Optimization
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Lecture - 23
Nonlinear Programming with Equality Constraint

Today in continuation to my previous lecture on classical optimization, today we are dealing with a Non Linear Programming problem with the constraints.

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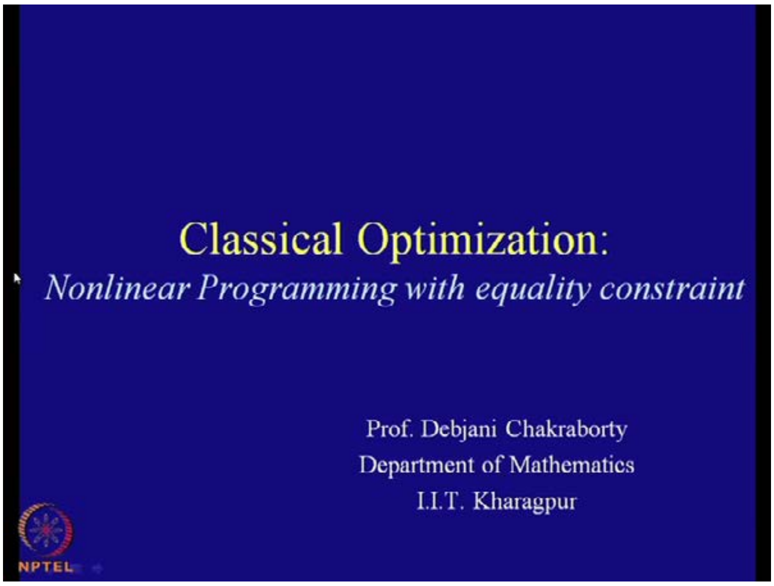

That is why, this can be said as a constraint optimization technique with equality and inequality constraint and today in specific, I will deal with the equality type of constraints.

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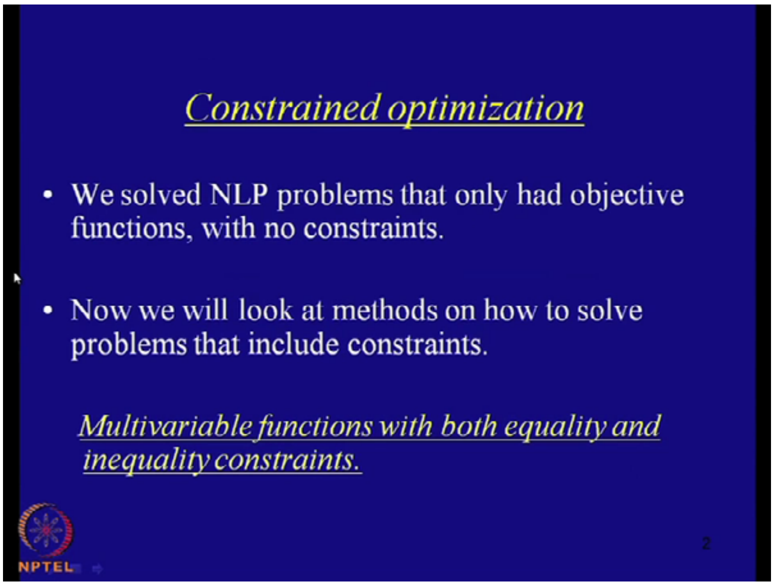

Now, a general non linear programming problem may have the equality constraint, it may have inequality constraints and as well, it may have equality and inequality constraints both. Now, whenever we are applying the classical optimization technique to get the optimal solution for this non linear kind of problem for the equality type and for the inequality type of constraints, there are methods to solve it. And today in specific, I am just explaining the Lagrange method and some other technique to solve the non linear programming problem with equality type of constraints.

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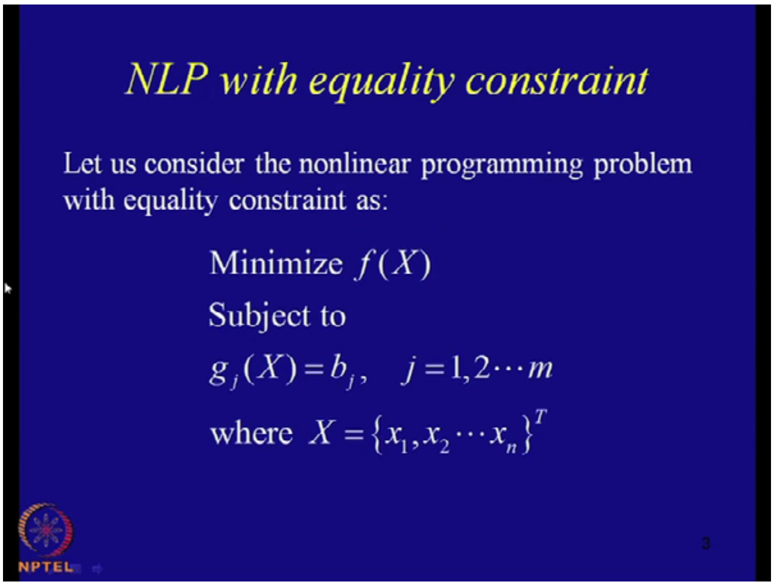

Now, before going into the detail of the technique, classical optimization technique, to get the optimal optimum solution for this problem, let us first try to explain, how the non linear programming problem with the equality constraint looks like. Now here, a general model can be considered that is, minimization of f X subject to g j X equal to b j, where m number of constraints are there and all the constraints are of equality type and it involves n number of decision variables.

Now, one thing it is clear, this is the non linear programming problem, because the functions, which are involved in this problem that is, function f X objective function and the constraint g j X, m number of constraints are there and these are all non linear in nature.

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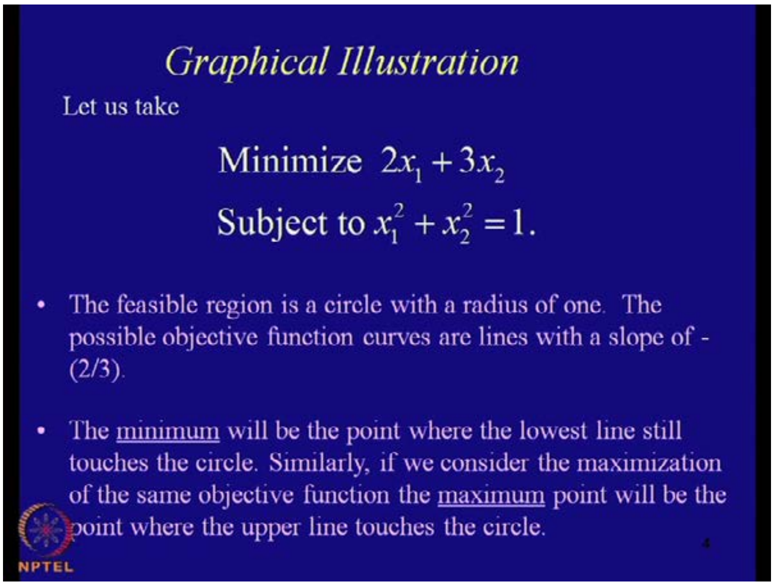

Now, before going into detail about the classical technique, I am just, first let me explain the problem in general and I will just solve the problem graphically. For that thing, let us consider a simple non linear programming problem, where two number of variables are involved x 1 and x 2 and objective function is minimization of 2 x 1 plus 3 x 2, subject to x 1square plus x 2 square is equal to 1.

Now, this is the equality constraint we are having here, one thing it is clear from the equation given that, this is a circle of unit radius and the centre of the circle is 0 0, that is the constraint of us. And we are having the objective function that is, the linear function that is, a line and this line is having the slope minus 2 by 3. Now, that is why we can say, the feasible region of this problem is a circle of unit radius and objective functions are the contours, these are the lines with a slope 2 by 3.

Now, the minimum point would be that point, where the lowest lines touches the circle. And if we just maximize the problem, the same problem then, it will be the point, that will be declared as the optimal one if the point, which will be the point with a upper line touches the circle.

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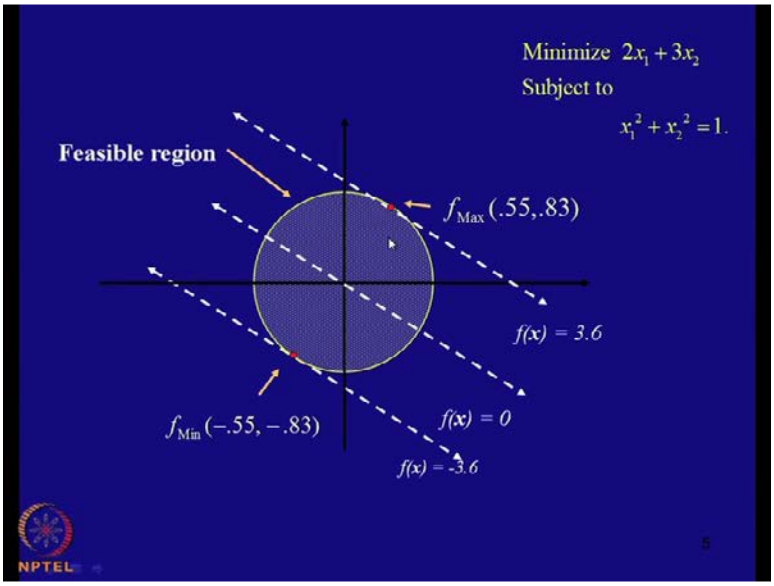

What even I say, let me just explain with the graph below and this is the problem for us, minimization of 2 x 1 plus 3 x 2 subject to x 1 plus x 2 is a x 1 square plus x 2 square is equal to 1. If we just draw the graph of it then, that would be the circle, that is a feasible region, the boundary of the circle is the feasible region for it, because this is the equality constraint type. Now, what about the line, the line is 2 x 1 plus 3 x 2, that is why the line, that is a objective function is moving on the space, in the x this is the x 1 axis and this is our x 2 axis.

Now, objective function is moving along this space, now if this is so then, at this point, where x 1 equal to 0, x 2 equal to 0, the functional value of the objective function is 0 and this is the slope of the objective function. If this objective function is moving through this way then, objective functional value will increase. And if the objective function is moving to the reverse side, if it takes the reverse path then, the functional f x value will decrease.

In this way, the maximum value would be that line, which maximally if the function is moves like this then, it will just touch at the last at this point. And after that, if I just increase the function, it will be beyond the feasible region, that is why we can say that, possible maximum point may be here. Similarly, we can explain the minimum point as well, whatever I say, just let me just explain through the function, this is the objective function is moving, now this is another state of the objective function, where f x is equal 1.

Now, in the next, again the objective function is moved in another place and here, the objective functional value is coming minus 3.6. If you see here that, the objective function as I said, if it is coming through this way, the functional value will always decrease further and further. And at this point minimally, the function can move at the most, because that is the point, where at the last the function is touching the feasible region.

That is why, we can declare this point as a point of minimum and this minimum point is, the coordinate of this point is minus 0.55 minus 0.83. And the objective function is minimum and the corresponding objective functional value is minus 3.6 and what about the maximum point. Similarly, if the function is moving further and further, the function is giving the maximum value here and this is the maximum objective functional value, that the value is 3.6 and the corresponding optimal point is 0.55 and 0.83.

Now, this is we have considered in circle, now from here graphically we could visualize that, function is having a maximum point here, function is having the minimum point here. The given optimization problem, we have considered the minimization type, that is why the solution of this given optimization problem would be is equal to minus 0.55 minus 0.83.

But, if we analyse further, if we just want to maximize the same function subject to the same equality constraint then, maximum will be at the point 0.55 and 0.83. And where the maximums are occurring, there are certain things to be noticed, actually the point here, we need to consider the gradient of the function.

