Optimization
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Lecture - 12
Integer Programming- II

In the last class, we were doing the Gomory's cutting plane method for solving the integer linear programming problems, where you have seen that, whenever I am obtaining an optimal solution, but in the optimal solution in the basis, if some basic variable contains an non integer value, in that case how to handle the fractional part and how we are handling it. Today in the second lecture, we want to develop the other algorithm that is Branch and Bound algorithm for IPP, Integer Programming Problem.

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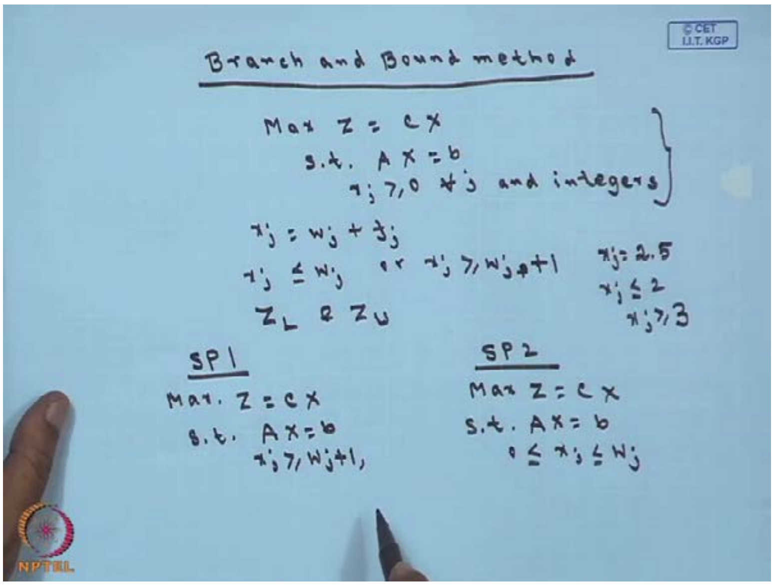

So, we are going to discuss today, the Branch and Bound method for solving the integer programming problem. You have the problem, maximise Z equals C X subject to A X equals b, your x j greater than equals 0 for all j and they are integers, this is your problem. Now, suppose in the earlier method, whatever you have seen, that you are finding out the fractional part, you are solving the problem initially.

And after that, you are seeing, if there is any non integer value then, the highest fractional part variable you are picking up then, you are adding one more constraint over there that is, which we are calling as Gomorian constraint. And you are reformulating the problem, you are deformulating the table and you are solving it and in this process, if required, you are using dual simplex method, whenever the value of the basic variable becomes negative.

Now, if you see, suppose your x j is the integer, non integer, if x j is non integer obviously, it will contain the integer part and the fractional part. So, if x j is non integer then, I can write down x j equals w j plus f j, where w j is the integer part and f j is the fractional part. So, whenever any feasible integer solution of the problem will satisfy this one, x j should be less than equals w j or x j should be greater than equals w j plus 1.

So, whenever you are having x j equals w j plus x j, where w j is the integer part and x j is the fractional part, always x j either will be less than equals w j or x j will be greater than equals w j plus 1. Or in other sense, you are getting some lower bound and some upper bound for the x j, you are getting this lower bound as less than equals w j and upper bound is greater than equals w j. And using these values, you can calculate what is the lower bound for the objective function Z and what is the upper bound for the objective function Z, so say that is, Z L and Z U.

So, if this was the original problem, now this problem whenever I am having the fractional part after obtaining the optimal solution, this problem can be divided into two sub problems. I am writing S P 1, in S P 1 the problem we will formulate like this, maximise Z equals C X, subject to your A X equals b, which we have written in the standard form. And your, one will be x j greater than equals w j plus 1, this is additional constraints you have put and obviously, the variables are greater than equals 0 and x j are integers.

So, we will solve one problem S P 1, another problem S P 2 we will formulate like this, maximise Z equals C X subject to A X equals b and x j lies in between 0 to w j. This is coming from these two conditions, one is x j less than equals w j, from here you can write down 0 less than equals x j less than equals w j, because x j is always greater than equals 0 and it is upper bound is w j. Whereas, for these case, x j will be greater than equals w j plus 1 or in other sense, if your x j is taking some value 2.5 say.

So, in that case, one value of x j will be less than equals to, the other value can be taken as x j greater than equals 2 plus 1 that is, 3. So, like this way, you are getting two different constraints for the non integer variable x j and using these two constraints, you are reformulating two problems. Or in other sense, you are breaking the original problem into two sub problems, S P 1 and S P 2, where for the non integer value, you are taking two constraints, one is 0 less than equals x j less than equals w j, the other one is x j greater than equals w j.

Now, solve both the problems, if both the problems have optimal solution, whatever problem gives you the maximum profit, that will be your optimal solution. And now, if again in one case, you are obtaining that, the non integer value in the basis of the optimal solution then, repeat this process. That means, break that problem into two parts and follow the same procedure, whatever we have told.

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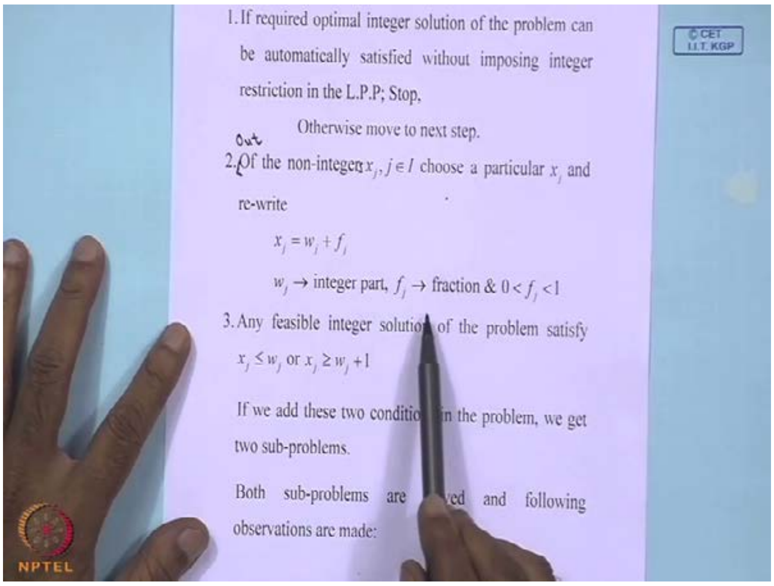

So, in general, what you can say that is, for the method of this one, the Branch and Bound algorithm can be written something like this. If the number 1 you see, if the required optimal integer solution of the problem can be automatically satisfied without imposing integer restriction in the LPP then, you stop. That means, you have the original problem, the original problem what you are doing, the original problem you are putting it in the standard form first.

Then, using the simplex algorithm, you are solving the problem without bothering about the nature of the variables, whether the variables are integer or non integer. Once you obtain the optimal solution, now check the values of the basic variable. If the values of all the basic variables are integers, you got your optimal solution, obviously you do not that to do anything and you stop the algorithm. Otherwise, in the next step you see, next step is, out of the non integers x j that is, you may have more than one variable, whose values are non integers.

So, out of the non integers x j, choose a particular x j and rewrite x j equals w j plus f j, now choose a particular x j means, I want to say here, which is having the maximum fractional part. Please note this one, whenever you are having the non integer value in that case, the non integer value means, you are having the fractional part. So, if more than one variables value is non integer, you choose that variable, which is having maximum fractional part and if there is a tie, you can choose arbitrarily any one of them.

As I have told you earlier, you see here, x j is equals to w j plus x j, where w j is the integer part and f j is the fractional part and obviously, 0 less than f j less than 1. So, any feasible integer solution of the problem will satisfy, either x j less than equals w j or x j greater than equals w j plus 1, as we have told earlier, just the other slide we have shown this one.

