Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 8
Tutorial – II

Hello, in the last class we have reviewed some exercises. I mean while solving exercises we have reviewed some of the single step methods. So, let us continue to do that and we missed may be solving system of equation and then... Of course, we have learnt R K methods, but only explicit. So, may be with reference to some implicit nature. So, let us try to attempt some problems.

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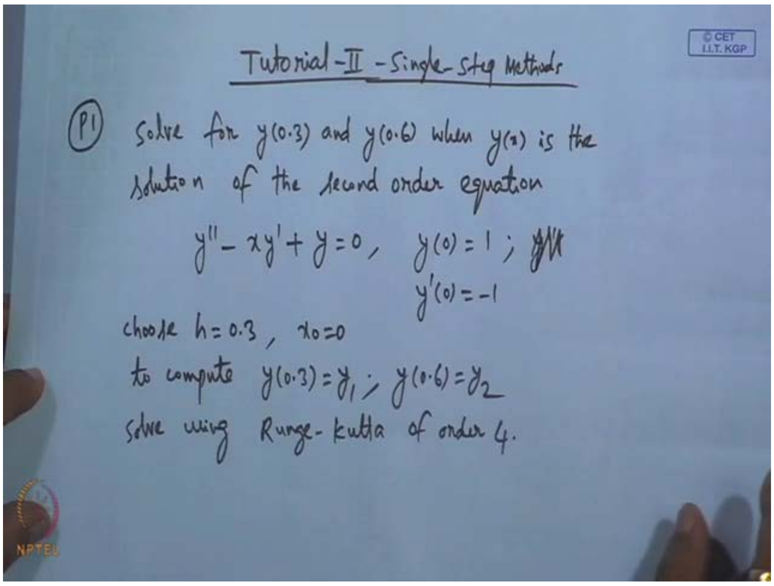

So, this is tutorial two still continuing single step methods. Solve for y of point 3 point 6 when y x is the solution of second order equation. So, we need two mutual conditions sorry. Choose h is point 3. So, solve for y of point 3 and the y of point 6 when y x is the solution of the second order equation this initial value problem and choose h point 3. So, that means to compute y of point 3 with this. So, x 0 is 0, so y of point 3 is y 1 y of point 6 is y 2.

Now as we discussed so earlier so this is a second order. So, what do we have to do convert it to couple system of fist order equation and then try to adopt one of the methods. The story is not over, which method we have to mention? Solve using runge -

kutta of order four right. If we solve runge - kutta order four then lot of general idea we may get.

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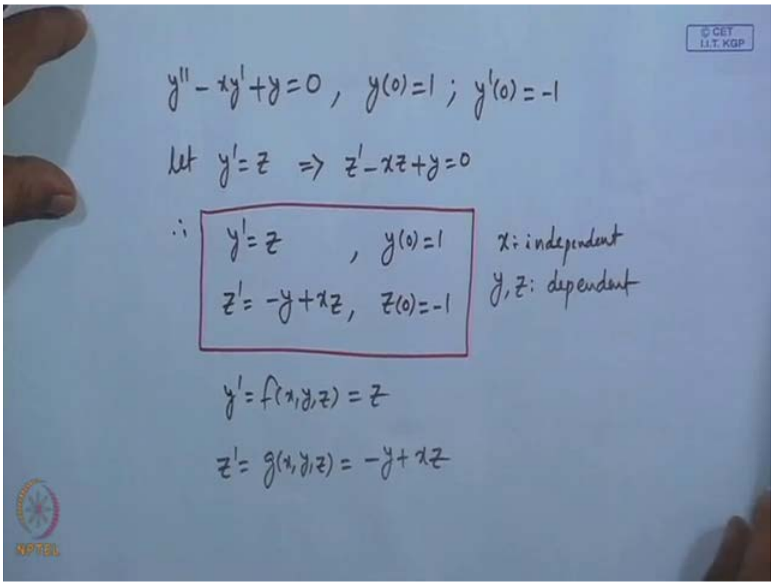

So what was our equation? Y double minus x y dashed plus y equals 0 and y of 0 is 1 and y dash of 0 is minus 1 right? Now we have to convert this to system. Let y dash is z. So, then this becomes this implies z dashed minus x z plus y equals to 0. So, therefore, we have the following system y dash equals to z, z dash d is minus y plus x z, y of zero is 1. And y dash of 0 becomes z of 0 is minus 1. So, this is our system, so this our system that we have to solve.

Now since it is a remark x is independent variable y and z dependent variables. So, therefore, when we use R K fourth order we have to be careful and we have to define two different functions. So, how do we define? So, the general with reference to general system y dash equals to f of x y z, which is z it is z dash these g of x y z, which is minus y plus x z. So, corresponding to y we define a specific functions and correspond z we define different functions. So how do we do so how do we do?