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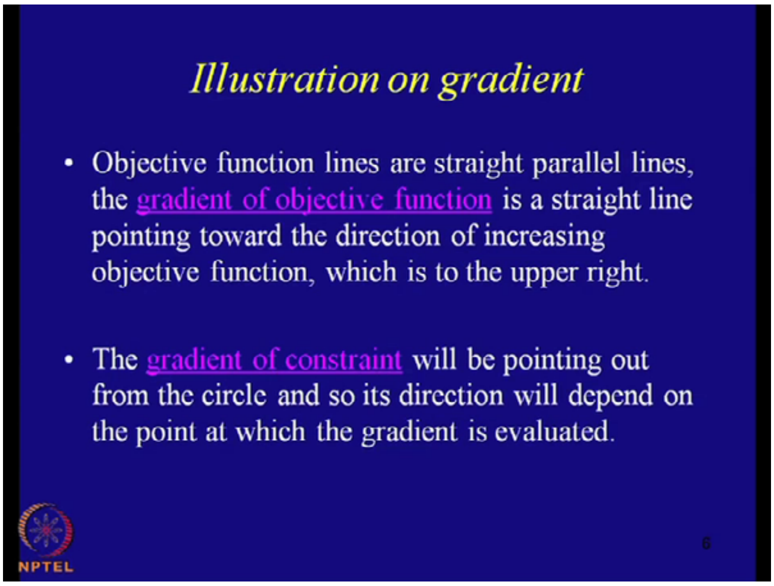

I am coming to that gradient of the function concept, now what is gradient of a function, generally in vector calculus, gradient of the scalar field is a vector field and gradient is the direction, in which direction the function is increasing. Now, in the gradient direction only, function has the fastest rate of change of the functional value, change in the positive sense.

Now, what is that direction, that is why if we say, the objective function lines are straight parallel lines and the gradient of the objective function is the straight line, pointing towards the direction of the increasing objective function, which is the upper right. It means that, if we consider a scalar field, here the objective function is thus, objective function is only the straight line, it could be thus non linear surface even.

If we consider the non linear surface then, at any point if we consider the tangent plane, at that point if we consider another direction, that is orthogonal to the tangent plane, that would be the gradient direction. In specific, I would like to say that, if this is the curve for us then, at this point, this is the tangent line and the gradient direction would be the orthogonal to this tangent line, that is why always, whatever objective function we are having, gradient direction would be the perpendicular direction of the tangent lines.

Since here the objective function is the straight line, that is why the gradient direction would be in the upper right. And another important fact as I said before that, in the gradient direction, functional value increases further and further and the magnitude of that vector gives us the rate of increase of that functional value. And we will see, there is a relation of the gradient of a objective function, gradient of the constraint with the optimal value of the given optimization problem, that part now I am going into detail to you.

Now, this is the definition for the gradient of a objective function, similarly we can define gradient of the constraint, that will be pointing out from the circle here in specific. Because, again in the circle, if we just consider a tangent plane at tangent line at every point then, if the normal direction to the tangent line would be the gradient direction. And that direction would be the gradient direction, we will see that, there is a relation between the gradient of the objective function and a gradient of the constraint at the optimal point.

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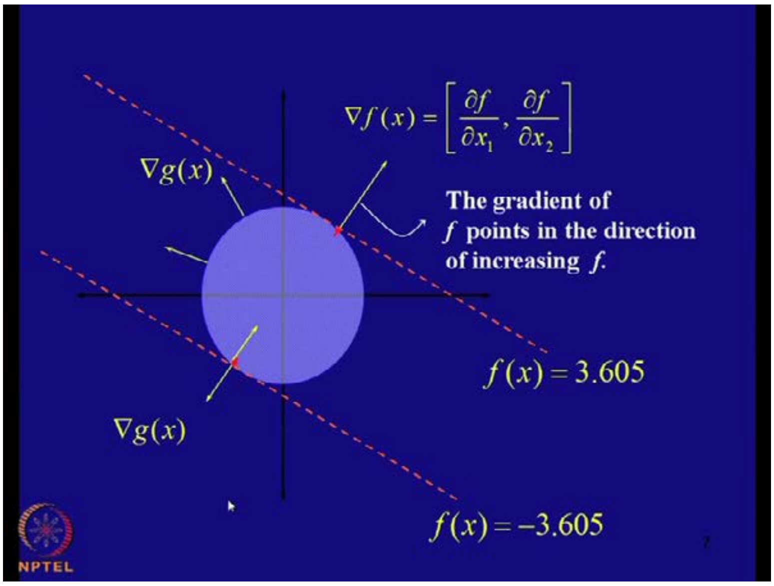

For defining that thing, let us go back to the problem again and let us try to explain the problem once more with this given graph. This is the feasible, the boundary of this circle would be the feasible region of the given problem. Now, this is the maximum point, where the objective function is having the maximum value that is, the value is 3.605. Now, if we consider gradient at that point, that would be directed to the normal to the tangent line, that is why this is the yellow line that is directed upward right, that would be the gradient direction.

That means that, if the function moves in this direction then, functional value will increase further and further, but what is the gradient. As I said, gradient is just an operator and is defined in the vector calculus, gradient is operating on a scalar field, that is the function here f x. That is why, gradient is the operator, that is the del operator we are considering and del of f x would be del f by del x 1 and del f by del x 2, this is the vector we are considering.

And if you want to have the magnitude of this gradient then, it would be the magnitude of this given vector. Now, this is the gradient of the objective function, now what about the gradient of the constraint. Similarly, we can see the gradient of the constraint as well and this is the minimum point, where the function is having the minimum value. This is the gradient direction of the constraint, g x is our constraint for here, here g x equal to x 1 square plus x 2 square equal to1 and at every point, we can define the gradient in this direction.

And at this point, where the function is having the optimal value, gradient of the constraint would be the direction here. But, what about gradient of the objective function, gradient of the objective function would be the, at this point it would be just reverse to the gradient of the constraint. But, what is happening in the maximum point, in the maximum point, gradient of the objective function the direction is this way, the direction of the corresponding vector.

And if we consider the gradient of the constraint, that would be again at this point will be in the same direction. That is why it has been seen graphically that, gradient at the maximum point and gradient at the minimum point, there is a relation to it and we are trying to find out those relations. Now, these are the gradient directions as I said, at every point it is the normal to the tangent at that point, these are the gradient direction. From the minimum point, if the gradient vectors are moving in this way and it is reaching at the maximum point here.

What we see, we see that, at the maximum point, the direction of the gradient of the constraint would be the similar to the gradient of the objective function. Why I said direction, because as we know, for every vector function there are two components, one is the magnitude and another one is the direction. Direction tells here, for the optimization problem it tells us that, gradient of the objective function and the gradient of the constraint at maximum point, both are having the same direction, but the magnitudes are different.

And in the similar way, if we consider the gradient at the minimum point, gradient of the constraint and the gradient of the objective functions are parallel, but these are reverse in direction. And one thing again this is here, there is a relation between the magnitude of the gradient of the constraint and the magnitude of the gradient of the objective function. From this fact only, if this is the basis for development of Lagrange multiplier method, which I will explain you in the next.

And there we will see that, the Lagrange multiplier will give us the, that only gives us the relation between the magnitude of the gradient of the constraint and the gradient of the objective function.

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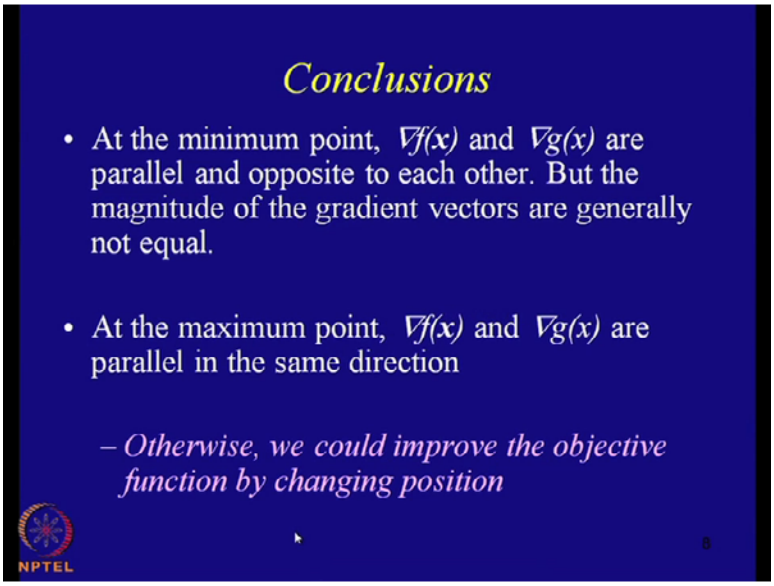

That is why from here, we can conclude something, we can conclude that, at the minimum point, grad of f x, grad is the del operator, del l x del l y of f x and grad of g x, these are parallel and opposite to each other, but magnitude of the gradient vectors are generally not equal. And at the maximum point, the gradient of objective function and gradient of constraint, these are parallel in the same direction, again the same thing the magnitudes are not equal, one is the multiple of the other. And in other points, in other points other than the optimal points, there is a scope to improve the objective functional value further and further.

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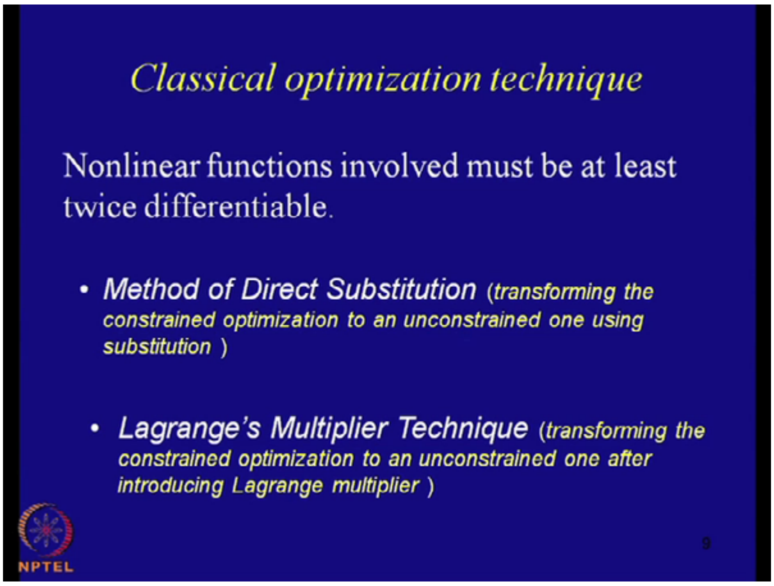

Now, whatever I say it, that is the graphical illustration for a non linear programming problem. Now, where we have considered the non linear programming problem with equality constraint and containing two variables. Now, we are coming to the classical optimization technique and where we will just show you two methods, one method is the method of direct substitution and another method is Lagrange multiplier technique. Both the techniques will give us the value for the maximum of the objective function and minimum value of the objective function, under certain constraint.

Now, here one assumption we need to make that, all the non linear functions involved in the non linear programming problem, this must be twice differentiable. Otherwise, this technique is not applicable, rather that necessary condition can be applicable, but the sufficient condition cannot confirm the optimal conditions further. Now, the first method is the method of direct substitution, this is a very much rather combination of the algebraic and the classical optimization technique.