Now, if we add these two conditions in the problem, we get two sub problems, so on the original problem, in one problem you are adding the constraint x j less than equals w j, 0 less than equals x j less than equals w j and in the other sub problem, you are adding the constraint x j greater than equals w j plus 1. By this way, you are obtaining two different sub problems and you have to now solve both the sub problems. Whenever you are solving both the sub problems, some different cases can occur, what are the different cases, let us see.

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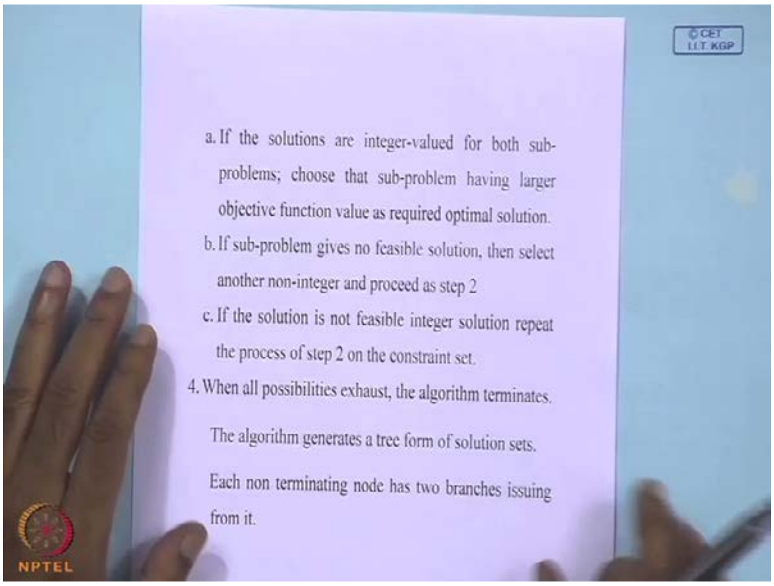

If your solution are integer valued for both sub problems, choose the sub problem having larger objective function value as required optimal solution. Please note this, if the solutions are integer valued for both sub problems, choose the sub problem having larger objective function value as required optimal solution. So, note this one, whenever you are solving two sub problems, you are obtaining two different optimal solutions. If both are optimal solutions and in the basis, the basic variables contains only integer values that means, both are optimal and the basic variable values are integer.

In that case, from the two sub problems, you will take that optimal solution, for which the value of the objective function Z is maximum, this is the easiest case that is, straight case. Number 2 b part, if one sub problem gives no feasible solution then, select another non integer and proceed as step 2. So, if your sub problem does not give any feasible solution, both the sub problems in that case. So, select another non integer means, we are assuming that, there will be more than one basic variables, whose values are non integer in the original problem.

So, I have taken the first integer, first non integer variable, from that I am posing two sub problems, S P 1, S P 2. So, by solving we are finding that, there is no feasible solution then, you take the next highest variable, whose fractional value is next variable, whose fractional value is highest. And again break it into two sub problems and proceed as step

2 means, you just break it into two sub problems and again solve it. Number c, if the solution is not feasible integer solution, repeat the process of step 2 on the constraint set.

Number c means, if the solution is not feasible integer solution, so you got the solution which is feasible, but the solution is not giving you the integer solution. So, in that case, you have the fractional values, so again repeat the step 2 means, you choose one variable, whose value is fractional, non integer and again break it into two sub problems by considering the lower bound and upper bound of the variable. As we have told, x j less than equals w j or x j greater than equals w j and then, solve the problem.

Number 4, when all possibilities exhausted then, the algorithm terminates that means, you are going on breaking the problem, a situation may occur when you are not in a position to break it into two sub problems again. But, you have not obtained any feasible solution, so you have to say that, feasible solution does not exist in that case, it is not possible to derive the feasible solution for that particular problem. So, basically if you see, the algorithm generates a tree form of solution sets.

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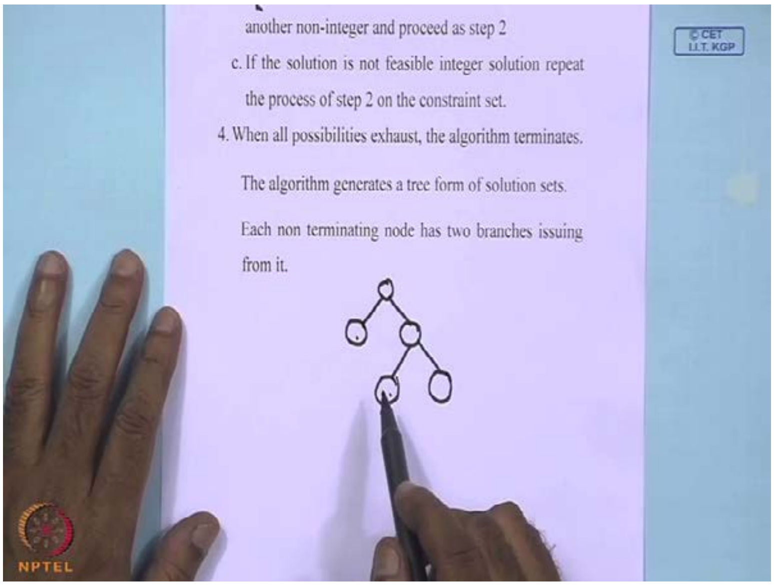

That means, you have the original problem, this original problem you are breaking it into two sub problems. It may happen that, you are breaking this into two another sub problems and like this way, you are going on in the form of tree and each non terminating node has two branches issuing from it. If some more problem is having no feasible solution, so you will terminate here just like I have terminated this one over here. So, this is the basic structure of the Branch and Bound algorithm that means, you use the bounds for the non integer variables, break the problem intro sub problems and go on repeating the process, you can obtain the solution. Now, let us see, how to solve the problem using our examples.

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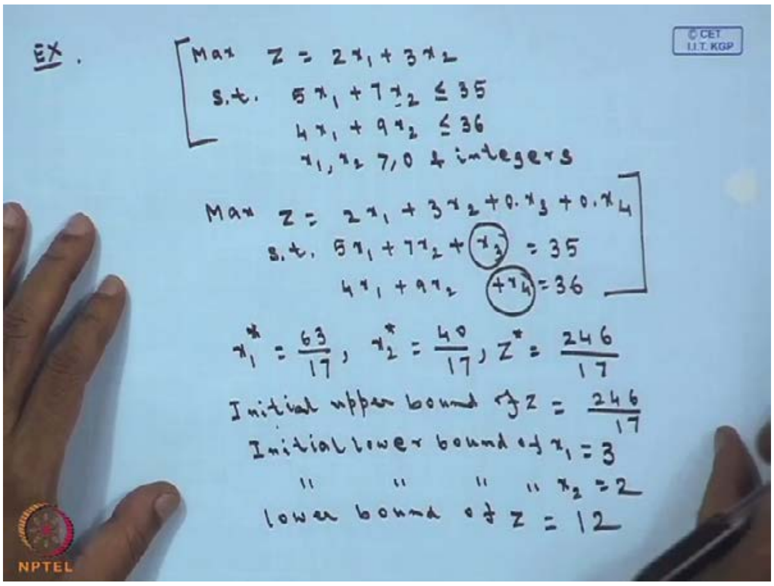

Let us take the first example say, first example is maximise Z equals twice x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2, this is less than equals 35, 4 x 1 plus 9 x 2, this is less than equals 36. Your x 1 and x 2 are greater than equals 0 and we are assuming both are integers. If you see, the problem is having two variables only x 1 and x 2, we have two less than equals inequality constraints. So, at first as usual, we have to write it in the standard form by using two slack variables, because we have two less than equals constraint.

So, let introduce two less than equals constraints x 3 and x 4 and let us write down the LPP in the standard format like this, maximise Z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2. You can write it plus x 3, this is equals 35 and 4 x 1 plus 9 x 2 plus x 4 equals 36 and obviously, we have a x 1 x 2 x 3 x 4 are greater than equals 0 and x 1 x 2 are integers. So, this is in the standard form of course, plus 0 into x 3 plus 0 into x 4, so using simplex algorithm, you can find out the solution of this particular problem and you can write down basically, what is the solution.

