our procedure in that way that, we will reduce the step length. Instead of taking 0.4 we can take it 0.04. In that way will move further and further, we can fine tune our optimal solution. In that way we can develop the unrestricted search technique very nicely. This is all together the unrestricted search. Now we are moving to the next that is the little bit complicated than the unrestricted search technique that is the exhaustive search technique.

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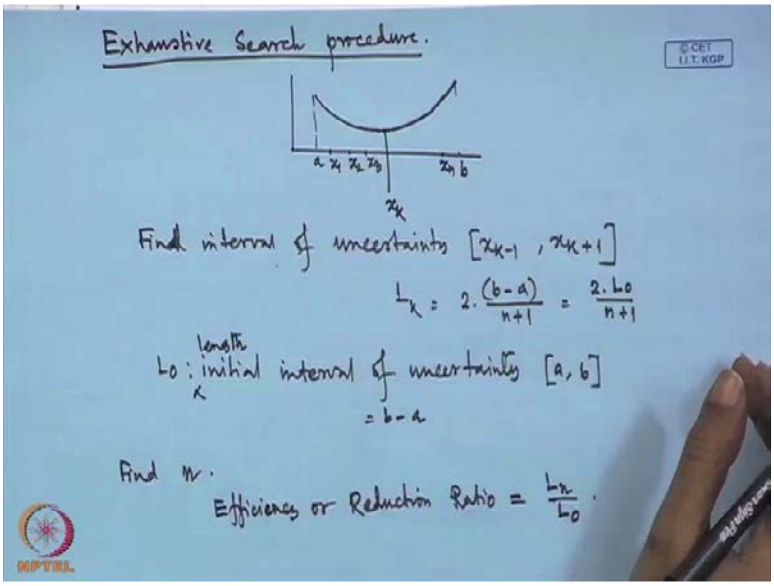

The exhaustive search procedure is mainly based on this assumption again, the function must be unimodal within the domain of definition. This is my functional value. Certainly, the minimum is lying here. This is the domain of definition a to b, and the process is such that, we will just divide the whole interval of uncertainty, initial interval of uncertainty into n part, n plus 1 equally spaced sub intervals. That is why, I will go to x 1, then x 2, then x 3, in this way we will go to x n.

How many parts are there? 1, 2, 3, 4 up to n plus 1. So, initial interval of uncertainty is divided into n plus 1 equally spaced sub intervals. Then we will see the functional value at each and every point within the given range. If minimum occurs, here is the minimum for us, minimum occurs at x k, then we will say that final interval of uncertainty will be x k minus 1, to x k plus 1.

Final interval of uncertainty would be from x k minus 1 to x k plus 1. Then what is the corresponding length for it? Certainly the corresponding length would be 2 into b minus a divided by n plus 1. Otherwise, if I say that L 0 is the length of the initial interval of uncertainty that is from a to b, as equal to b minus a, then L k can be written as 2 into L 0 divided by n plus 1.

Next thing is that we have to find to out the functional value at each and every point, where the functional value is minimum, we will stop our thing. There are few things to be discussed in this regard that is in the exhaustive search, we have seen that we have to sub divide the interval into n plus 1 equally spaced sub intervals. So, our next task is to find n that is more important. The value of n totally depends on the fact that what accuracy I want regarding the optimal solution.

Shall I allow the 10 percent error or shall I allow 5 percent error on the initial interval of uncertainty? Depending on that we will select the corresponding n. That is why there is a major for any region elimination technique that is called the efficiency of the searching technique, rather the major of efficiency. This is also named as the reduction ratio, which equals to L n by L 0. That means after n iterations, what is the interval of uncertainty and what was the previous interval of uncertainty?

That is why there is part I have to discuss further. Though we are discussing several searching procedures, we will discuss in the next that which method is more efficient and that will totally depend on the reduction ratio value. Here I am just concluding my lecture today. I will start my next lecture with this exhaustive search procedure with some specific example. Thank you.