And this direct substitution method transform the original constrained optimization problem to an unconstrained optimization problem using the substitution procedure. And Lagrange multiplier technique is another technique, in the next I will just explain to you and that is a more powerful tool to handle the constraint optimization problems. And here also, though we are transforming the constraint problem to an unconstraint one and the rest procedure would be the same, as I told before for the unconstraint classical optimization technique.

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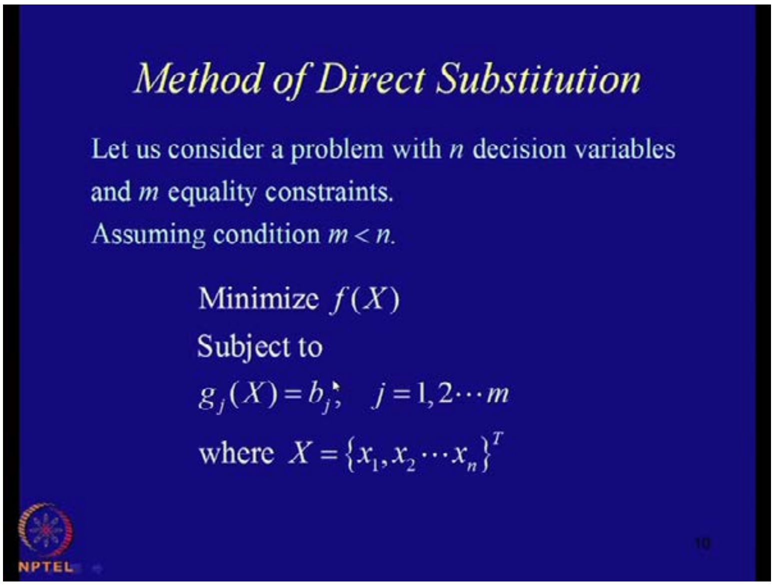

Now, let me come in specific to the method of direct substitution, now again I have considered the same problems, same non linear model, general model, where we have considered the minimization of the objective function and subject to m number of constraints. This problem is having m number of constraints and n number of decision variables and for this problem, we are assuming the condition that, number of constraints are lesser than the number of decision variables that is, m lesser than n.

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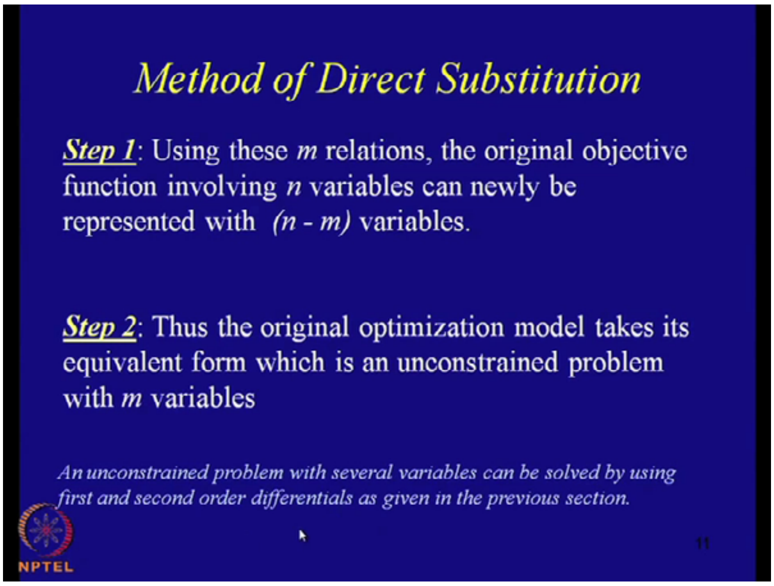

With this consideration, I am moving to the method of direct substitution. As m is lesser than n, that is why from the given constraint set, step 1 would be we can easily just make n number of variables we can transform to n minus m number of variables from the given set of constraints. Once again I am repeating, that step 1 suggest that, use the m number of constraint relations and the original objective function, which involve n number of variables, that will be transformed with the n minus m number of variables.

Then, once the step 1 will be completed then, we have taken care the constraint set well, that is why the original constrained optimization problem will be transformed to unconstrained optimization problem, where it will have only m number of variables, that would be the step 2. And after that, the unconstrained optimization technique whatever we did before, the same will be applied to get the optimal solution for the given optimization problem, let me explain this method with the example.

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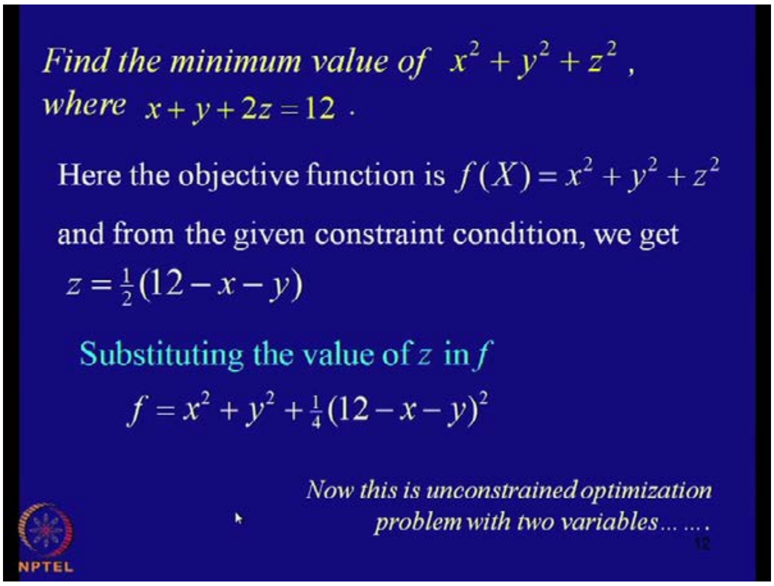

This is the example let us consider, minimization of x square plus y square plus z square subject to the constraint x plus y plus 2 z is equal to 12. Here, we have considered the one constraint is involved and constraint is of equality type, three variables are involved, that is why solving this problem graphically will be little difficult for us. And we need to apply certain classical optimization technique for solving this and here, we are applying the method of direct substitution to get it.

As I said, the step 1 would be the, in the method of direct substitution that, whatever relation we are having one constraint, from there we will just make the objective function in terms of two variables only that is, n minus m number of variables will be involved, how to do it. Just see, from the constraint x plus y plus 2 z is equal to12, this can be written as z can be expressed in terms of x and y then, it will be half 12 minus x minus y. Once we are getting z, we will substitute the z value in the objective function f x that is, x square plus y square plus the square of these right hand side term.

Then, the function objective function will become the objective function of two variables only. And not only that, this is the problem, the original this is the problem, which is equivalent to the given problem, because we have taken care the constraint part well and constraint is nothing, but where we are giving the restrictions on x, y and z. These are the decision variables and these we have considered already and now our task would be the minimization of f, where f is equal to x square plus y square plus 1 by 2 into 12 minus x minus y whole square that is, 1 by 4, etcetera.

Now here, we want to minimize f, if I want to minimize f, again let me apply the classical optimization technique, which I did before. That for necessary condition, we will take the first order partial derivative of f, del f by del x we will equate to 0, del f by del y we will equate to 0, from there we will get the value for x and y and the corresponding z as well, which will give us the possible extreme points of this function.

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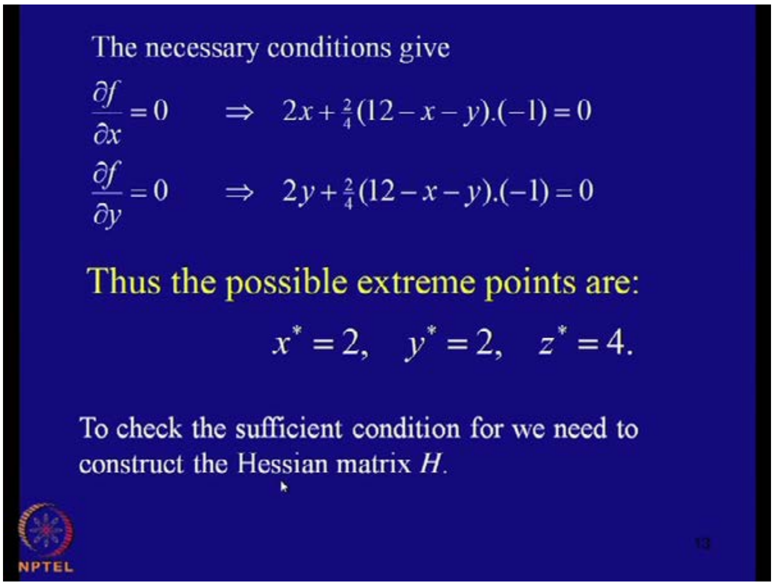

Now, as I said, I have just did it, 2 x plus 2 by 4 12 minus x minus y minus 1 equal to 0 and this is the next equation we are getting. Now, we are having two equations and two unknowns, we want to get the value for x and y from this. For getting this, we will just equate the first equation with the second equation and we will get x is equal to y. Once we will get x equal to y, we will substitute x equal to y in one of this equation and after that, we will get the value for x as 2, value for y as 2 and corresponding z would be as it is half into 12 minus x minus y then, z would be 4 only.

Now, this is the possible extreme point for the given optimization problem, now we want to confirm that, whether this point is a minimum or not.

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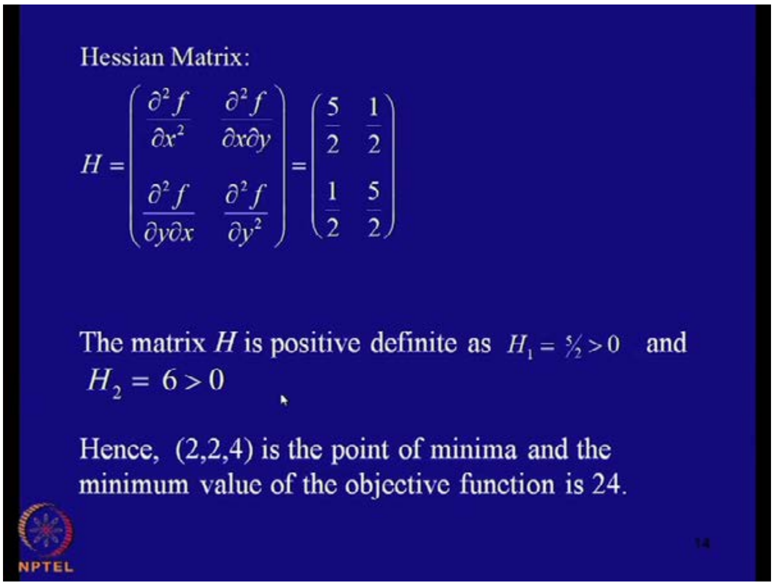

For that thing, we need to go for the second ordered partial derivative, rather we need to check the property of the Hessian matrix H, that is the sufficient condition for us. Now, this is the del f by del x, del f by del y, how to construct the Hessian matrix, Hessian matrix would be del 2 f by del x 2 del 2 f by del x del y del 2 f by del y del x and del 2 f by del y 2, I have done it in the last. And if we just do the second ordered successive differentiation for the given expression, we will get the value 5 by 2 half half and 5 by 2.

We need to check the property of this Hessian matrix, because it will ensure, whether at the given point, the problem is having minimum or maximum. As I said, there are different method for solving it, one of the method is to check the corresponding minors, sign of the minors. As we see that, the first minor is positive, second minor is again positive then, we can conclude the Hessian matrix is positive definite. As we know, if the Hessian matrix is positive definite then, the corresponding point must be the point of minima.