Now, if you note one thing here, why should I go for this one, for simplicity I am developing the simplex algorithm for this particular problem. Because, this problem consisting of only two variables, I have chooses the problem like that so that, I do not have to draw the simplex algorithm. Graphically also, I can find out the solution of this problem, because this problem consist of only two variables.

So, basically we will find out the feasible region for these constraints x 1 x 2 greater than equals 0 and this is equals 35 this first constraint equals, second constraints equals 36. Not this one, just x 3 x 4, these we will not take, I wrote it, but we will not use the simplex algorithm. We will draw the lines for 5 x 1 plus 7 x 2 equals 35 and 4 x 1 plus 9 x 2 equals 36, x 1 equals 0, x 2 equals 0.

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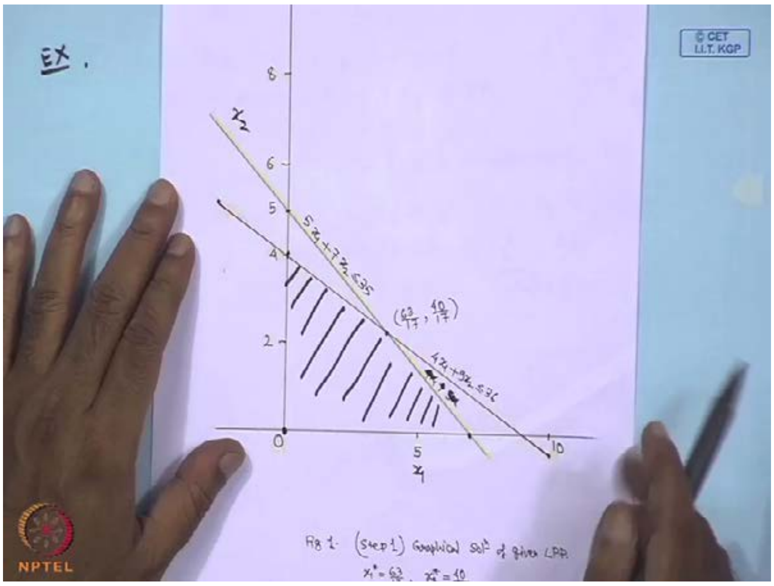

So, if you see the solution, I think it is visible, if you see your constraints one is 5 x 1 plus 7 x 2 this is equals 35, so your first line from the figure itself, this is cutting by 5, it is coming here. So, this is your 5 x 1 plus 7 x 2 less than equals 35 and this is 4 x 1 plus 9 x 2, this is less than equals 36. So, and you are having x 1 equals 0 x 2 equals 0 so that, your feasible region will become this thing. So, how to obtain the optimal solution, I think it is known to you now, at the corner points, extreme points we will find out the value of those.

The extreme points are this one, one is this point, one is this point, another is this point, another is this point, already I have written. Now, if you calculate, that you can do it of your own, you will obtain the optimal solution at the point this one, at x 1 equals 63 by 17 and x 2 equals 40 by 17, you will obtain or you will get the maximum value of Z. So therefore, you can write down the solution of this problem, not this not the standard problem please note this one, I am solving this particular problem graphically.

So, I have drawn the graph, the solution of the original problem is equals to, you see here x 1 equals 63 by 17 and x 2 equals 40 by 17. So, x 1 equals 63 by 17, x 2 star this is equals 40 by 17 and your Z star if you calculate, your Z star will be equals to, this is your 2 into 63 plus 120 into 126, so it is Z star is 246 by 17. Note, this is the solution of the original problem when you have not imposed any restriction on the variables, only restrictions you have imposed that is, x 1 and x 2 are greater than equals 0, they are positive, but they may take integer value, they may take non integer value.

If you remember, in the last class also I have told that, whenever you are not imposing integer condition, in that case when you are obtaining the optimal solution, you are getting maximum value of Z. But, by imposing the condition that the variable must be integer, your maximum value is been reduced, but you have to impose these conditions as I have told you, the example of the sort problem yesterday, number of sorts cannot be fractional.

So, my profit may be reduced little bit, but the variable values cannot be changed, so they must be integers. So, now you see here, your x 1 star x 2 star are fractional values, so therefore, the solution is not having the fractional part, they are having non integer values, which one is having the maximum fractional part, x 1 star will have the maximum fractional part, because 63 by 17, x 2 is 40 by 17. So, x 1 we will work with x 1, because x 1 is having the largest fractional part.

So, initial upper bound of Z, you can write down initial upper bound of Z is equals to this 246 by 17. When you have not imposed any condition, what is the initial lower bound of x 1, this is equals will be 17, so the fraction will come 3 plus some fraction, so lower bound of x 1 is 3. So obviously, one part is lower bound of x 1 is 3 and another one is initial lower bound of x 2, that is equals 2, because here you are having the fractional x 2 star equals to 2 plus some fractional value.

So, you got the initial upper bound of Z by imposing no integer restriction and that is the upper bound of Z. Lower bound of Z you are obtaining like this, x 1 equals 3 and the initial lower bound of x 2 is from here, 40 by 17, so lower bound is this. So, you are going to impose condition on which one, another one is the lower bound of Z in that case, you can calculate from here. Lower bound of Z will be, if you put the lower bound of x 1 value and x 2 value in this equation.

So, lower bound of Z will be 6 plus 6 that is, 12, so you got a lower bound of Z that means, you cannot go below this value, your profits should not be below 12. Similarly, your profit will never be above 246 by 17 and lower bound of x 1 and x 2 also you have obtained. Now, which variable you have to use, as I have told, the variable which is having the maximum fractional value.

Since x 1 is having maximum fractional value, therefore I will use the restriction on x 1, so this is my original problem. My original problem, now I will break it into two sub problems like this.

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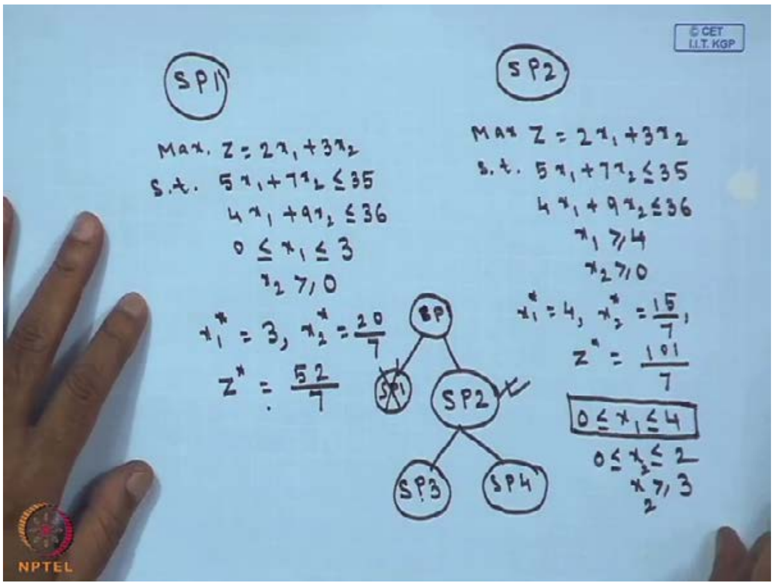

One problem I am writing as S P 1, the S P 1 will become just maximise Z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35. The next one is 4 x 1 plus 9 x 2 less than equals 36, this is your original problem. If you see, maximise Z equals twice x 1 plus thrice x 2 subject to 5 x 1 plus 7 x 2 less than equals 35 and 4 x 1 plus 9 x 2 less than equals 36. So, I have written this, now I have to impose condition on x 1, the lower bound of x is, x 1 is 3, x 1 less than equals 3, so therefore, my one condition will be 0 less than equals x 1 less than equals 3, x 2 greater than equals 0.