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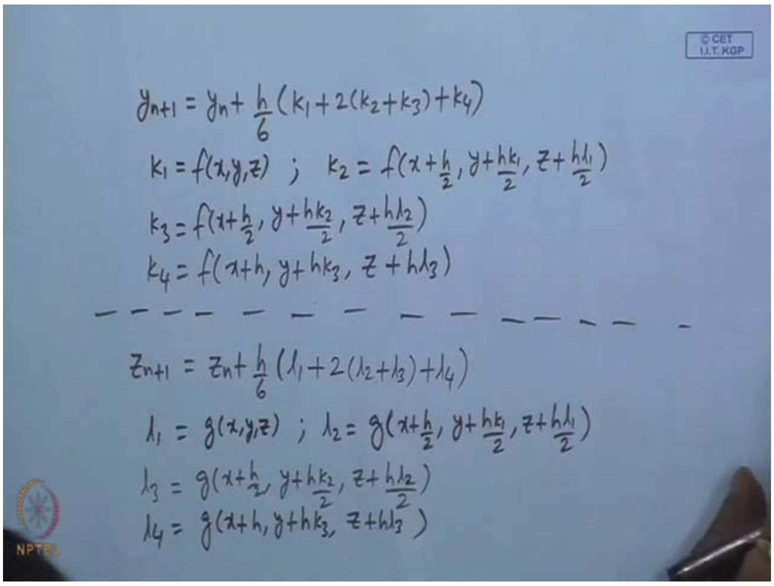

y n plus 1 equals y n plus h by 6 k 1 plus 2 k 2 plus k 3 plus k 4 right? Then so we define. So, let us have simultaneously z n plus 1 z n plus h by 6. So, we have to use a different notation. So, let us use a different notation l 1 plus 2, l 2 plus l 3, l 4. So what is k 1? K 1 is f of x, y, z and what is l 1? G of x, y, z then k 2, therefore l 2 and k 3. l 3, k 4, l 4. So, these are the two different coupled so unless you compute k 1 one cannot compute l 2. So, unless we compute l 1 one cannot compute k 2. So, they are coupled right. So, let us proceed further.

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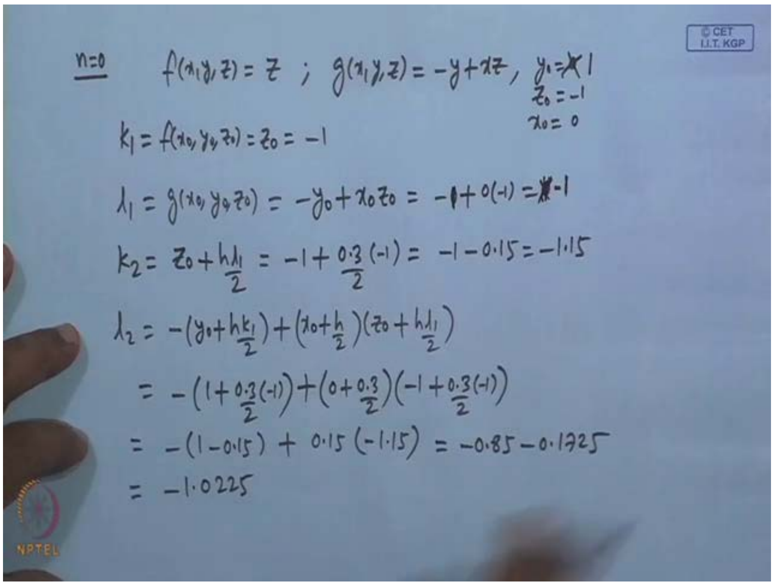

Now n is 0 so we have f of x, y, z is, f of x, y, z is z. g of x, y, z is minus y plus x z so k 1 is. So, let us write down the given that as well y 0 is 0 z 0 is minus 1 x 0 is 0. So z 0 is minus 1. And what is l 1? So, this is 0 0. So, l 1 is y 0 sorry y zero is 1. So this will be 1 minus 1. So this is minus 1. Then k 2 is z 0 plus h l 1 by 2 minus 1 plus l 1 is minus 1. So, this will be this l 2, l 2 will be minus. So, this on we have defined l 2 is… so we have a all three involved so the increments are h by 2, h k 1 by 2, h l 1 by 2. So from here minus y 0 plus this plus x 0 plus h by 2 z 0 plus h l 1 by 2. So this will be minus k 1 is minus 1 plus x 0 is 0, z 0 is minus 1, l 1 is minus 1. So this will be minus 1 minus so this will be so this is our l 2.

