Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 9
Multi-Step Methods (Explicit)

Good morning, till last class we have discussed single steps methods to solve initial problems. And we have reviewed some of the methods and then try to attempt some problems. So, in the single steps methods what we have seen to compute value at a particular lid point. We need one value at one past point. So this is single steps method. That is means to compute y of x n plus 1, we need a y of x n. So, as the name suggest as in multi step method maybe we could expect that instead of one past point we need a several past points. So, the further these multi steps methods can be classified into different varieties. So, let us look at into these multi step methods.

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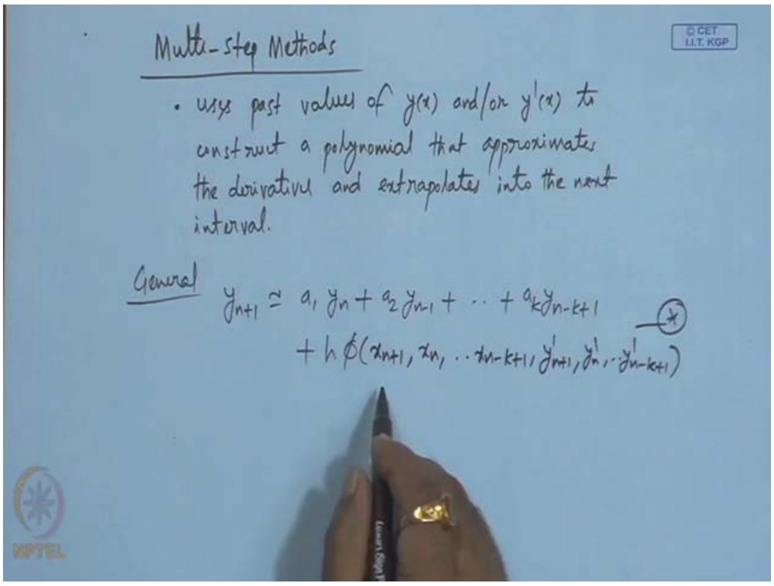

So, multi step methods. So, what it does this uses past values of y of x and or y dashed of x to construct a polynomial that approximate the derivatives, and extrapolates into the next intervals. So, what it does? It uses past values of y x and or. That means could be sometimes y dash or sometimes just y alone. To construct a polynomial that approximates the derivatives and extrapolates into the next interval.

So, what is the general, the general methods? y n plus 1 plus h times some processor phi x n plus x n k plus 1 then the derivative terms. So, look at it the value of n plus 1 stage demands values at n, n minus 1, n minus k plus 1. Plus also demands the derivative values these points. So, this is more general case right? So, this can also be put it in a little different sums as follows.

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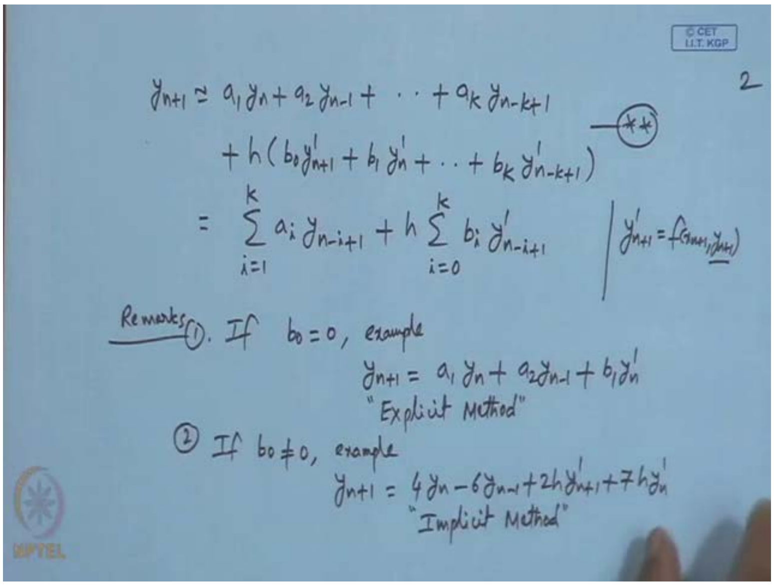

So, this is functional use at past points plus h. So, explicitly I am writing the phi. So, which can be written as… So, here i is 1 to k, here we are including 0 as well. So, for the derivative. So, this the more general method. So, once we write such a general method few remarks. If b 0 is 0. Example if b 0 is 0, example y n plus 1, a 1 y n plus a 2 y n minus 1 plus b 1 y n prime, if b 0 that is b 0. So, look at it what is b 0 is coefficient of y n plus 1 prime. And we are trying to compute y n plus 1. Now what is y n plus one prime? X n plus 1.