And so, I have not imposed, although I know the lower bound of x 2, but I have imposed any condition on x 2. We will see later why, because we have to take one variable at a time, we have to impose the restriction on one variable at a time only and that is the reason, we are not taking x 2 here. We are just saying x 2 is greater than equals 0 and obviously, x 1 and x 2 both are integers. So, this is your S P 1, what is the sub problem 2, sub problem 2, the first part will remain same that is, maximise Z equals twice x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35, 4 x 1 plus 9 x 2 less than equals 36.

What is the other condition on x 1, the other condition on x 1 is, either x 1 can take any integer value between 0 to 3 or x 1 can take any integer value, which is equals to 4 or above. So, the other condition is x 1 greater than equals 4 and your x 2 is greater than equals 0 obviously, x 1 x 2 again both are integers, which I have not written. So, I am solving this not by simplex method, because it is the easy one.

So, here these two lines, if you see the first two lines will remain same, 5 x 1 plus 7 x 2 equals 35 and 4 x 1 plus 9 x 2 this is equals 36, this two lines will remain same. Only thing, x 2 is greater than equals 0 that is, x 2 equals 0 is there, the other one you have to take is x 1 less than equals 3.

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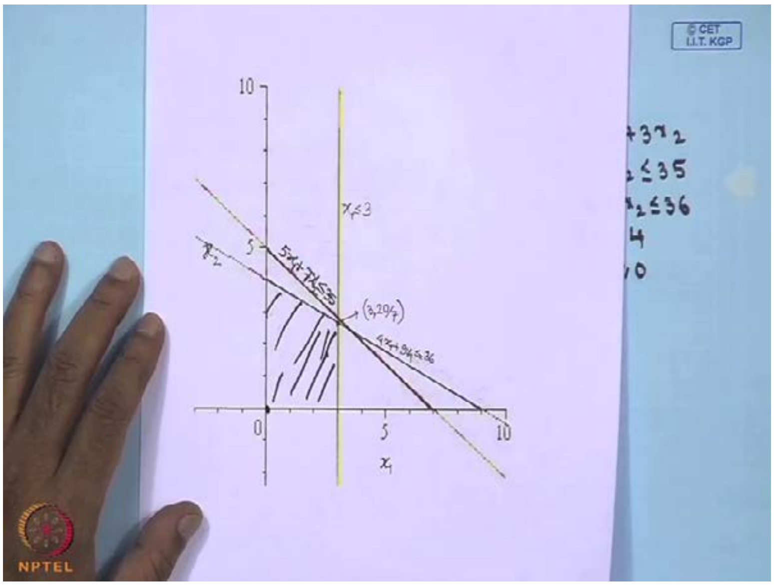

So, if you draw the graph, you will see it is coming something like this, this is your x 1 less than equals 3, so this is your condition. According to the conditions, this will be the your feasible region is coming, this is less than equals and x 2 equals 0. So, again you are calculating, since you are doing graphically the solution at the extreme points, you will calculate the value of the objective function Z and you will choose the point, which gives you the maximum value of Z.

Now, if you calculate these of your own, you will find that, you are obtaining the solution, maximum value of the Z at the point 3 comma 20 by 7 that is, x 1 star. So, this is your graph, graphically we are showing now, x 1 star is for the S P 1, your solution is then x 1 star equals 3 and x 2 star, this is equals to 20 by 7. You can calculate the Z star, Z star will be 6 plus 60 by 7, x 2 is 20 by 7, so Z 1 star if you calculate in that case, it will come as, how much it is coming, 52 by 7.

Now, on the same way, am I correct, because it is coming 6 plus 60 here that is fine, let us see. Next one is your S P 2, for S P 2 if you see, one is x 1 greater than equals 4 that is, this is the line, so again feasible region is this one. So, whenever you are calculating the feasible region, you are obtaining the solutions at this points only and you will see that, you are obtaining the solution at 4 and 15 by 7 that is, x 1 equals 4 and x 2 equals 15 by 7, you are obtaining the solution.

So, for S P 2, your x 1 star equals 4, your x 2 star is becoming 15 by 7 and your Z star will be 101 by 7. If you note one thing, in the earlier solution, in the original problem if you see, when you are doing the original problem, the x 1 value, the value of the variable x 1 was non integer. Now, when you have broken into two sub problems and after that, you are finding the optimal solution, you are finding that, one out of the variable x 1, on which you impose the restriction, now it has become integer value.

So, this is the beauty of the Branch and Bound algorithm, you have to simply breaking into smaller sub problems and you are imposing conditions. So, here your x 1 variable in both the sub problems if you see, have taken the integer values. For the sub problem 1, the value is 3 for x 1, whereas for the sub problem 2, value of the variable is 4 for x 1. So, both are having now if you see, the non integer value in another variable that is, one x 2, so the solution are optimal, but not feasible.

So, what I have to do, out of these two sub problems, I have to choose now one problem, which one I should choose, I should choose the sub problem, for which the value of the objective function is maximum. Please note this one, if I have broken the sub problem, the original problem into two sub problems, whenever I am breaking it, I am finding the optimal solution. Now, both the solutions are optimal solutions, but they are not feasible, in that case I will discard one sub problem and I will take only the other sub problem.

I will discard the sub problem, which is having the lower objective function value or less objective function value. So, for this problem, we will discard the S P 1, because it is objective function value is 52 by 7, whereas I will take the value, the second problem S P 2 whose objective function value is 101 by 7. So now, what I have to do, I have to now check S P 2, for S P 2 I have only one variable, whose value is non integer.

So, I have to impose condition on S p 2 only or in other sense, basically now what I will do, I will break this S P 2 into two sub problems like this, S P 3 is one and another one is S P 4. So, if you see, you was having S P original problem, this you broken into two parts, S P 1 and S P 2. Now, you are discarding this S P 1 and you are choosing S P 2 and you are breaking it into two, you have to break it into two parts, S P 3 and S P 4. So, Z, now your x 2, I have to impose restriction on x 2 star.

So, note one thing, your x 1 value is 4 that means, x 1 cannot take any integer value beyond 4, it cannot take any integer value beyond 4 So, the restriction on x 1 will be, it must lie between 0 to 4 and what is the bound for x 2, x 2 is 2 plus some fractional value, 2 plus 1 by 7. So, lower bound of x 2 will be equals to 2 or in other sense, your x 2 will be this one, lower bound of x 2 is this and upper bound that is, either x will be lying in between 0 to 2 or x 2 will be greater than equals 3.

So, for x 1, you have fixed a limit now, which has become integer after breaking the original problem into two sub problems. Then, you got one variable as non integer, on that you are imposing the condition from the value of the lower bound. So, 0 less than equals x 2 less than equals 2 or x 2 can take values, which is greater than equals 3. So, once we have done, now I can write down, what is my S P 3 and S P 4.

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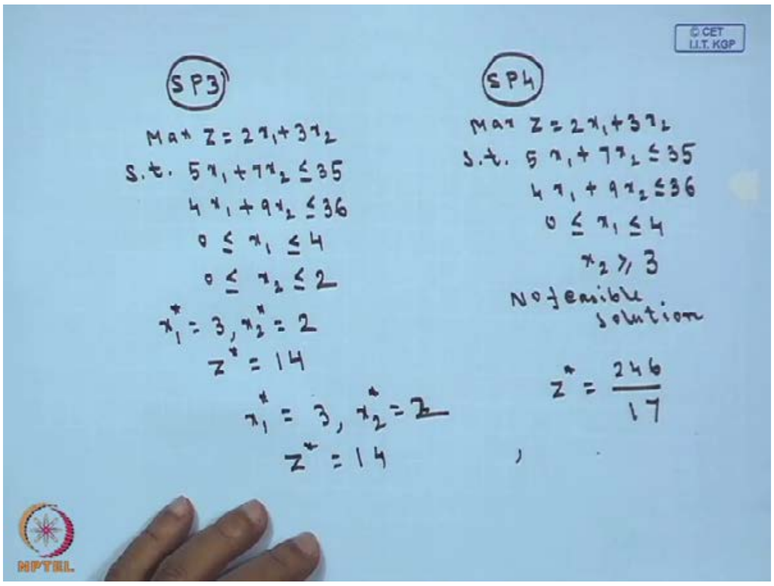

What is your S P 3 now, S P 3 is maximize Z equals 2 x 1 plus 3 x 2 subject to 5 x 1 plus 7 x 2 less than equals 35 and 4 x 1 plus 9 x 2 less than equals 36, this was the original constant. Along with this, now you are imposing one restriction on the x 1 that is, x 1 lies between 0 to 4 and your x 2 lies between 0 to 2 and obviously, x 1 and x 2 are integers. So, this is the sub problem 3, which you have broken from S P 2 and what is the sub problem 4.