Thus, we can say that, 2 2 4 is the point of minima for the given function and the corresponding functional value would be 2 square plus 2 square plus 4 square, that would be 24, that is all about the method of substitution. Now, moving to the next method, that is the most powerful technique for solving non linear optimization problem with equality constraint. The method is in the name of scientist, Lagrange and the method is method of

Lagrangian multiplier and this method involves some unknown parameter, that is called the Lagrange parameter.

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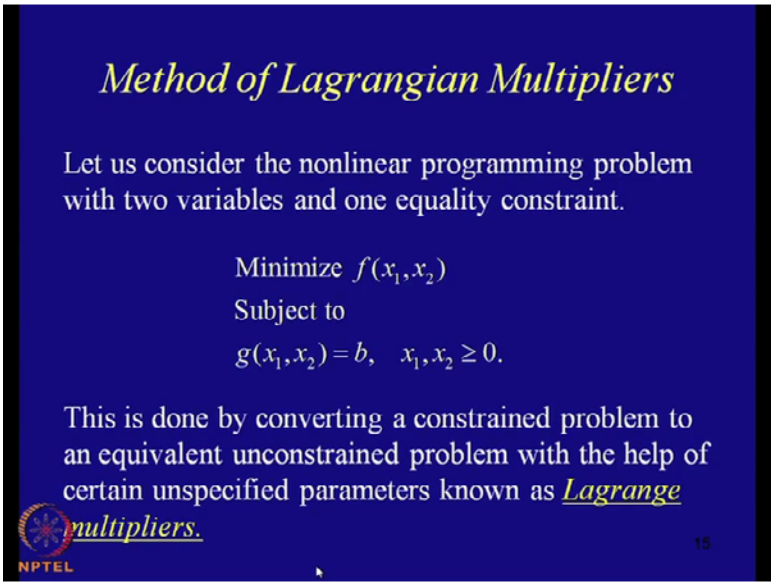

And I am just going to detail this method in the next, method of Lagrangian multiplier, here also we have considered the same non linear programming problem. And first we are considering the simplest one, a non linear with two variables x 1 and x 2 and we are having only one equality constraint that is, g x 1 x 2 is equal to b. Let me explain the Lagrangian multiplier technique for this problem first then, we will generalise this method for n number of decision variables and m number of constraint.

Now, this is the method, what is the method involves, the method involves in this way that, we will convert this constraint optimization problem to an equivalent unconstrained optimization problem with the help of some unspecified parameter, the name of this parameter is the Lagrange multiplier. What is the way of doing it, we will convert the constraint optimization problem to an equivalent by introducing the Lagrange function. That is why in the next, I will show you how to construct the Lagrange function and after that, we will just minimise the Lagrange function for getting the optimal solution for this.

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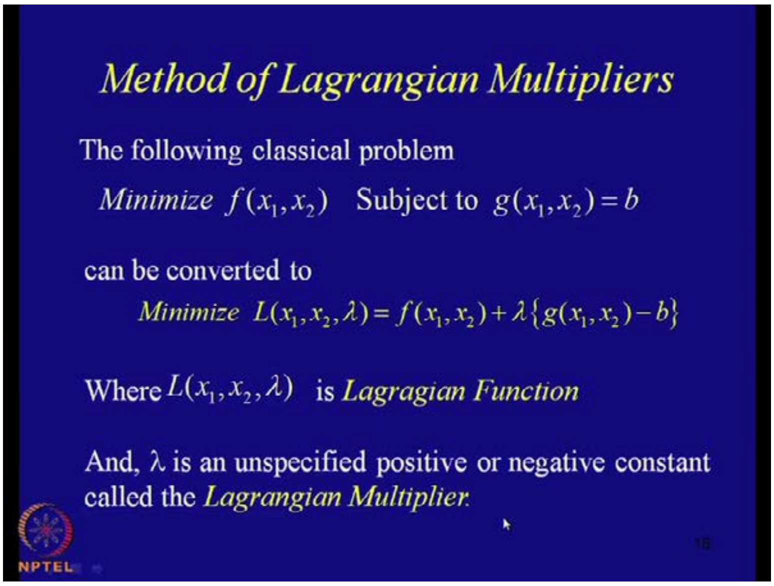

Now, this is the problem for us and as I said that, we will convert this constraint optimization problem to an unconstrained one with introduction of the Lagrange multiplier lambda. Just see, we have constructed the function L and the function L involves two functions together, one function is the f objective function, f of x 1 x 2 plus lambda g of x 1 x 2 minus p. Now, if we minimise this Lagrange function then, the problem comes down to the same problem that is, the unconstrained optimization problem.

As you know, how to solve the unconstrained optimization problem, the first order partial derivatives will give us the necessary conditions and second order will tells us about the sufficient conditions of the optimal values. But, just look at this Lagrange function, the L function, this L function involves three variables, two are the decision variables x 1and x 2 and there is another variable is being involved, that is the lambda, it is called the Lagrange multiplier.

This is the unspecified positive and negative value it can take, thus lambda is an unspecified positive and negative constant and it is being named as the Lagrange multiplier. Now, we are trying to minimise the Lagrange function in the next.

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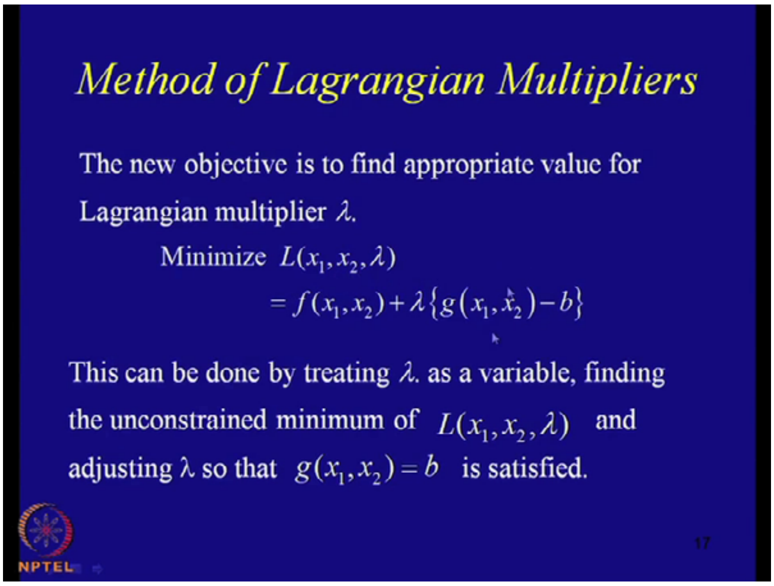

Now, this is the problem, now the new objective function would be the minimization of the Lagrange function. Now, here the lambda is a variable for us then, what would be the necessary condition for this, necessary conditions would be the gradient of L must be is equal to 0.

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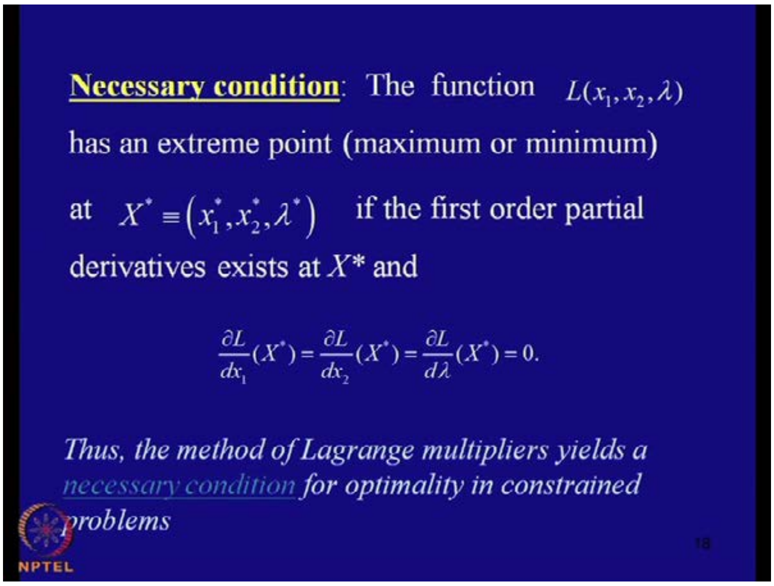

What is the meaning of it, it means that, del L by del x 1 at the optimal point x star equal to 0, del L by del x 2 at the optimal point, this must be equal to 0. Similarly, del L by del lambda at x star, it will be 0, that is the necessary condition can be framed in this way.

The Lagrange function L, which involves three number of decision variables x 1, x 2, lambda has an extreme point that is, maximum or minimum at x star that is, x 1 star, x 2 star, lambda star.

If the first order partial derivates exist and at the optimal point, all are equal to 0 and this gives us the possible extreme points for a given non linear optimization problem with equality constraint.

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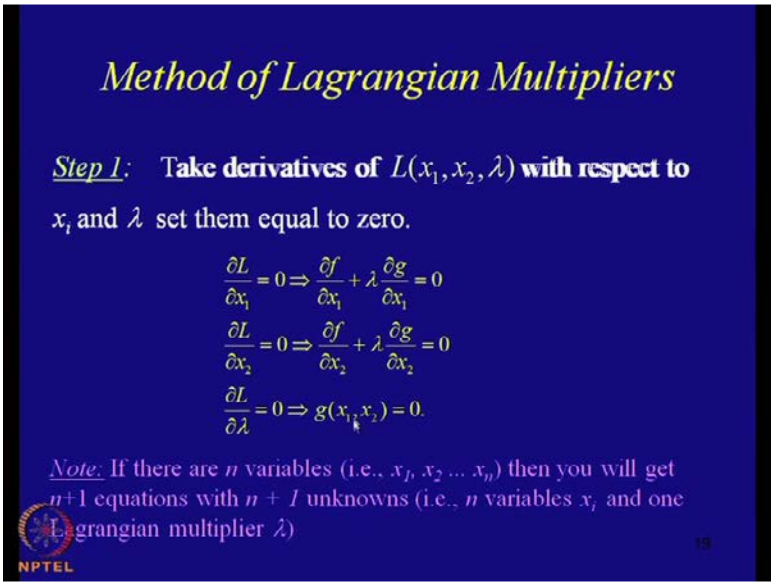

Now, as I said, let me just write it down in detail, L is a function, where L is equal to f plus lambda g, that is why we can write del L by del x1 equal to 0 with the similar equation del f by del x 1 plus lambda del g by del x 1 equal to 0. Del L by del x 2 equal to 0 gives us the equation del f by del x 2 plus lambda del g by del x 2 equal to 0. What about del L by del lambda equal to 0, del L by del lambda will give us g is equal to 0, that is the original constraint for us.

Now, that is what we get from step 1, we get three equations and three unknowns, unknowns are x 1, x 2 and lambda. Therefore, solving these three equations, we will get the values, possible extreme point for the given optimization problem, that is the x 1, x 2, x 3. Now, one thing it is to be noticed that, if we are having n number of decision variables and m number of constraints, at least here we have consider only one constraint and this necessary condition will give us n plus 1 equations with n plus 1 unknowns in general.