So, it demands y n plus 1 that means to compute y n plus 1 right hand side demands y n plus 1. So, now if b 0 is 0, then the right hand side does not demand y n plus 1. So, to compute y n plus 1, y n, n minus 1, n minus k plus 1. And since b 0 is 0 we need from y n prime up to y n prime n minus k plus 1. So, that means if you know the past values n to n minus k plus 1, 1 can compute y n plus 1.

Hence this method is explicit method, this method is explicit method. Now if b 0 is non 0. Say example y n plus 1 is say some 4 y n minus 6 y n plus 1 plus 2 h y n plus prime plus 7 h. So, look at that. We are computing y n plus 1 and right hand side demands y n 1 1, because to compute y n plus 1 prime we need y n plus 1. So, that means this is implicit method why? Because right hand side… See you are computing y n plus 1, but right hand side also demands y n plus 1. So, therefore, this is implicit method. So let us try to discuss explicit methods. So let us start explicit method and try to understand.

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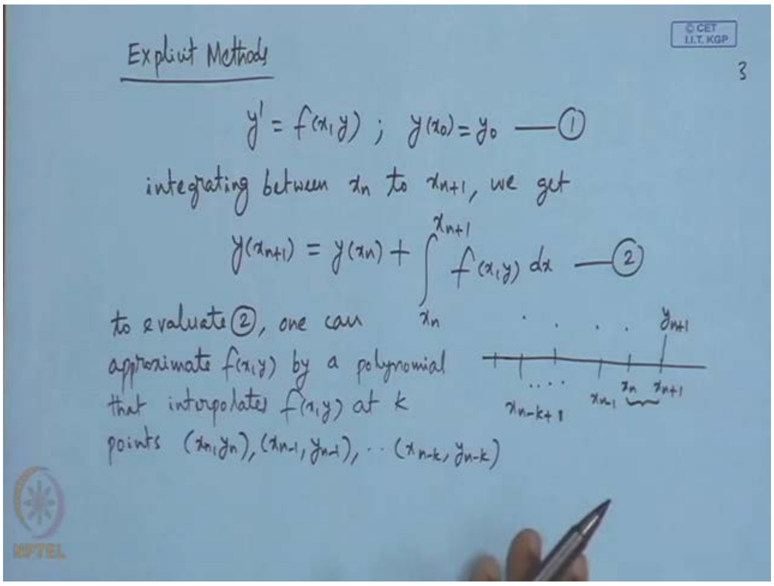

So, explicit methods. So, consider our initial value problem with. Now integrating between one interval that is x n to x n plus 1 we get. So, what we are done? So, we have integrated between just one interval. So, the method is. Suppose this is x n, so x n minus 1 n minus k plus 1. So, the method we are talking about explicit methods, that means to compute y here, so we would like to compute y there. So, we need we need the past points y. x n, x n minus 1, x n minus k plus 1 right? So, for that what we are doing we are just integrating between only one step.

Now where are we trying to use the past points? So, the past points look at this. Now we have to approximate f by a suitable polynomial. So, here we will use this past points. So, what we do? To evaluate two one can approximate f of x y by a polynomial that interpolates f of x y at k points. What are they? x n, y n, x n minus 1, y n minus 1, x n minus k.

So look, we have to evaluate this integran the integration so integration. So, one can approximate by a polynomial that interpolates f at k point. And what are they? x n minus

1, x n minus k. So, these are all past points so there is nothing implicit everything is explicit. Now how do we approximate? One you can use forward or backward or difference formulas right? So let us try to use one of them to approximate f.

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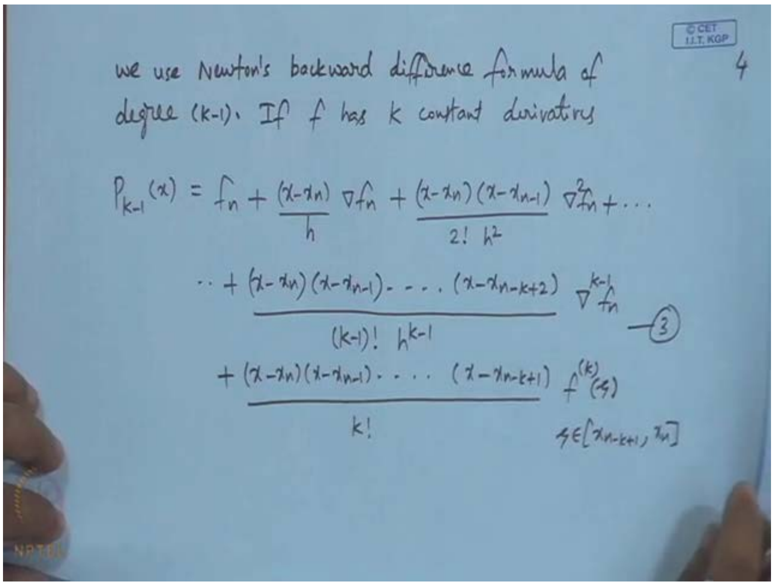