Your sub problem 4 is maximize Z equals 2 x 1 plus 3 x 2 subject to again first two constraints will remain same, 7 x 2 less than equals 35, 4 x 1 plus 9 x 2 this is less than equals 36, condition restriction imposed on x 1 will remain same that is, x 1 lies between 0 to 4 only. But, the restriction on x 2 will be in this case greater than equals 3, x 2 is greater than equals 3 and again x 1 and x 2 are both are integers. So, again graphically, I can obtain the solution of this particular problem.

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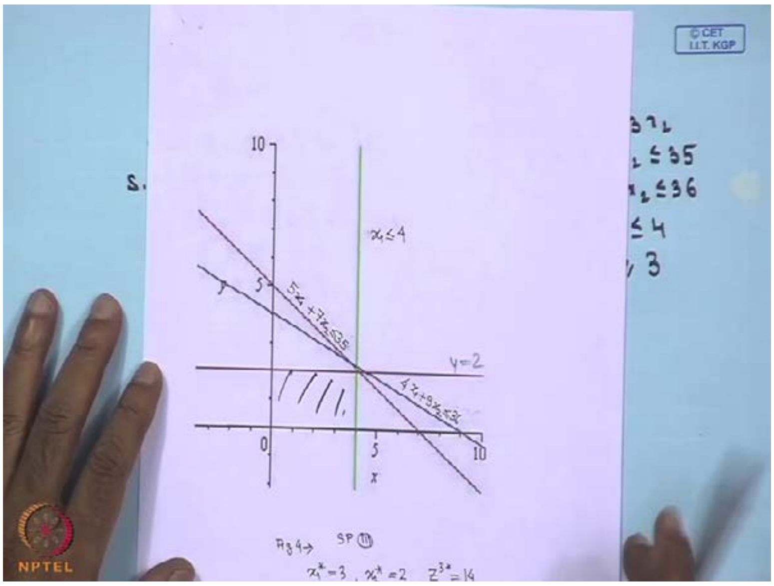

For the sub problem 3, I can find out the solution something like this, these two lines are there, x 2 less than equals 4 and y equals 2. So, the solution space obviously will come in this region, here for this case the solution point will be x 1 equals 3 and x 2 equals 2. Just see the figure, from here you can check it, at the point where you are getting the solution. So, your solution for S P 3 will be x 1 star this is equals 3, x 2 star this is equals 2 and your Z star equals 14.

Whereas, if you take the S P 4, where this one is lies between x 1 is fine less than equals 4, but your x 2 that is y, that is greater than equals 3 that is this one, here not y, this is x 2 greater than equals 3, that is above this, you see this one now.

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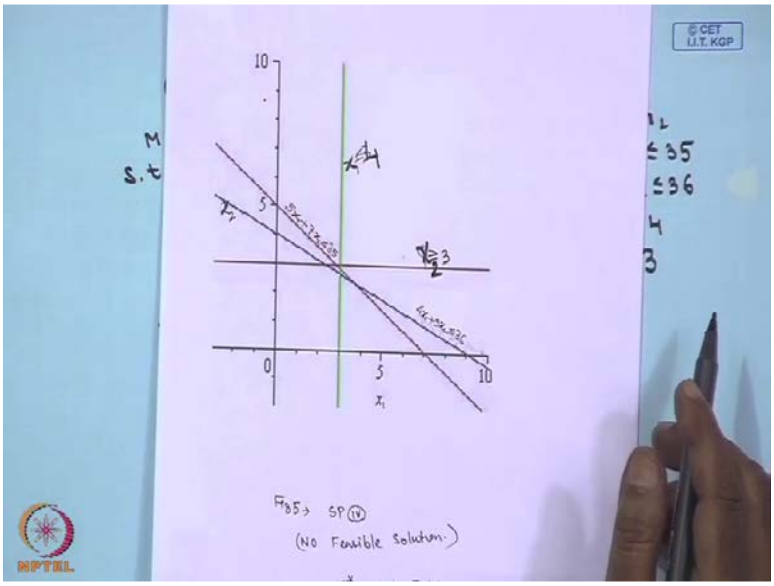

So, one is x less than equals 4 and above this, can you find out any feasible solution here from the graphically, I am not discussing this spot again, there is no feasible solution for this one. Or in other sense, Z star does not exist, from the graph it is very clear, you can check it of your own also. Since we have discussed this things earlier, so I am not going into details of this, so for this one I can say that, no feasible solution. And for the problem S P 4, the objective function does not exist, so what you are finding now, you see here, you broke the sub problem 2 into two sub problems, S P 3 and S P 4.

Now, for S P 3, you are getting integer solutions, whereas for S P 4, you have no feasible solution. Therefore, ultimately your solution you can write down, your solution of the problem is x 1 star equals 3 x 2 star equals 2 and Z star this is equals 14. Whereas, if you do not impose the condition of integer values, you got the value of Z star as 246 by 17. So, if you see this one, in that case you are obtaining the less value for this case, but as I told you, there are situations where you cannot afford to take non integer values, because that cannot be non integer.

So, although your profit is little bit lower, but you are taking the values of the decision variables as integer only.

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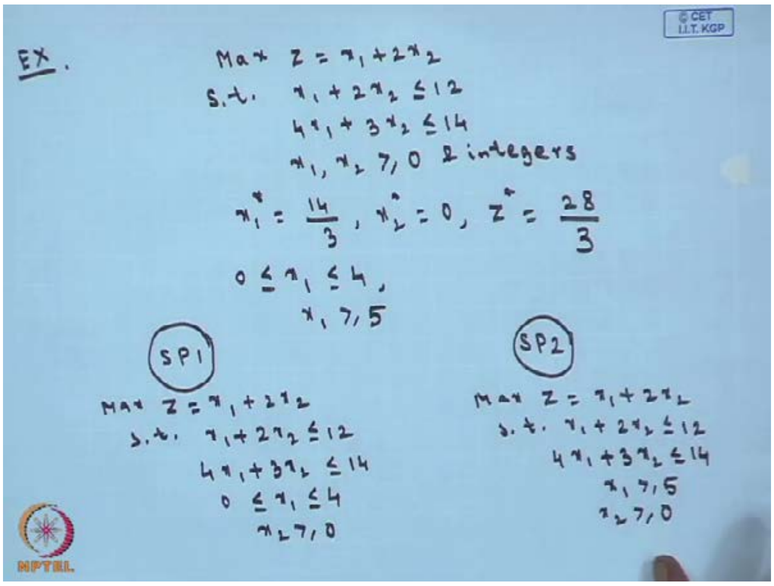

I think it is almost clear, just let me do one more problem on this, that also I will solve it graphically so that, it becomes easy for us and we can solve it very easily. Maximise Z equals x 1 plus 2 x 2 subject to x 1 plus 2 x 2 less than equals 12, 4 x 1 plus 3 x 2 less than equals 14 and x 1 and x 2 both are greater than equals 0 and obviously, and they are integers, this condition we are imposing over here, they are integers.