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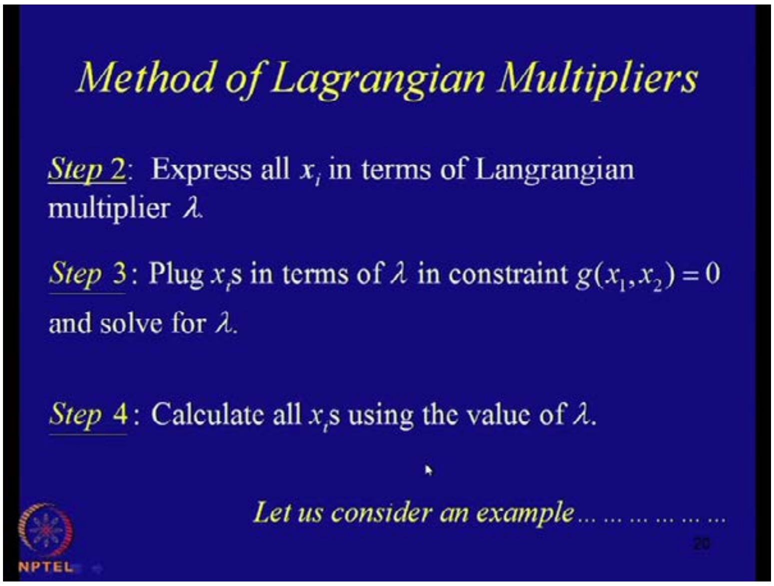

Now, going to the next step, once we get the three equations and three unknowns, we will express all x i s in terms of Lagrange multiplier for the given equation, from first two equations. And once we will get x 1 and x 2 in terms of lambda, we will substitute that one in the third equation in step 3 and we will get the value for lambda. Once we will get the value for lambda, from the given necessary conditions, we will get the values for x1 and x 2 as well. Whatever I said, let me explain through the numerical example in the next.

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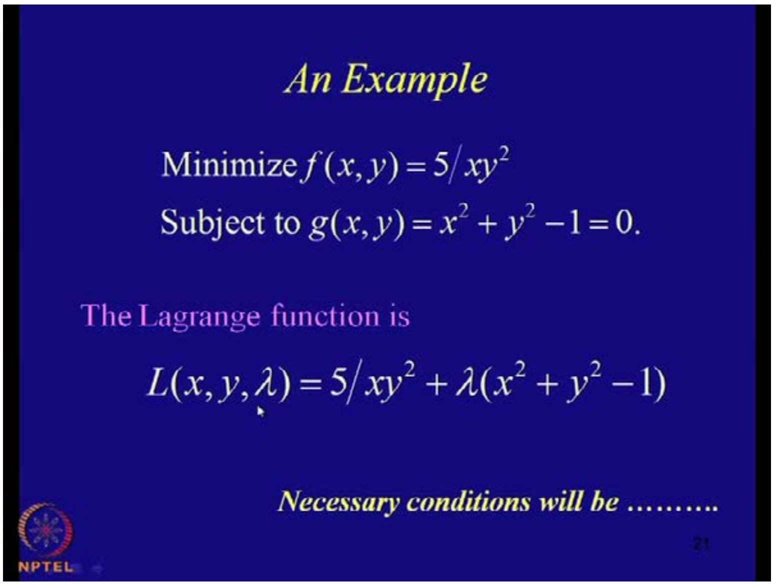

This is the example for us, here also we have considered two variables and one equality constraint and the problem is of minimization type. Minimization of f x y, 5 by x y square subject to the constraint x square plus y square is equal to 1. Now, for solving this problem, first we need to construct the Lagrange function, after introducing the Lagrange multiplier lambda. How to construct Lagrange function as I said, Lagrange function would be the function of three variables here, the decision variables x and y, as well as the unspecified Lagrange multiplier that is, lambda.

And that would be is equal to objective function plus lambda into the constraint, that is the original problem. Constraint optimization problem will be transformed to an unconstraint optimization problem, where the problem is having three number of variables.

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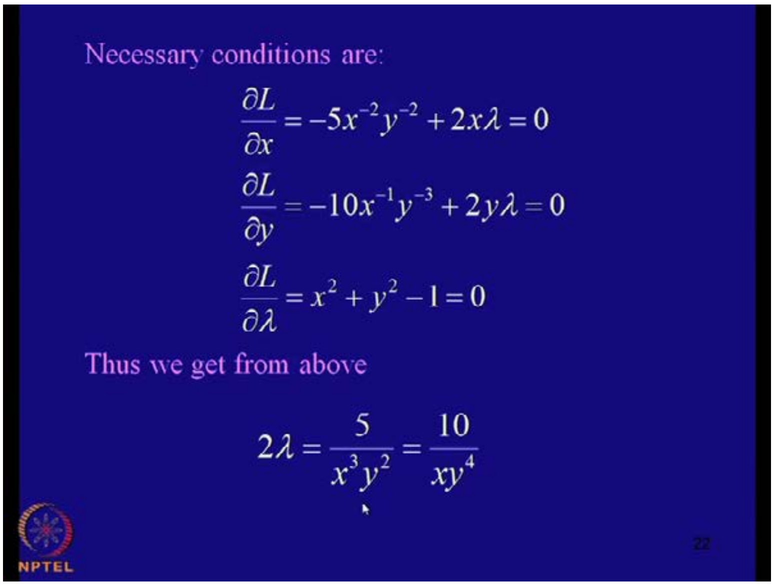

Now, we will go for the necessary conditions for the given equation, we will get from the given equation del L by del x equal to 0. We are getting this equation minus 5 1 by x square y square plus 2 x lambda equal to 0, del L by del y equal to 0 gives us the equation minus 10 by x 1 y cube plus 2 y lambda is equal to 0 and the third equation that is, del L by del lambda equal to 0, will give the original constraint that, x square plus y square minus 1 equal to 0.

Now, these three equations we are having and three unknowns we are having, from these three equations, we will just transform lambda in terms of x and y from first two equations. And we will substitute those values in the third equation we will get the corresponding value of lambda and in sequence, the values of x and y. From the first equation, we get 2 lambda is equal to 5 by x cube y square and from the second equation, we get 10 by x y to the power 4.

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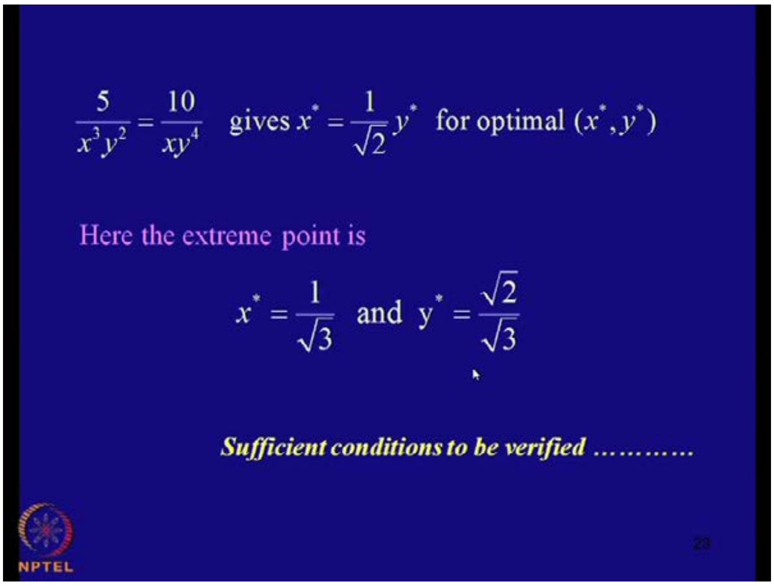

Once we have done so, we are substituting this one in the next and we will get the value for lambda as well as x star and y star. And we get the optimal value as x star is equal to 1 by root 3 and y star is equal to root 2 by root 3 and this is the possible extreme point for the given constraint optimization problem. Now, we need to verify the sufficient condition for this problem, now for verifying the sufficient condition for this problem, we need to know what are the sufficient conditions for the given Lagrange function, that will be explained in the next.

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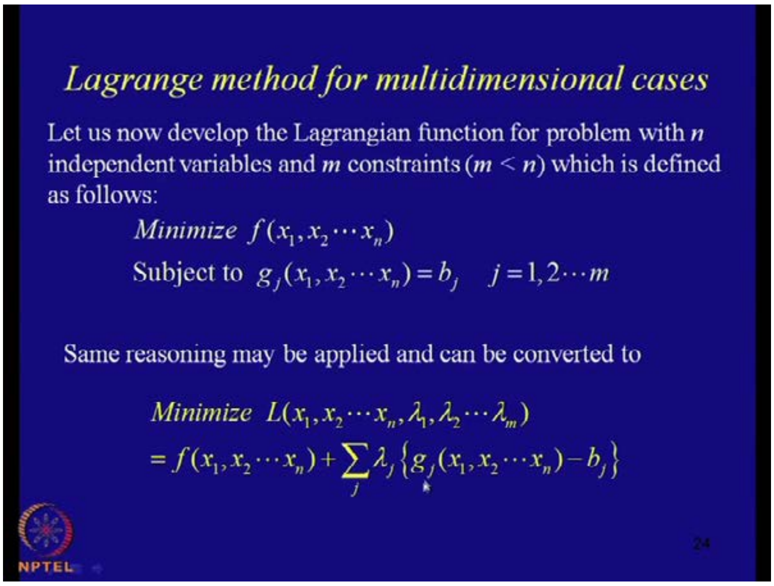

But, before to that, let me explain the necessary condition for general non linear programming problem, where the function is having n number of decision variables and m number of constraints. Now, here also the same reasoning can be applied, we can introduce the Lagrange multiplier and we will construct the Lagrange function for this given optimization problem. Now, one thing is clear that, for this problem, we are having m number of constraints, that is why for constructing the Lagrange function, we need to use m number of Lagrange multipliers.

And we will construct the Lagrange function in this way f plus lambda 1 into g 1 minus b 1 plus lambda 2 into g 2 minus b 2, in this way j will run from 1 to m and this is the corresponding Lagrange function. First what we see in the Lagrange function, Lagrange function is a, this problem is again an unconstraint optimization problem and this problem involves m plus n number of decision variables.

Because, the n number of original decision variables and we have introduced m number of Lagrange multipliers due to m number of constraints and this is the form of the Lagrange function that is, f plus summation lambda j g j minus b j, j is running from one to m. Now, again we will apply the necessary condition for these, necessary condition is as I said, gradient of the objective function we will just equate to 0 that is, del L by del x 1 equal to 0, del L by del x 2, in this way del L by del x n equal to 0.

Similarly, del L by del lambda 1, del L by del lambda 2, del L by del lambda m is equal to 0 and that will give us total m plus n number of equations, that I am going to detail in the next.

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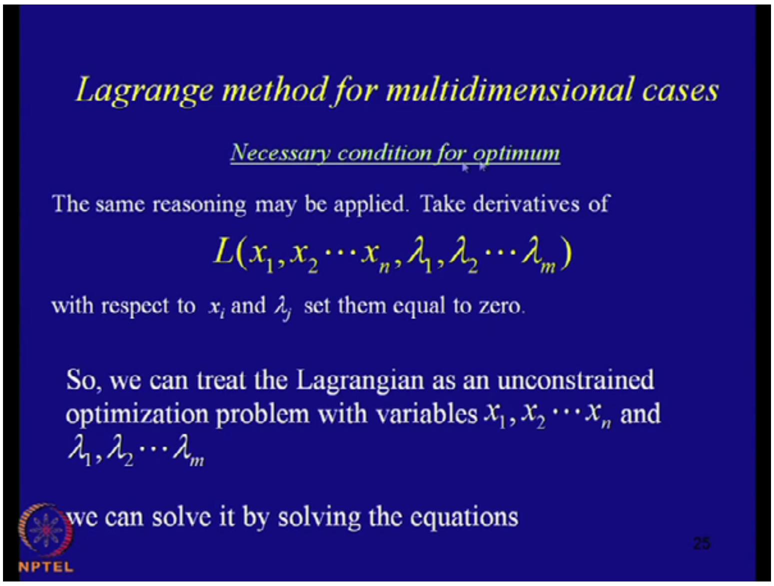

That is, we can say, this is the necessary condition for getting optimum for multidimensional non linear programming problem, where n number of decision variables are involved.