So, we use Newton's backward difference formula of. Well if there are k past points what will be the degree of the polynomial that could be approximated k minus one? Now in order to use this we have to assume the that. If f has k constant derivatives or polynomial of degree k minus 1 will be f n plus x minus x n by h delta f n, x minus x n, x minus x n minus 1 factorial 2 h squared. Then the second differences plus. x minus x n minus 1 n minus k plus 2. This k minus 1 turn h power k minus 1 and this. So this is backward operator where, this is backward operator, so that is this on f n backward operator. Now plus k factorial and this, this is the reminder term. So, this where f k is the k th derivative of f evaluate at some zeta in the interval.

So look at it, we have used Newton's backward difference formula and this is given by, this is given by f n plus x minus x n by h. This is a first differences second order differences that ((Refer Time: 19.43)) order differences and this is the k minus 1 term. Then k th term is the reminder term that involves derivatives of f k thought derivative of f. Now our task is that to simplify further and substitute this in the integrant. So before we substitute let us try to simplify this further. So note that you have... See p minus k 1 is a function of x right. So you have this variables x minus x n your variable is now here x that is x minus x n, x minus x n, minus 1. So, 1 so far so we try to introduce change of variable as follows.

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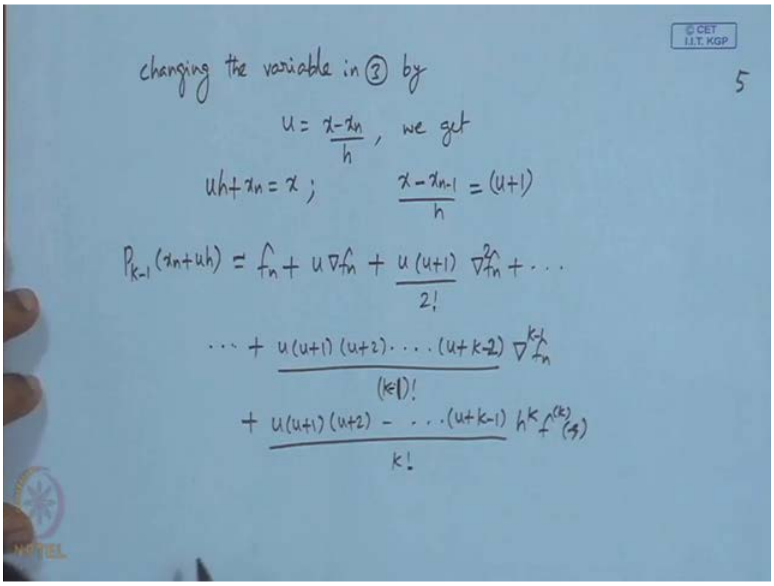

So, what is the change of variable? Changing the variable in three by u equals to x minus x n by h. Look at that. So you have backward Newton difference formula, Newton's backward different formula. So this is x minus x n by h we are transforming it to new variable u. Hence we try to convert everything in terms of u. So if, note that if this is u what will be x minus x n minus 1. What will be that? Let us see this is… that means u h plus x n is x. So, we need a next term is x minus x n h. So we need to compute this.

So what will be this to this notation? This will be u plus 1. So, you can verify. So, hence with this our polynomial become p k minus 1 x n plus u h because x is this. Equals f n approximate u times first different. So, what could be the next term? Again we go back. See this is x minus x n by h is u, so x minus x n by h is 1 u. Now we have x minus x n minus 1 by h. So that must be u plus 1.

So u, u plus 1 by 2 factorial second order differences. Now the next term u plus k minus 1 k minus k minus 1 factorial, k minus 1 factorial, this will be k minus 2 because we need a one previous factorial term. So u, u plus 1, plus 2, plus k minus 2, k minus 1 factorial, k minus 1, f n plus the next term k factorial, two h power k. Look what did we do?