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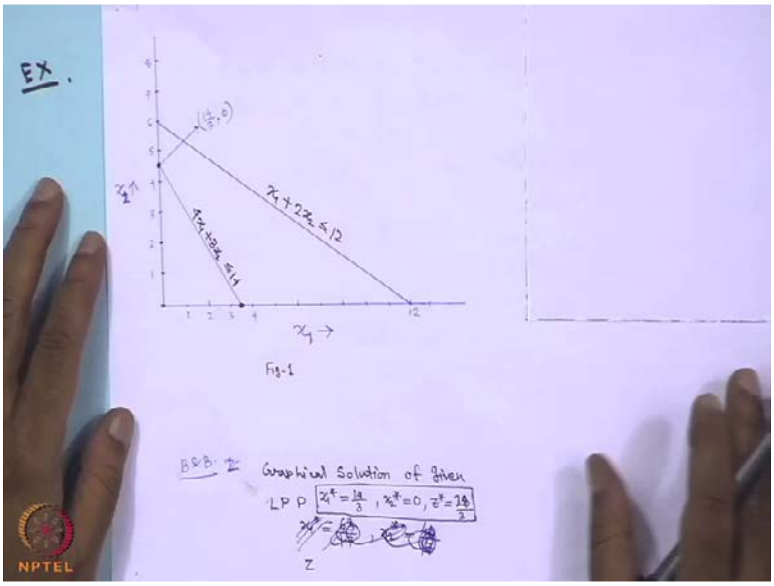

If you see the graphical solution, again I am going to the graphical solution, I have written over here, this is the line 4 x 1 plus 3 x 2 less than equals 14, x 1 plus 2 x 2 less than equals 12. So, these are the extreme points and you will obtain the solution at the point 14 by 3, 0. So, the solution of this is x 1 star equals 14 by 3, x 2 star equals 0, so what you are obtaining basically. For this problem, your x 1 star is 14 by 3 and x 2 star is equals to 0 and Z star this is equals to 28 by 3.

So, since x 1 star is 14 by 3, therefore what you can do now, since it is non integer, therefore for x 1, we will have a lower bound that is, for x 1 what you are getting, the lower bound is 0 less than equals x 1 less than equals 4 and the other one can be x 1 greater than equals 5. So, once you are obtaining these two, so you can break now it into two problems, S P 1 and S P 2. Your S P 1 will remain, the first condition will remain same that is, Z equals x 1 plus 2 x 2 subject to x 1 plus 2 x 2 less than equals 12, 4 x 1 plus 3 x 2 less than equals 14.

And you are imposing one condition, 0 less than equals x 1 less that equals 4, x 2 greater than equals 0, both are integers, this is the sub problem 1. Sub problem 2 will be x 1 plus 2 x 2 subject to x 1 plus 2 x 2 less than equals 12, 4 x 1 plus 3 x 2 less than equals 14 and x 1 greater than equals 5 x 2 greater than equals 0. So, again you can solve these two sub problems and you can repeat the process, graphically you can see it and ultimately you can obtain the solution.

I am not going to the solution, because you can check it of you own, but I think now the problem or the methodology is very clear to you, I am not completing this particular problem, you can complete it of your own. Now, let us come to the other part that is, we were talking about the variables, which can take integer values. Now, there may have situations, where the variables can take the real value as well as integer value, this type of problem if you remember, yesterday we have told in the last class that, mixed integer programming problem.

So, let me briefly tell you, how to solve the mixed integer programming problem that is, when the decision variables x j can take integer value as well as it can take non integer value. Now, the Branch and Bound method what I have discussed just now, I can use the Branch and Bound algorithm to solve a mixed integer programming problem also by the mechanism we have discussed. Using Gomorian cut also, we can solve the mixed integer programming problem and little with some modifications. So, I am now telling you briefly, the solution procedure of mixed integer programming problem by Gomorian cut method, let us see what happens.

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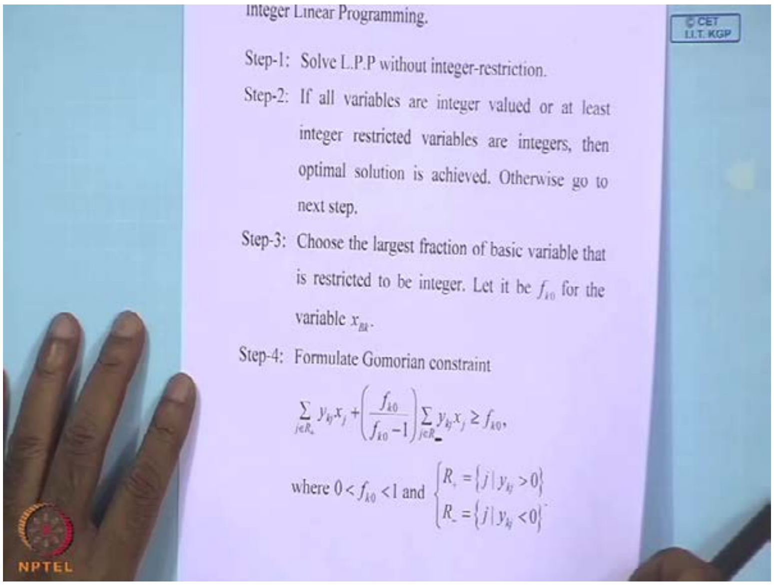

If you see here, if only a proper subset of decision variables in an LPP is restricted to integer, the problem is known as mixed integer linear programming, only a proper sub set, not all the variables, please note this one. How to solve it, step 1, solve the LPP without integer restriction that is, you are solving it without any restriction that some of the variables must be integers. Step 2, if all the variables are integer valued or at least integer restricted variables are integer then, the optimal solution has been achieved.

Means, if for the all the variables, optimum value feasible value is integer then, no problem or the variables, whose value must be integer. If you obtained the optimum solution, their value has integer only then, you got the optimal solution and you should stop now. Otherwise, you have to go to the step 3, I am doing please note the Gomorian cut method. In step 3, choose the largest fraction of the basic variable, that is restricted to be integer that is, you may have some very basic variables, whose values should be the integer, from that you choose the variable, whose fractional value is largest.

Now, I am telling, let it be f k 0 for the variable x b k, since already I have discussed the Gomorian constraints, so I am directly writing here summation over j belongs to r plus y k j x j plus f k 0 by f k 0 minus 1 into summation j belongs to r minus y k j x j greater than equals your f k 0. This is the value of the, f k 0 is the value of the basic variable, obviously f k 0 lies between 0 if f k 0 lies between 0 and 1, where your r plus is j such that, y k j greater than 0 that is, in the row value of the variable is greater than 0 and r minus denotes value of the variable, which is less than 0.

If you remember, in the pure integer problem, we have not considered the case or we have not considered the variables, for which the restriction was the greater than 0. If the value of a variable is greater than 0, in that case we have not considered this, we have only took the coefficients of the variables, whose values are less than 0 that is, one restriction we have imposed over here, one modification we have made. Another condition we have imposed that is, f k 0 by f k 0 minus 1, this has come.

Now, in the last case, we derived how you are getting it, now we are not deriving you can go through or you can see the books, you will find it that, this constraint we have added over here. So, whenever you are finding a variables x b k, whose corresponding b value is f k 0, you are formulating Gomorian constraint by this formula.