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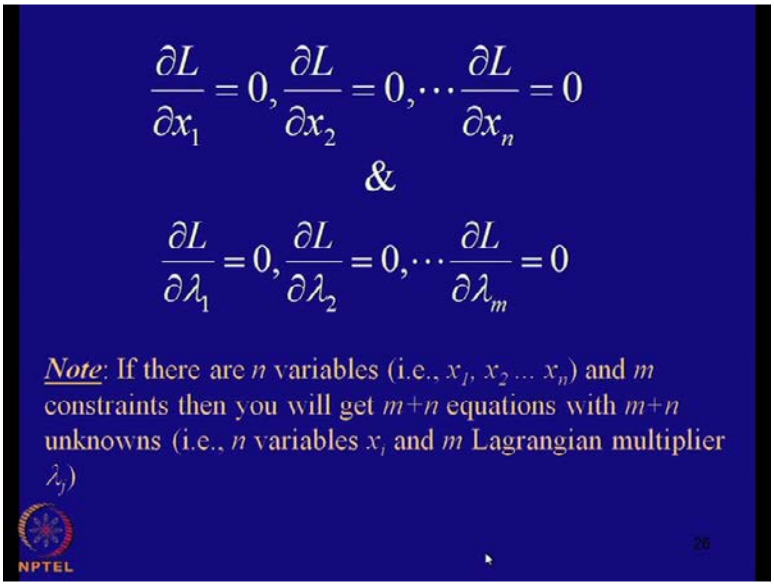

Now, what are the equations we are getting just see, del L by del x 1 equal to 0, del L by del x 2 equal to 0, del L by del x n equal to 0, similarly del L by del lambda 1 equal to 0 this way. Now, whatever I say it, the same thing we will apply for the numerical example and we will see one thing it is to be noted here that, since the problem is having n number of decision variables and m number of constraints, necessary condition gives us m plus n number of equations and we need to find an m plus n number of unknowns. Thus, we need to find out the possible optimal value from this m plus n number of equations.

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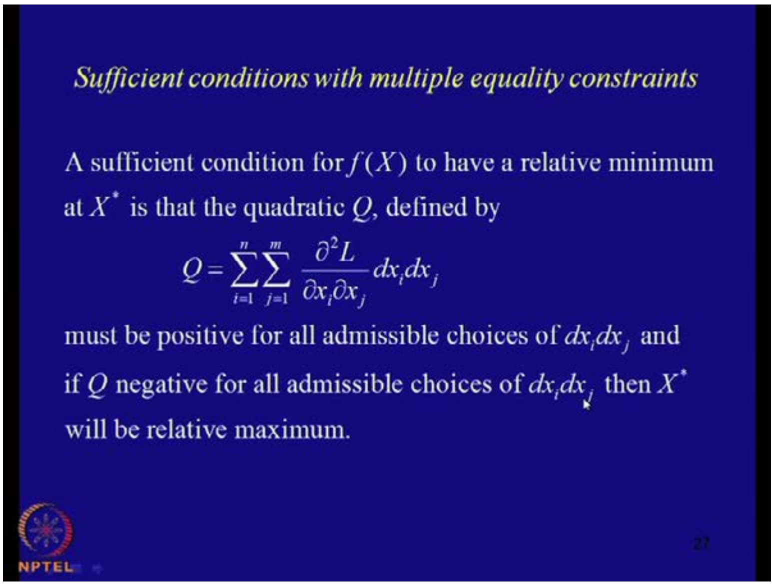

Now, before going into detail of the application of this technique for more than two variable non linear programming problem, let us first introduce the sufficient condition for dealing with the equality constraint, where we are having m number of constraints and n number of decision variables. Here also as I said that, L is a function here, where m plus n number of decision variables are there, for applying the sufficient condition, we need to construct the Hessian matrix with L.

Now, the Hessian matrix will be positive definite for minimum value, negative definite for maximum value and for saddle point, neither positive definite neither negative definite. Thus, we can say that, the sufficient condition for f x to have a relative minimum at x star is that, quadratic Q. That is, the summation over i and over j del 2 L by del x i del x j d x i d x j, this quadratic must be positive that is, the quadratic form must be positive for all admissible values for d x i d x j and this would be negative for maximum for all admissible values of d x i and d x j.

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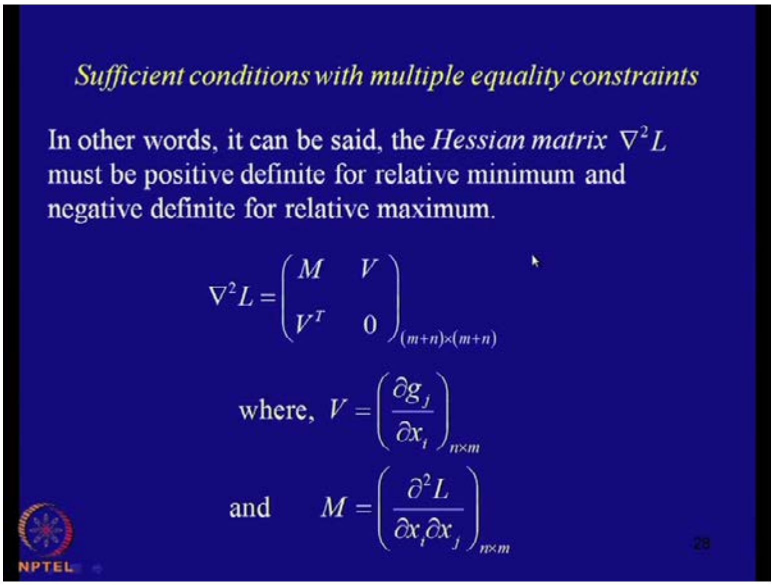

Now, how del 2 L by del x j i del x j looks like, this can be formed with this Hessian matrix. Now, whatever matrix has been formed here, let me explain the same with this, let me just see, what are the forms of M, V, etcetera in the next.

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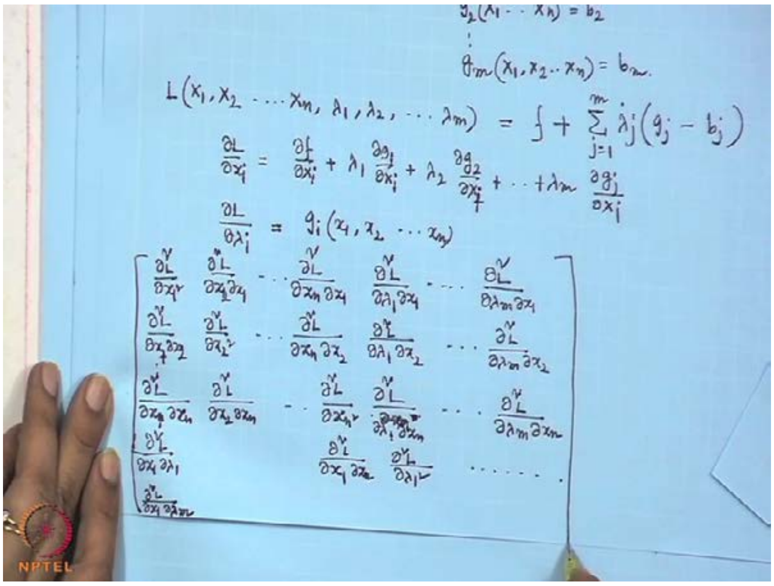

As I said, we have considered the function minimization of f x 1 x 2 x n subject to g 1 x 1 to x n is equal to b 1, g 2 x 1 to x n is equal to b2 and upto g m x 1 to x 2 to x n is equal to b m. Now, our corresponding Lagrangian function would be L, that will involve n number of decision variables x 1, x 2, x n and we are having m number of Lagrange multiplier lambda 1, lambda 2 upto lambda m and this would be is equal to f plus summation j lambda j g j minus b j, j will be from 1 to m.

If we just see here that, del L by del x 1 would be is equal to del f by del x 1 plus lambda 1 del g 1 by del x 1 plus lambda 2 del g 2 by del x 2 upto lambda m del g j by del x 1, this should be x1. Similarly, if I make del L by del x 2, this should be del f by del x 2 plus this, that is why we can make it in general del L by del x i is equal to del f by del x i plus lambda 1 del g1 by del x i plus lambda 2 del g 2 by del x i upto lambda m del g j by del x i.

Similarly, if we just do the first order derivative of L with respect to lambda 1 then, this would be is equal to, the first one would be is equal to g 1 only and what about del L by del lambda 2, it would be g 2. In this way, if I just make, it would be del L by del lambda i would be g i of x 1 x 2 x n. Now, we are constructing the Hessian matrix, how Hessian matrix looks like, Hessian matrix would be this, the matrix would be this one, del 2 l by del x 1 square del 2 L by del x 1 del x 2 del x 2 del x 1.

In this way, del 2 L by del x n del x 1 and in the next, del 2 L by del lambda 1 del x 1, in this way if I just move further, the last element of this row would be del 2 L by del lambda m del x 1. And what would be the next second line, second line would be del 2 L by del x 1 x 2 del 2 L by del x 2 square del 2 L by del x n del x 2 del 2 L by del lambda 1del x 2 del 2 L by del lambda m del x 2. In this way, the last term would be here, it is not the last term, that is the n th row would be, del 2 L by del x n del x 1del x n del 2 L by del x 2 del x n.

And the last term would be del 2 L by del x 1 square, here del x 1 square and this would be del lambda 1 del x n and here it would be del 2 L by del lambda m del x n. If I move further, because L is a function of n number of decision variables, not only that m number of m number of Lagrange multiplier as well. That is why, the next term would be del 2 L by del x 1 del lambda 1 and the last one would be del 2 L by del x 1 del lambda m.

And this term would be del 2 L by del x 1 del x n and this would be del 2 L by del lambda 1 square. And if it will be go further and further, whatever I have written in the matrix, let us try to evaluate all the values in the next, if you see the values here, we will see the value should be like this.

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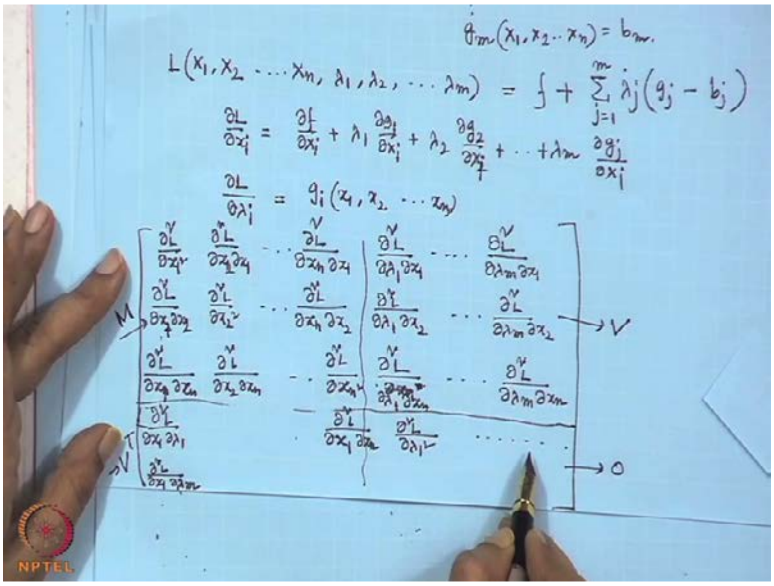

What is del 2 L by del x 1 square, del 2 L by del x one square from the given equation it would be, del 2 f by del x 1 square plus lambda 1 del 2 g 1 by del x 1 square. In this way, lambda m del 2 g by g j by del x 1 square, rather it would be is equal to del 2 L by del x 1 square. Similarly, the next term let us see, del 2 L by del x 2 del x 1, del 2 x by del x 2 del x 1 would be is equal to, if we just see this one, it would be is equal to del 2 f by del x 2 del x1 plus summation lambda j del 2 g j divided by del 2 x 1, that would be is equal to del 2 L by del x 2 del x 1.