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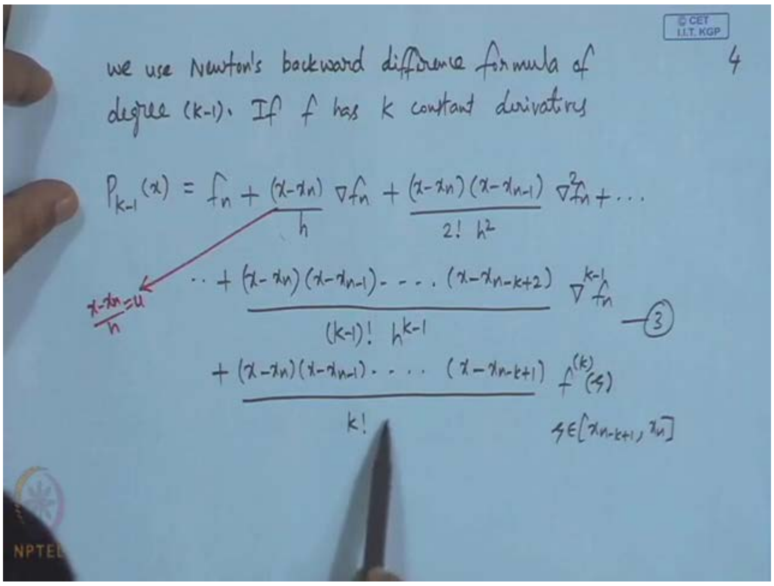

We have transform this using variable u so correspondingly one h goes to x minus x n by h another goes to this. But if you observe there is no h here in this term in the denominator. So, therefore, we have to supply 1 h 2 and n minus k plus 1 further this term. So that means total how many k? So, accordingly we get a numerator h power k and the denominator h power k. So, that denominator got adjusted into u, u plus 1 u plus 2 this under the numerator determined.

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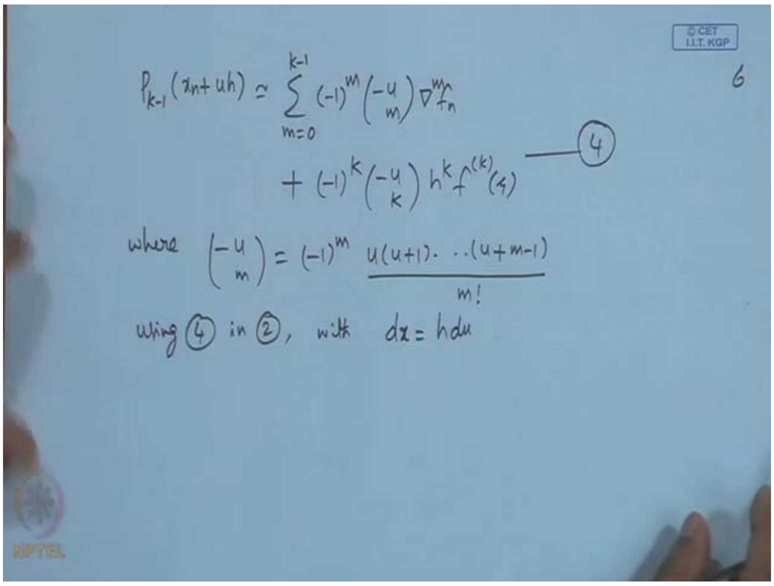

So, now further this can be simplified as follows, P k minus 1 x n plus. So what is our new variable now? Its u, u is our new variable. So, this equals m 0 2, k minus 1 minus m power m, minus u m plus u c k h power k. So we have to define where minus u c m is? So what we have done we have approximated and we have put it in a simpler form and this is a reminder term. Now what is our concern?

Our concern was to substitute the polynomial that has been approximated into this integrant, which was two. So we have approximated f by using k past points and that was four. So, now what we have suppose to do now using four into with this notation b x will be h d u. This is because of the change of the variable. Now when we substitute this in the integrant the expression two reduces as follows.