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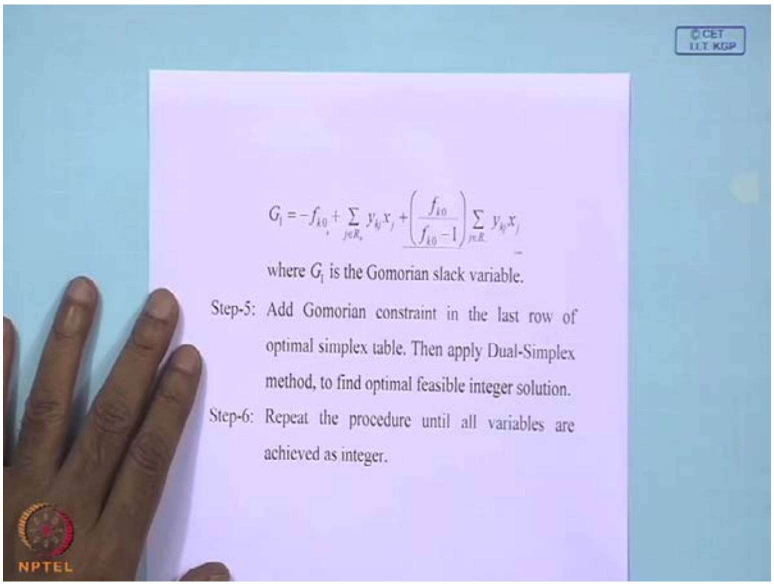

And once I have done this, now using the Gomorian slack variable, this is greater than equals 0 is there and I can write it into equality constraint G 1 equals this, where G 1 is the Gomorian slack variable. Now, like earlier in the last lecture we have discussed, in the optimal table at this Gomorian constraint in the last row of the optimal simplex table that is, this Gomorian constraint will come in the last row of the optimal simplex table, f k 0 will be the value of the b value, this will be the b value others will come.

And since b value is negative, obviously you will use the dual simplex method to find the optimal feasible integer solution and this process will be repeated until all variables are becoming integers. So, the other part will remain the same, as we have done for the Gomorian case for pure integer programming problem. Only thing, the Gomorian constraint is being changed or modified over here like this, let us take one example and let us see, how we can solve the problem.

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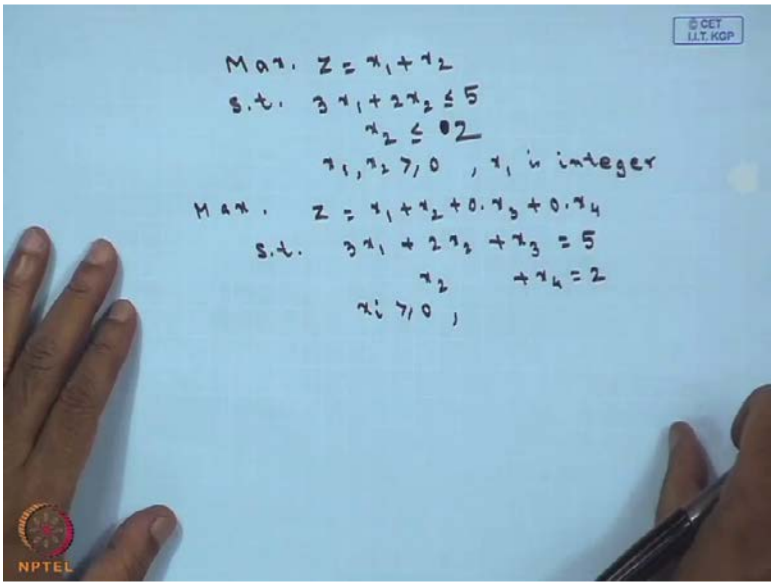

Let us take the problem maximise Z equals x 1 plus x 2 subject to 3 x 1 plus 2 x 2, this is less than equals 5. One condition is there, x 2 less than equals 2 and x 1 x 2 greater than equals 0 and x 1 is integer. Please note this one, x 2 is less than equals 2, x 1 x 2 greater than equals 0, x 1 is integer that means, x 2 must be positive, but it can take any value that is, it might take integer value, it might take real value also. So, we call this type of problem as the mixed integer programming problem.

Now, if you see, you can use or you can solve this problem by Branch and Bound method also that is, since it is a objective function, is a function of two variables, graphically also I can do it without using the simplex algorithm. But, already you can check it of your own, I will solve it by the Gomorian cut method, so I have to use the simplex tables. So, at first, we are writing it in the standard form like this by introducing the slack variables, maximise Z equals x 1 plus x 2 plus 0 into x 3 plus 0 into x 4 subject to 3 x 1 plus 2 x 2 plus x 3 this is equals 5 and x 2 plus x 4 this is equals 2.

Obviously, your x i greater than equals 0 and x 1 is integer, I am not writing that, x 1 is integer, so this is the initial or standard form. So therefore, in the basis, x 3 and x 4 will enter, they are forming the identity matrix.

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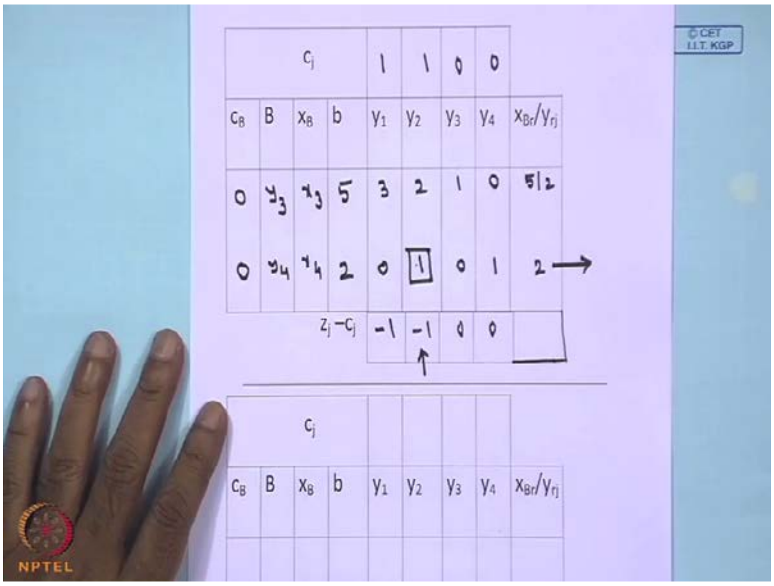

So, if you see here, this is your problem, so in the b value, you will have y 3, you will have y 4, so here you are having x 3 and x 4, c j values are 1 1 0 and 0. So, you are writing 1 1 0 and 0 so that, your C B value will become y 3 corresponding to y 3 and y 4 is 0, b values for these variables are 5 and 2 respectively. Now, write down the rows that is, 3 2 1, x 4 is 0, so it is 3 2 1 and 0, the next one is only you have x 2 and x 4, so x 1 and x 3 will be 0, and x 2 and x 4 corresponding coefficient is 1.

So, from the problem we are forming the first simplex table, once you are forming it, now you know the standard process that is, you have to calculate z j minus c j. z j minus c j means, 0 plus 0, 0 minus 1, this is again 0 plus 0 minus 1, this is 0, this is 0, so both are having same maximum value, I am choosing these, we are not going into details, how we are choosing it. So therefore, for this one it is 5 by 2 and this is your 2, since it is 2, therefore your outgoing vector is x 4 and incoming vector is x 2. Therefore, you are having this one, your pivot element is this that means, I have to make this one as 1 and this one as 0.

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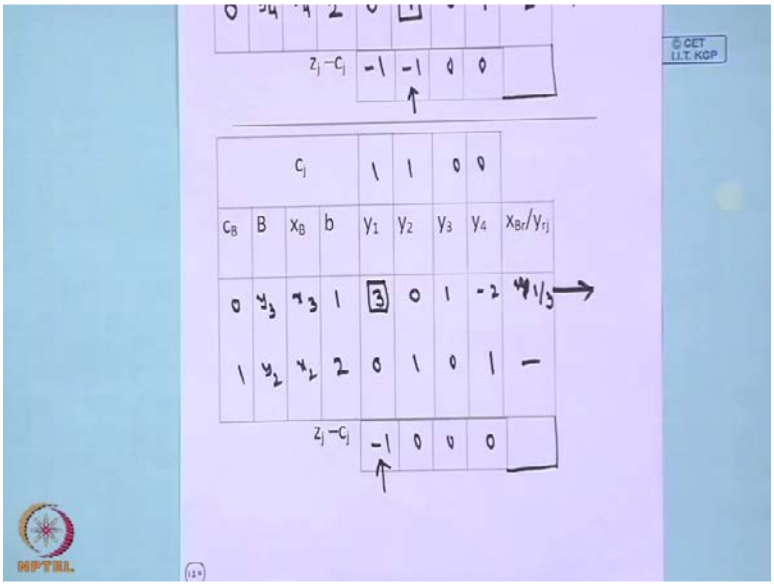

This is again the standard form, so this is your y 3, this is your y 2, so x 3 x 2 this is 1 1 0 0, so y 3 0, y 2 1, directly I am writing, what values you will get, you will check it of your own, 1 3 0 1 minus 2 and this is your 2 0 1 0 1. So therefore, now calculate your z j minus c j, your z j minus c j will be minus 1, all three will be 0. So, your y 1 will enter, for this one this is 1 by 3 and this is your infinite, so your x 3 will be out and x 1 will come, so your pivot element is becoming this one.