Now, in this way, if we just move further and further, we will see the last term, that is the term that is, del 2 L by del lambda 1 del x 1, lambda 1 del x 1 this would be is equal to del 2 L by del lambda 1 is equal to g 1 and here g 1 and del 2 L by del x 1 is equal to del f by del x 1 plus lambda 1 into g del g 1 by del x 1 plus the other terms, that is why we are getting from here, this term would be is equal to del g 1 by del x 1. In this way, if we just do further, the other terms as well we will see that, the function m, the term, it is not the function, the part of the matrix, whatever matrix I have written, the same matrix the m.

Here the same matrix if I just write down then, upto this, that if I just partition the matrix in this fashion, we will see that, this matrix corresponds to capital M, this corresponds to capital V, this corresponds to the transpose of V and here it will be 0, because del 2 L by del lambda 1 square will be 0. Similarly, for other terms as well, that is why we can say that, del 2 L is equal to M V V T and 0, where M is equal to del 2 L by del x i del x j and V is equal to del g j by del x i and this matrix would be is equal to 0.

And we will see in the next as well with example the same Hessian matrix, how it will be constructed for the given numerical example. Now, what we can conclude that, if the Hessian matrix is positive definite then, it will be relative minimum, if it is negative definite, it will be relative maximum.

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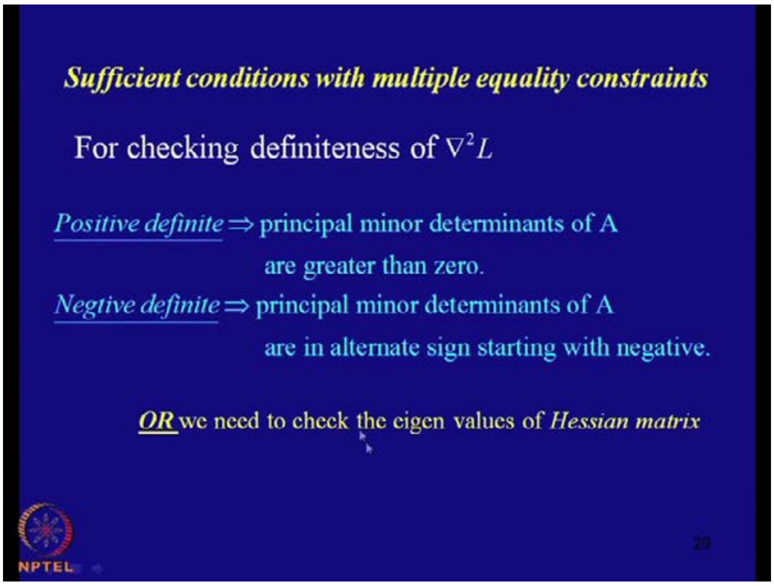

That is why, this is the sufficient condition we can say that, if the principal minors determinants of the given Hessian matrix are all greater than 0 that is, A corresponds to del 2 L. If these are all greater than 0 then, it is positive definite, otherwise its negative definite. Now, there is another method to check the Hessian matrix, whether this is positive definite or negative definite.

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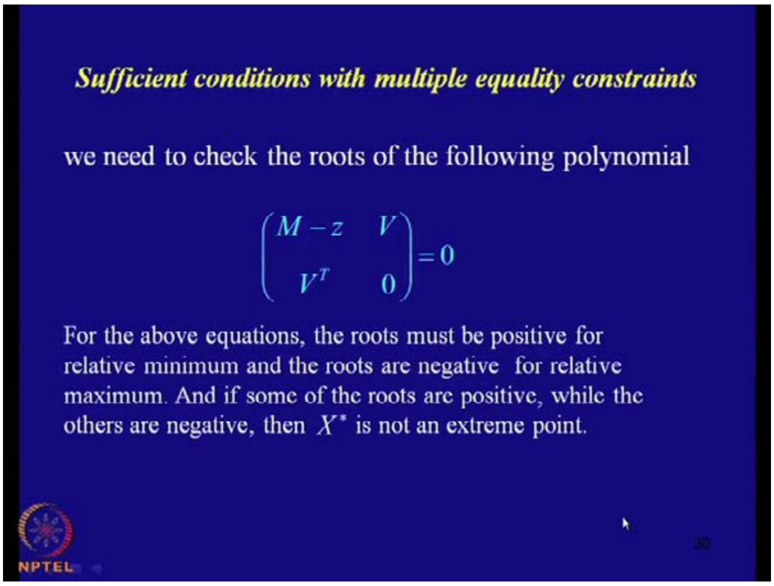

The method tells us that, that is called the Eigen value method and Eigen value method tells us that, if the Eigen values of the given matrix are all positive then, the corresponding matrix is positive definite. Now, for this thing, we can construct the characteristic equation, for the given Hessian matrix, this is the characteristic equation for us. And we have considered M minus z V V T and if this should be 0 minus z and if the roots of this matrix that is, the values of z, these are all positive.

Then, we can say the corresponding Hessian matrix is positive definite, rather we are having the relative minimum. Now, this is the alternative process, otherwise we can apply the, checking the sign of the minor, that will work fine for checking the positive definiteness or negative definiteness for the given Hessian matrix.

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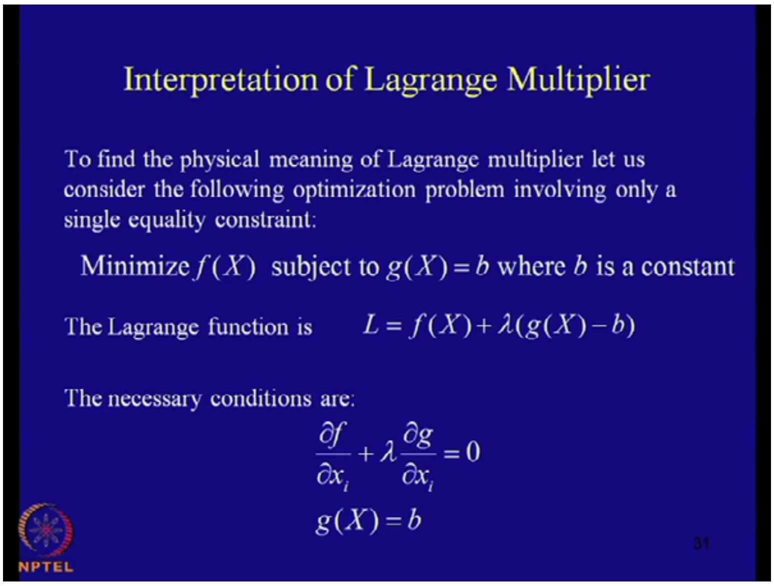

Now, before applying the same technique for the general non linear programming problem, let us try to understand further, regarding the method of Lagrange multiplier. As we see the method of Lagrange multiplier, this is a rather an extension of the classical optimization technique, only the new thing is involved here. We are introducing the unknown Lagrange multiplier and we are constructing the Lagrange function and we are going for the necessary and sufficient condition over the Lagrange function.

But, what is the meaning of the Lagrange function, what is the physical interpretation rather the economic interpretation of the Lagrange multiplier, that is the very important thing for a decision theory. That is why, I am coming to the next the Lagrange interpretation of the Lagrange function, Lagrange multiplier, the physical meaning to it and the economic interpretation of the same as well. We are considering a general problem that is, the optimization of f X subject to g X is equal to b, b is a constant we have considered here.

Now, the Lagrange function would be certainly L is equal to f X plus lambda into g X minus b and the necessary condition will be del f by del x i plus lambda into del g by del x i equal to 0. And the other condition would be g of x 1 x 2 x n would be is equal to b, that is why these are the necessary condition.

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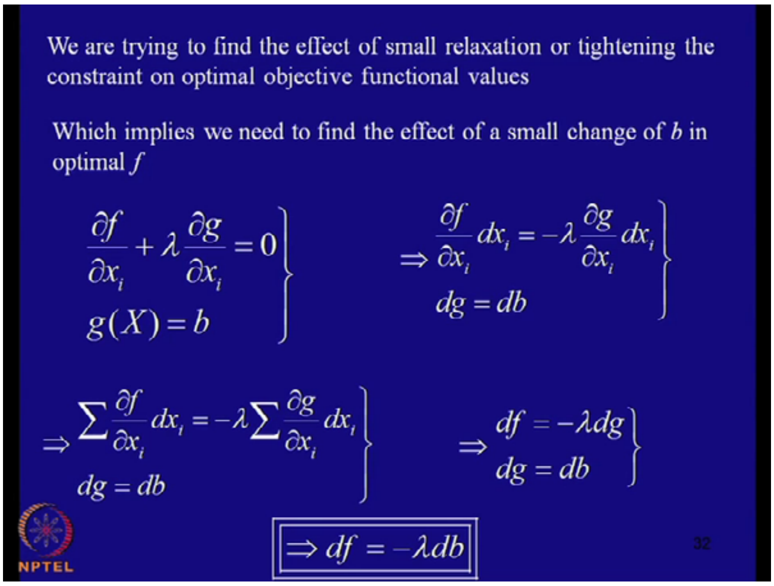

Now, if you analyse the necessary condition further, we will see that, from the necessary condition, we can do the sensitive analysis over these. How to do the sensitivity analysis over this, we will see that, if we just change the value for b that is, if we just change the feasible space, feasible region, if we either extend or relax the feasible region or we will just tighten the feasible region, we will see the objective functional value will change.

Not only that, the Lagrange multiplier will play a key role on these and from that mathematical explanation, we can generate the economic interpretation of the Lagrange multiplier. That is why, our next task would be to do the sensitivity analysis over these and we will change the value for b and we will see, how the objective functional value are changes with respect to the value of lambda, that is why we are doing certain simplifications of the necessary conditions.

These are the necessary conditions for us, from here we will get the relation between the optimal value of f and the Lagrange multiple lambda and the b as well, because we need to find out the effect of a small change of b in the optimal value of f, that means we are changing the feasible region. Now, if I just do certain simplification here, we can see that, del f by del x i d x i would be is equal to minus lambda del g by del x i d x i. One thing we can see here as well that, del f by del x i is equal to minus lambda del g by del x i.

What it gives us, it tells us that, if we consider the gradient of the objective function, gradient of the objective function is del f by del x 1 del f by del x 2 upto del f by del x n. And whatever the gradient of the constraint, gradient of constraint would be del g by del x 1 del g by, rather we are applying the del operator on the constraint as well as the objective function.

What we see, we see that, the gradient of f is equal to minus lambda gradient of g, it will gives us some idea in the optimal value, what is happening, how the gradients are parallel and what would be the magnitude of the gradient vector for the objective function and the constraint. I will come into detail to that, but for doing that thing, let me simplify the given necessary conditions further. The simplifications can be done in this way so that, we will get the total we will get...

From here, we are simplifying g x equal to b, rather d g is equal to d b, now here one thing it is clear that, generally b is a constant for the given constraint. But, you see if the b is something we want to change it, we want to do the sensitivity analysis over b, that is why we are taking the value for d b, we are considering the d b, the change of b value so that, the size of the feasible region will change. Now, in the next, we have taken the summation over all i’s, considering all decision variables together in the left hand side as well as considering the right hand side.