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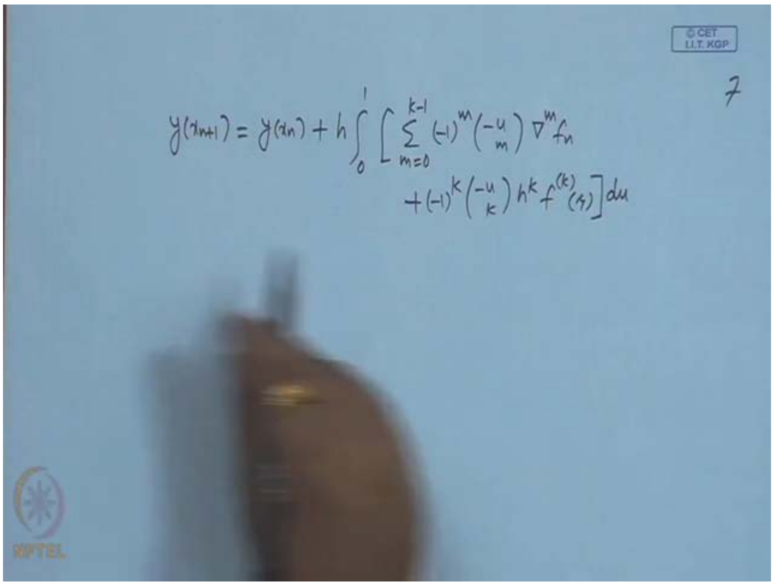

Plus h 0 to 1. So, I will explain why this 0 to 1. This is the reminder term and the variable is t u. So, why the…

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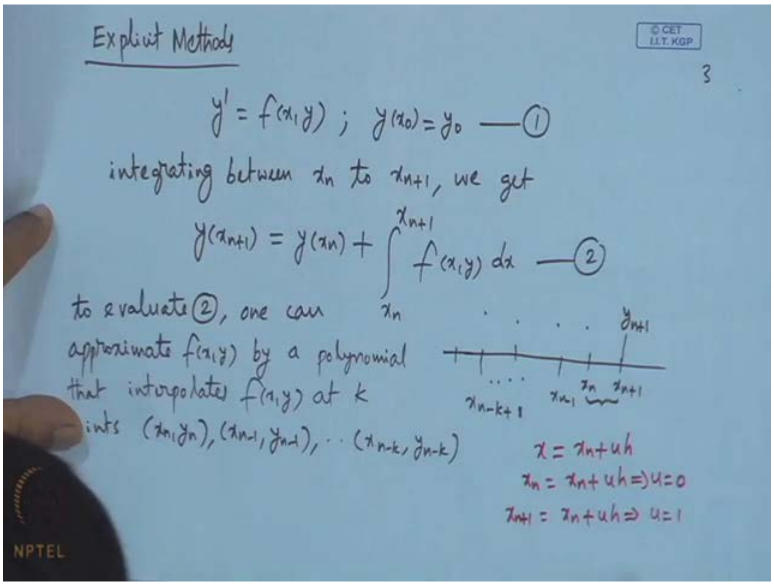

So, our limits x n 2 x n plus 1. That what was our transformation? Our transformation was x equals to x n plus u h. So, therefore, when x is an x u is varying from 0 and one x is, x n plus 1 so this will be. See when x is x n this implies u is 0. And when x is x n plus 1, x n plus 1 minus x n is h.

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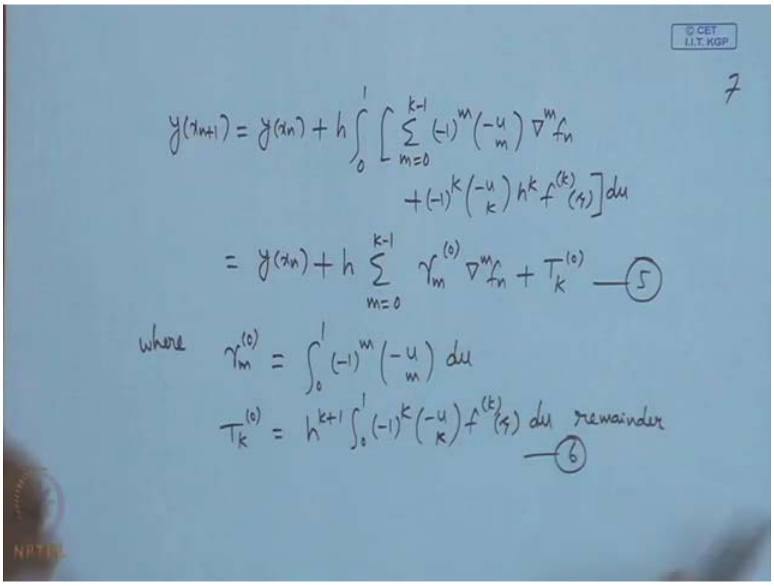

So, this is so the change of variable brings these limits. So, now this is equals plus h I take the integrant inside summation outside, which is permitted. So, 0 k minus 1 so some new notation. So I introduce a new notation look this same thing I am writing, where this has got in. So integrant, integral has gone inside and the differences have retained as it is and this is definitely t k. So that means when the summation has come out what is left integral 0 to 1 minus 1 power n and minus u c m. So that must be therefore, let us defined where.