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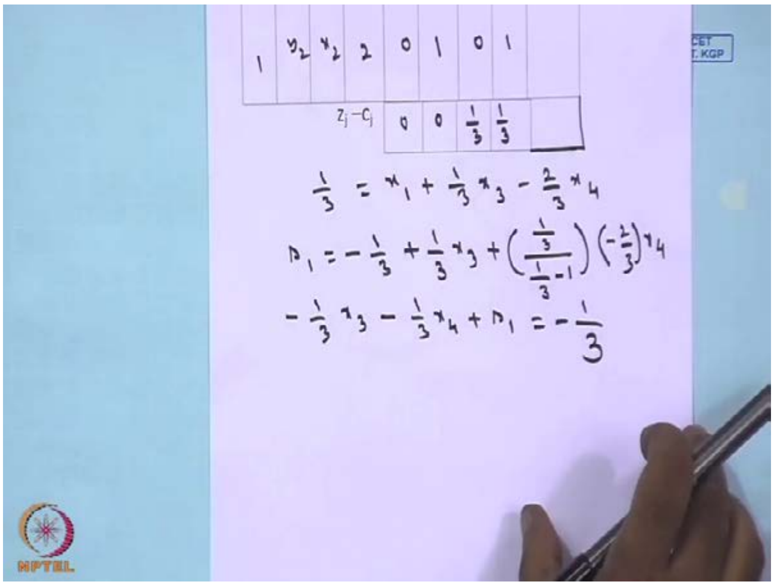

So therefore, from here you are obtaining, you can formulate the next table that is, y 1 and y 2, you are having x 1 and x 2. So, it is 1 1 0 and 0, so this is 1 and 1, again I am writing the values, one third 1 0 one third minus two third, this is 2 0 1 0 and 1. If you calculate the z j values then, it will be 0 0 one third one third, so z j minus c j is greater than equals 0, but the solution is not feasible, because x 1 is having fractional value. X 1 means, this is basically one third equals this row represents, one third equals x 1 plus one third, x 3 that is one third equals x 1 plus one third x 3 minus two third x 4.

So, what will be the, using the Gomorian cut again I am writing directly, s 1 will be equals to minus f 1 0. That is, minus one third will come over here, minus f 1 0, f 1 0 is one third plus f 1 3 x 3, f 1 3 x 3 is one third into x 3 plus your f 1 0 by f 1 0 minus 1 that is, one third by one third minus 1 into your minus two third that is, coefficient of this one, so this part has been added over here into x 4. So, if you write it as minus one third x 3 minus one third x 4 plus s 1 equals minus one third. So, this is the another constrain, Gomorian constraints you have to add on this table only.

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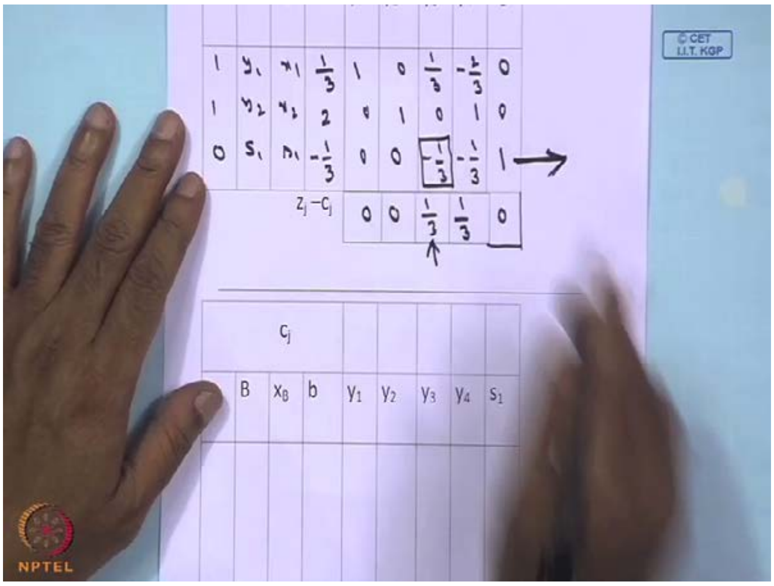

So, whenever you are adding this, you will obtain the next table as you will have three variables in the basis, y 1 y 2 and S 1, so it will be x 1 x 2 and s 1. The values here your S 1 is coming coefficient is 0, C B will be 1 1 0, first two row will come from the last optimal table. So, just we are writing it, minus two third, S 1 will be 0 then, this is 2 0 1 0

1 and 0 and this coefficients now will come, your b value is minus one third. If you see here, your b value will be minus one third over here.

So, you are writing minus one third 0 0 minus one third minus one third and S 1 is one, calculate the z j minus c j value again, z j minus c j value if you calculate, 0 0 one third one third and 0. Again you are having, here if you see, you are having one negative value, so you have to use the dual simplex method. So, this will go out, if you compare this will be going out and from here, you are having only one negative value and therefore, you are getting this one as the entering vector. So therefore, your S 1 is going out and y 3 will enter over here.

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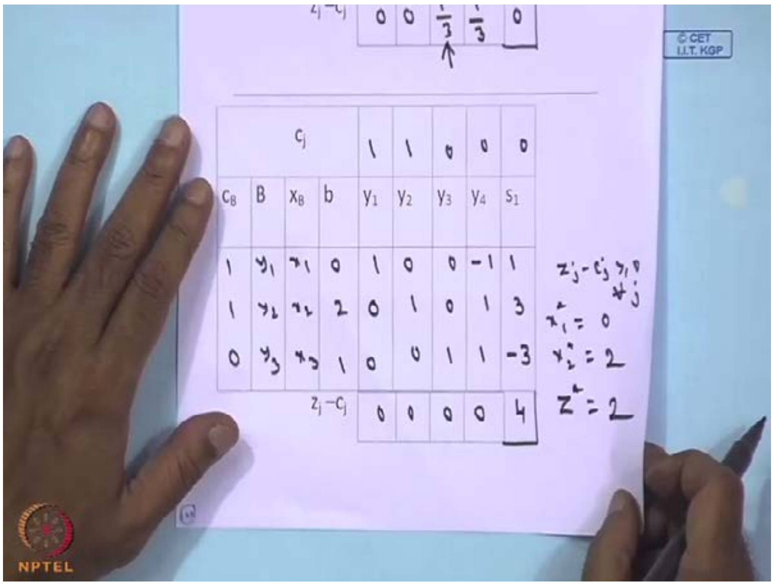

So, here you are having now y 1 y 2 and y 3 x 1 x 2 and x 3, c j values you are writing 1 1 0 0 and 0, this is 1 1 0. I am writing the solution directly, 0 1 0 0 minus 1 1 2 0 1 0 1 and 3 1 0 0 1 1 and minus 3. If you calculate the z j minus c j values, you will find 0 0 0 0, the last one will be 3 plus 1, 4. So, you see here, your z j minus c j are greater than equals 0, for all j and the value of the decision variables x 1 star, this is equals 0, x 2 star equals 2 and both are integer values.

So, you obtained your desired solution, your Z star if you calculate, it will be 2, so by this Gomorian constraint, by modification of the Gomorian constraint, if you are having the mixed integer problem, you can solve it like this way. Otherwise, you can use the Branch and Bound algorithm also to solve the problem, I think it has been understood.

Thank you.