And we can generate that d f is equal to minus lambda d g, rather we can see that, d f is equal to minus lambda d b, because as we know, d g is equal to d b, we have just done. Now, this relation gives us some more ideas about the Lagrange multiplier, let us see what is happening. As we known, in any business application for any optimization real life situation, the objective function f often represents the profit that has to be maximize or the revenue has to be maximized or the problem may involve the cost function, that has to be minimized.

And what are the constraints, constraints are the available resources like availability of manpower, availability of the raw material, the capacity or the budget constraints are there. Now, whenever we are trying to say that, we are relaxing or we are changing the value of b. We are relaxing or tightening the feasible region, it means that, we are changing the available resources. And we will see, if it change the available resource, if we extend the available resource, we need to incur certain cost due to it. And we want to change for that extra unit of limited resource, how much gain I am taking out of it, that gives me the value of d f, that I am coming into next.

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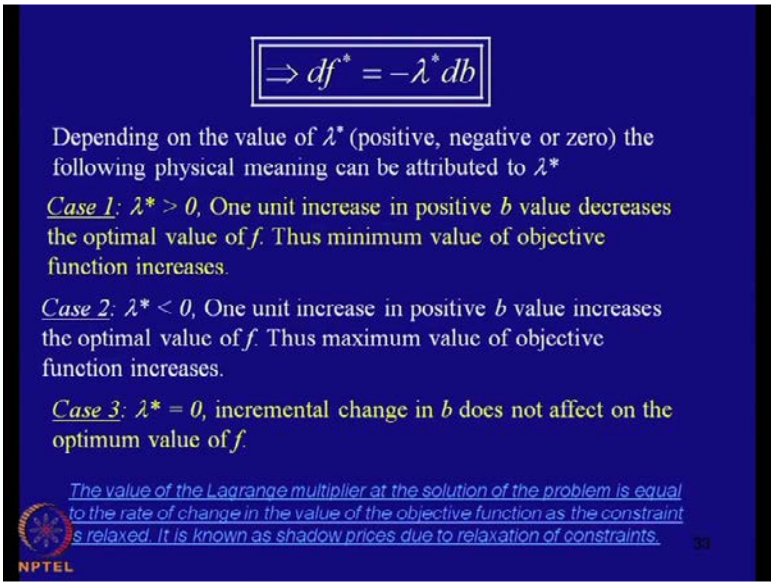

That is why we can say, at the optimal stage, where lambda star is the corresponding optimal value then, we can say that, d f star is the change in the optimal value, that would be is equal to minus lambda star d b. Then, what we can conclude here that, the values for lambda star, it could be positive, it could be negative, it could be 0 as well, that is why when lambda star would be 0, what is the meaning of it let us try to analyze. When lambda star would be negative what is the meaning of it, when it is 0 what is the meaning of that as well.

Let us try to analyze that in the next, now the case 1 if lambda star is positive then, we see, if d b is positive, that is 1 unit increase in the positive value of b, that means d b is positive. Then, we are getting the d f value will be negative, because lambda star positive, d b positive and d f star is negative. Then, what is happening, we are getting a better minimum, because the change in the objective value, functional value is negative that means, we are getting a better minimum in that case.

That is why we will see in general, for we will consider the Kuhn Tucker condition further, we will see for the minimization problem, always the lambda star will have the positive value, just we will detail in the next. But, for the timing we want to say that, if we just increase the available resource 1 unit then, the objective functional value will be decreased, that is better for the minimization of the optimization problem. And if we consider lambda star is negative then, again here if b is positive that means, if we relax the feasible region, we will see that, optimal value of the objective function will again increase.

Because, lambda star is negative, altogether it would be positive, that is why we will get the increase in the objective functional value. That means, we are getting better objective functional value in the maximization problem, that is the maximum value of the objective functional value increases. But, if we see the lambda star equal to 0 that means, there is no effect on the optimal value for the given optimization problem. From here, we can conclude that, if we want to pay extra for the available resources then, that is the limited resource.

If I want to pay more on that then, we are gaining something and how much gain we are considering, that can be easily connected with value of lambda star. Because, lambda star if it is positive then, we will get the better gain in minimization problem, if lambda star is negative, we will get the better gain in the maximization problem. Thus, in economy, this is the economic interpretation for the Lagrange multiplier, the value of Lagrange multiplier at the optimal solution is equal to the rate of change of the value of the objective function, as the constraint is relaxed and thus, it is being named as the shadow price due to the relaxation of constraint.

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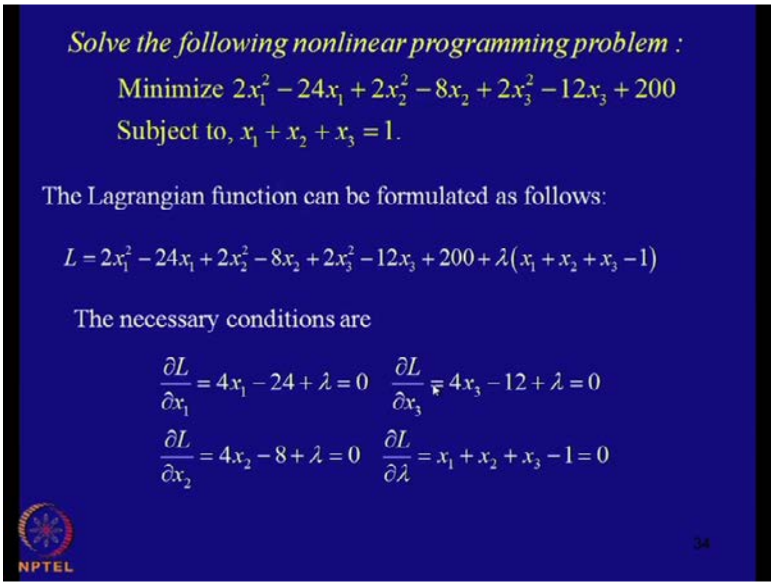

Now, this is the economic interpretation, now let us try to explain the necessary sufficient condition for the general non linear programming problem, where one constraint is involved, which is of equality type. Let us formulate the Lagrangian function, here the Lagrangian function would be the objective function plus lambda into the constraint. Again we are developing the necessary conditions by considering the first order partial derivates of the Lagrange function and we are equating to 0.

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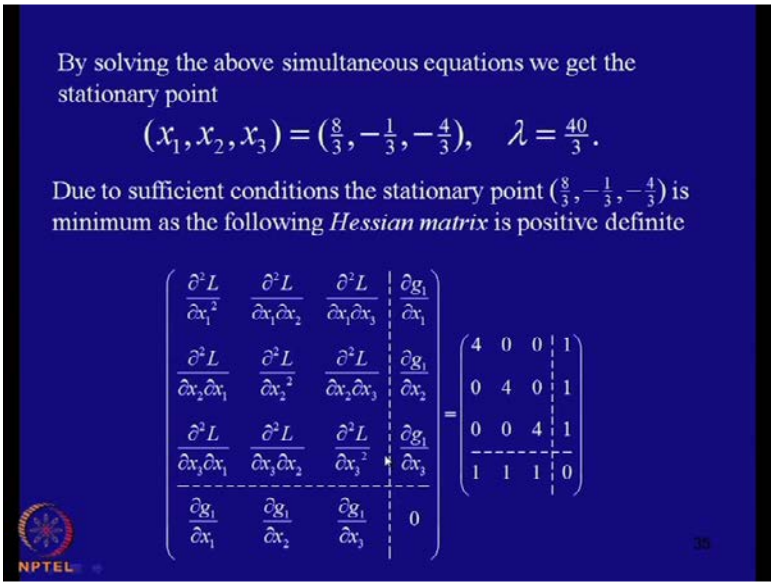

From here, we are getting the optimal values as 8 by 3 minus 1 by 3 minus 4 by 3 and corresponding lambda is 40 by 3, because we are having four equations and four unknowns, from here we are getting the optimal value for the given equation. Now, we need to check, whether this optimal value is the minimum or not. For that thing, we need to construct the Hessian matrix, for four variables this is the corresponding Hessian matrix for us, this corresponds to the matrix n as I referred before and this is the matrix of size 3 by 1, this is V, this is the matrix V T, transpose of the same matrix V.

And if we just substitute the corresponding values here, we will see that, this Hessian matrix is positive definite, because the principal minors all are positive. Thus, we can conclude that, whatever optimal solution we got for the problem, that was the minimum, that gives us the minimum value of the objective function and that is the solution for the optimization problem.

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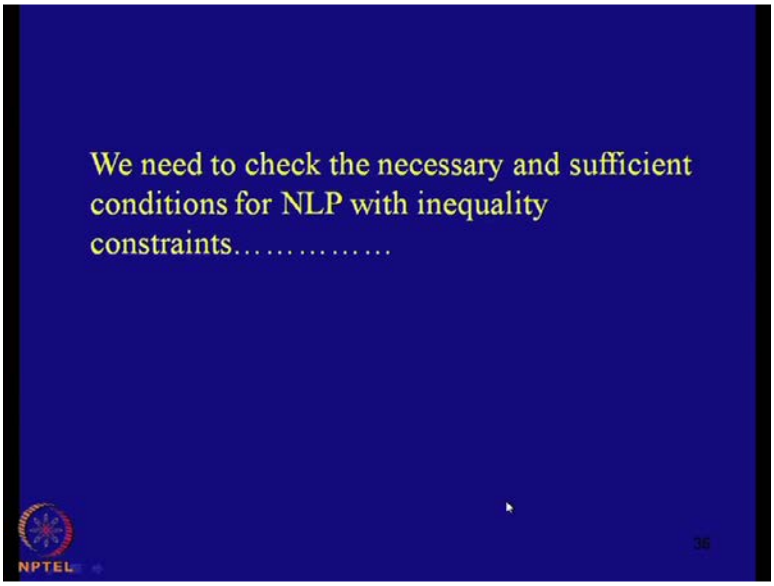

Now, we have considered the Lagrange function here, but something I would need to just mention here few things in connection to this Lagrange technique that, we have considered L is equal to f plus lambda 1 into g 1 plus lambda 2 into g 2 upto lambda k into g k. And as I said, necessary conditions will generate the values for the decision variables including the values for lambda. Now, we will see that, for the active constraints only, the Lagrange multiplier will give us the positive value for the minimization and negative value for the maximization problem.

And for the inactive constraint, the corresponding Lagrange multiplier value will be 0, we will talk on these further in the next topic when we will deal with a Kuhn Tucker conditions further. Now, in this connection, I want to say another thing as well, I want to mention that, some authors they prefer to write the Lagrange function in a different way, instead of x, instead of representing with f plus lambda 1 g 1 plus lambda 2 g 2, they prefer to write it f minus lambda 1 g 1 into minus lambda 2 g 2, etcetera.

Now, if you see, it is in that way, whatever analyses we have done, the similar analyses can be done here as well, but only changes will be there in this case, only in the sign of the Lagrange multiplier. For the minimization, we will see the lambda star would be negative in that case, positive for the maximization, that is the second thing. And another thing I just want to mention also that, as I said that at a maximum point, the gradient of the objective function and the gradient of constraint both are parallel, but the magnitudes are different.

And we will see that, the gradient of the objective function is the multiple of the gradient of the constraint, that multiplication constant is the Lagrange multiplier for this. That would be positive, that would be in the same direction in the maximum and it is in the reverse direction in the minimum, but the magnitudes are different, that I just wanted to mention here.

In the next, we will deal with the necessary and sufficient condition for the non linear programming problem with inequality constraints and in specific, the Kuhn Tucker conditions further ((Refer Time: 63:41)). And these are the references, you can refer for this kind of non linear programming problem and that is all for today.

Thank you.