So, these are some kind of codes equals this. So, where this and this h because h power k there is one h, h power k plus 1. So this is the reminder right? So, try to follow carefully so y x n plus 1 is we have approximated the integrant by polynomial using k past points. And we got a polynomial of degree k minus 1. And that has been substituted in the, in the integrant. So that has been substituted that has a simplified form and these are codes.

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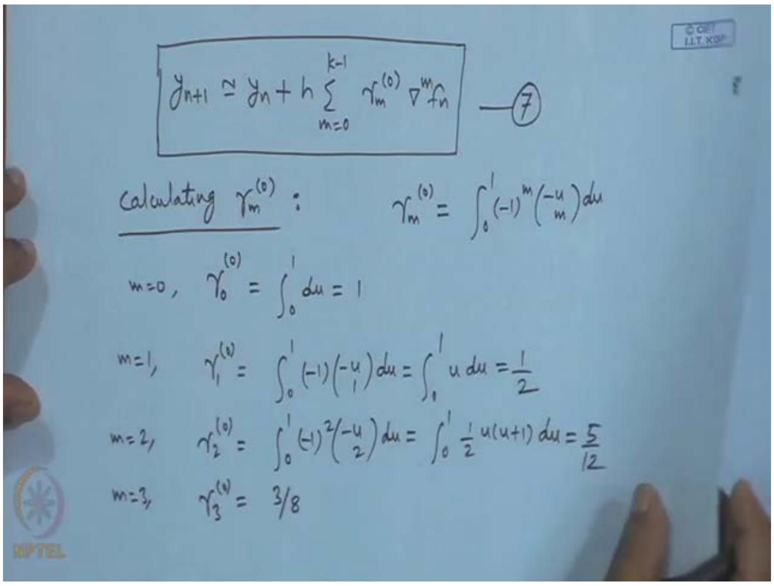

So now let us simplify further so the approximation is given by. So, after removing the error. I have removed the error time and this approximation. So, this our formula however these codes are to be computed right. So, then the next issue is calculating this codes. So, let us say m 0. So, m 0 because what was our formula our formula was n minus u c m also defined, so using that this will be 1. So, you can compute and to realize that it is this. Now these codes will be computed to get the formula. So, I am doing that we get the formula.

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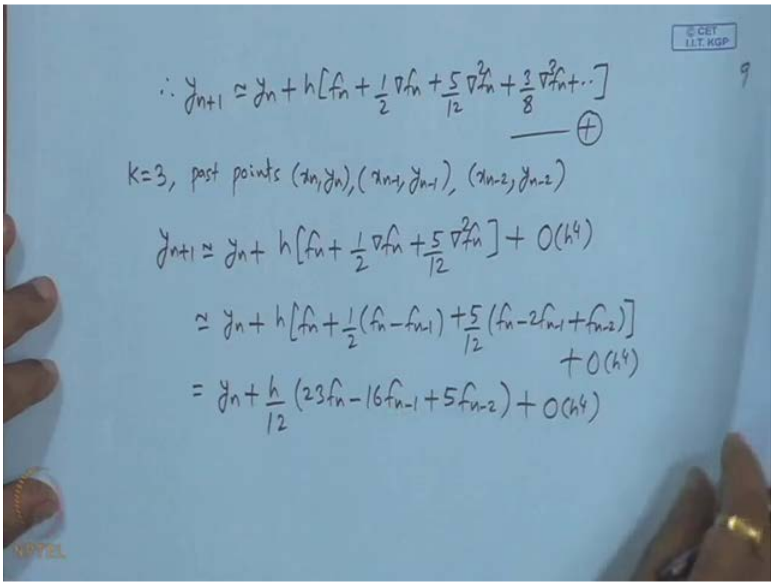

So, therefore, because what was the formula? 1 0 and 0 difference that is f n 1 0 is just 1 0 gamma 1 0 just 1 therefore, first term should be f n. This is then sorry gamma 0 0. So, then gamma 1 0 is half therefore, the next term should be half dell f n right? So, half of and 5 by 12 the second order differences 3 by 8. So, this is the general formula now why did we say this is explicit? Look to compute y n plus we need past points because you see these backward differences it will ask for n minus 1 so on.

Now let us come to specific, so this is more general. So, let us try to fix up the number of points. k is 3 and past points x n, y n, x n minus 1, y n minus 1, x n minus 2. So, suppose these are the past points that means we are approximating the polynomial using this past points. So, we have three points therefore, we expect a quadratic and that quadratic has been integrated to get the formula. So, in this case y n plus on1e is y n plus h.

So we have only three points, so we get differences up to second order plus there must be an error right? And what is a error t k, is if k is three h power 4 and this must be evaluated as reminder. So I can write h power 4. Now we have to expand so first all difference it has been backward operator has been expanded. So this is plus. So, we got the approximation and which is explicit and which is multi step. Why multi step? To compute y n plus we need n, n minus 1, n minus 2. So let us write down this explicitly.

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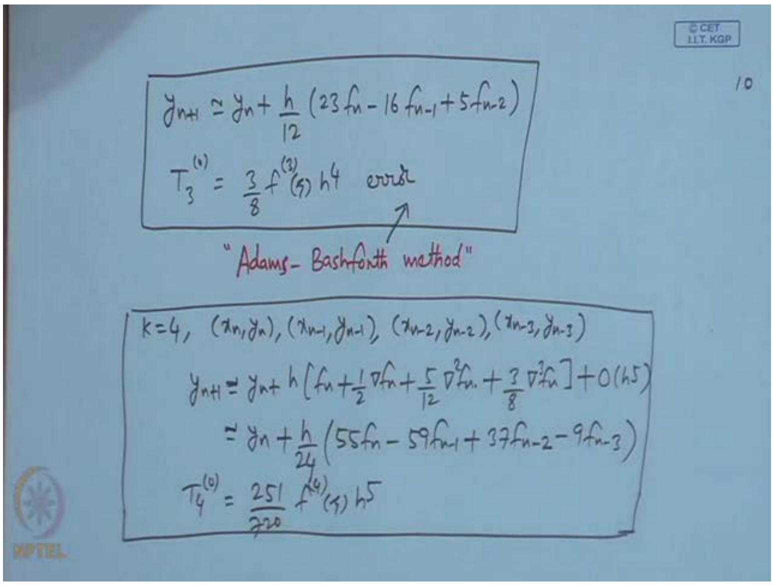

So, this error.... So, this an explicit method, which is third, order why? The error is h power 4. So, order of the method is less so that is third order and this has specific name literature Adams-Bashforth method. So this is called Adams-Bashforth method. Now naturally the question comes, can we reduce the error? So, what is the, see here we have derived multi step method and it is third order method. What did we do? We have used past points and how many? Three past points. So, we are approximated by a quadratic.

Now the natural question arises can we improve upon this method? So, what is the idea instead of three past points if you use more past points then we get better polynomial and then that that will be extrapolated right? So, let us try with a four points. Now if you try for four points. So, k equals 4 and the points are. Now I am not explaining the derivation because all that we have to do is simply take our general expression, take our general expression. And since we have one, two, three, four points then we can compute the differences up to thirds order. So by doing so we get, already we have done the codes after third order right? So, we get the following formula.

So this sum simplification 55 f n. 59 f n minus 1 coefficients are very big. So, competition will be little traduces and the error. So, this is another method. So, this Adams-Bashforth method. If you use more points we get the correspondence. Look at that the coefficient are completely different. So, you keep on using more past points and then you will get a different methods. So, let us quickly look at some problem so that we get some idea.

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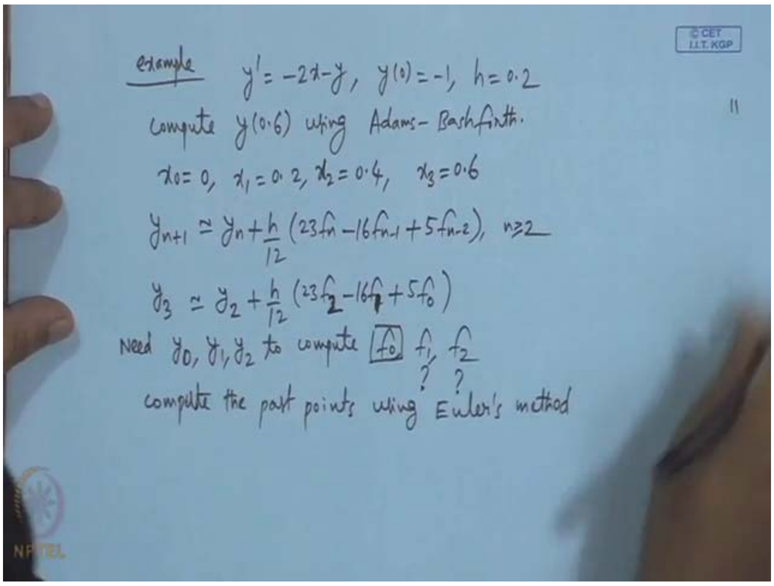

So, this is the u p compute y of point 6 using Bashforth. So, let us the formula is given. What are we asking to compute y of point six right? What was our x 0, 0 x 1, x 2 x 3. So, essentially we are asking to compute y 3, right? Now to compute y 3 your Adams-Bashforth is asking some past point right? y 1, y 2, y 0 of course, see let us look at your method. Adams-Bashforth so y 3 y 2.

f 3, f 2 sorry f 2 because n is 2, f 1 5 f 0. So, to compute f 1 and f 2 we need y 1 and y 2 right? So, remark need y 0 of course, y 1, y 2, to compute f 0, f 1, f 2. So, we have y 0 so there is no issue with this, but who will give us f 1 and f 2. So, in the question itself it should have been mentioned what are the past values require compute using something. Using which kind of method? Of course, using n explicit method then only we can compute. So, let us say it is mentioned compute the past points say using Euler’s method. So let us compute using Euler’s method.

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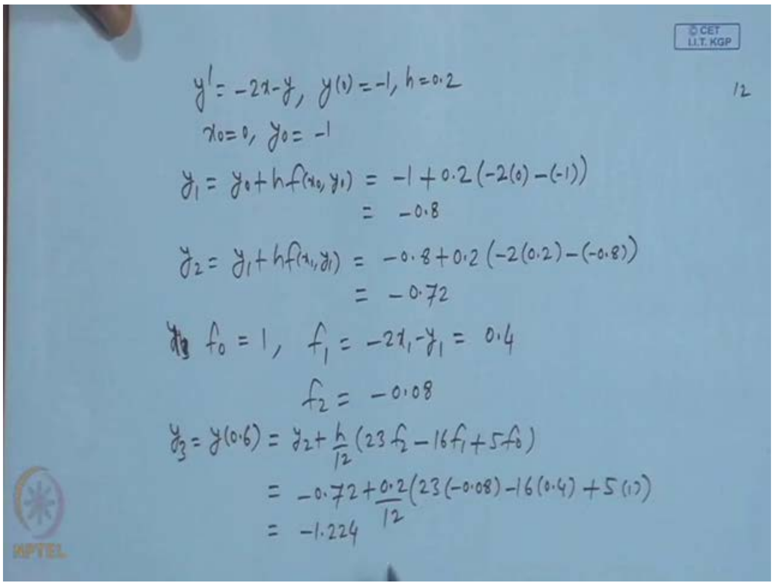

so, y dashed was… so x 0 is 0 y 0 is minus 1. Therefore, y 1 is so y 0 is minus 1 h is point 2 minus 2 x 0 is 0. So, this is then y 2 h times minus 2 x 1 then minus. So, this is minus point seven 2. So, we got y 1 y 2 using Euler method that means we have all the data, we have y 0, y 1, y 2. Now what was our? Of course we need immediately f 0 so f 0 is, f 0 is minus 1, f 0 is minus of minus 1. So, this is 1 then f 1 is minus 2, x 1 minus y 1. So, f 1 is point 4 and f 2, f 2 is minus point not 8. So we have to compute, then y 3, which is a f point 6. This is given by y 2 plus h by 12 23 f 2 minus 16 f 1 plus 5 f 0. So, this is h is f 2 minus 16 f 1 plus 5. So, we get approximate value, which is minus 1 point 2 2 4. So, what we have done? We need the past values.

So, to compute using Adams-Bashforth we need three past values n, n minus 1, n minus 2. And in the question it should be given using, which method one should compute the past values. In the present problem we have used Euler's method the simple. But if you really need better accuracy one should use may be higher order tailor city method or R K method. So, that will give more accurate results. So you have seen what is the logic we have integrated between one interval and then the polynomial has been fitted with a using k past point.

Now the larger the value of k the better approximation of the polynomial and hence the arrive will be less. So, we have tried k three and then k four and k three there is a specific name Adams-Bashforth, which is explicit method. And k four also is an explicit method.

Suppose instead of that if the right hand side demands y and plus 1 as well then we end up with something called implicit method. We, try to discuss in the next class these methods.

Thank you have a nice day.