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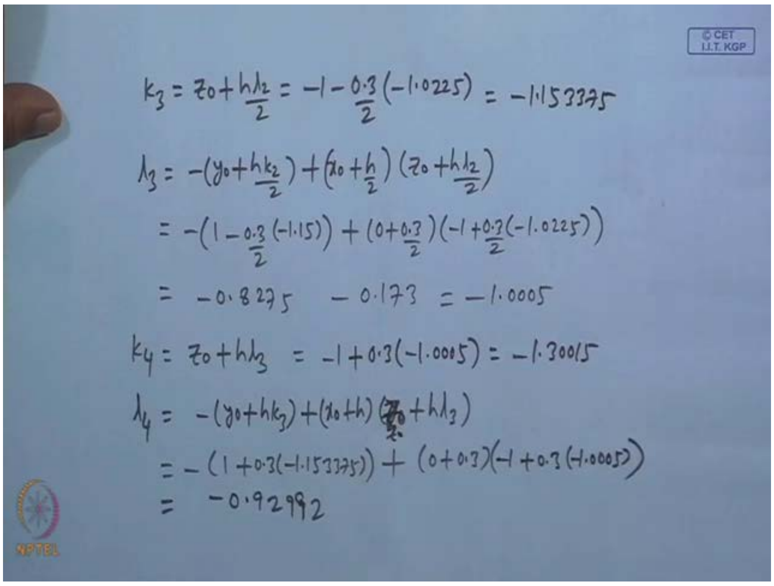

Now k 3 so l 2 so this will be, then l 3, k 2, l 2. So this will be, then k 4, l 3. So, this is then l 4, k 3 value minus y 0. This is z 0 so this is z 0. So, this can be simplified and we get minus point.

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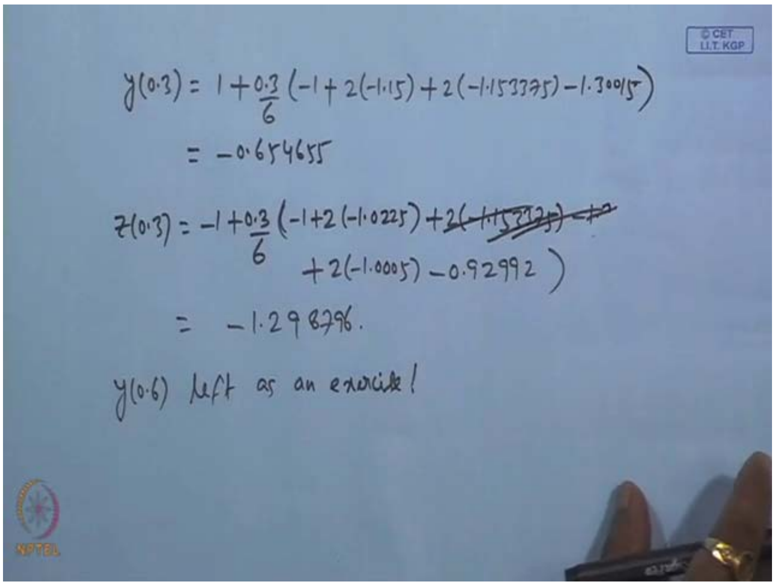

So, having obtained these values we have y 0, h by 6, then k 1 plus 2, k 2 plus 2, k 3 plus k 4, so this l 1 plus 2, l 2 plus 2. So, this can be. So, y of point 6 left as an exercise so similar method. So, you can try to do it. So, with a with R K method we could solve system. So that we get both simultaneously that is how to use R K method and simultaneously how to solve system of equations. And if you observe the R K method, which we have discussed both second order and then three stage and forth order etcetera.

So, these are all explicit in fact tell a series explicit method and the all that. So, there are some methods which are implicit. So, what do you mean by implicit? So, we will discuss in detail when we go to multi step methods, but just a brief what is implicit method and. Then we quickly look into implicit Runge - Kutta method. So let us do that.

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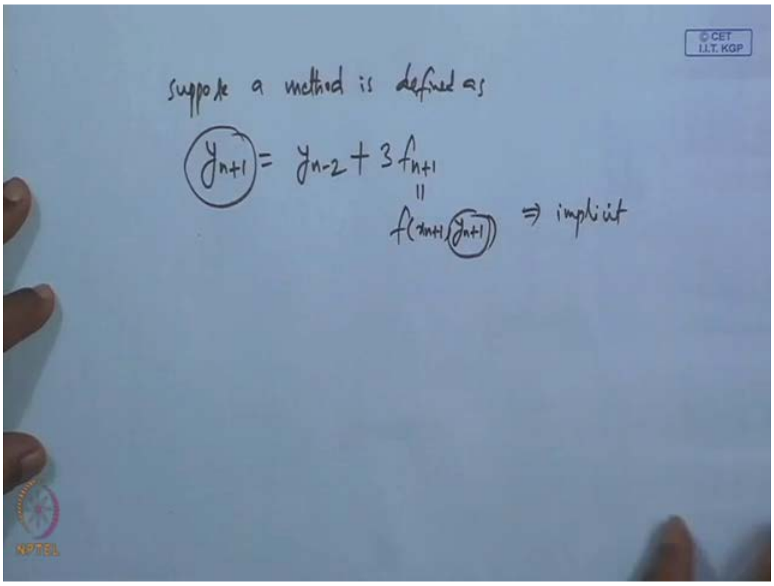

So, suppose a method is defined as y n plus 1 equals to y n minus 2 plus 3 f and plus 1. And this f n plus 1 is f of. So, that means we are asking for computing y n plus 1, but right hand side is also asking. So such method is implicit right?

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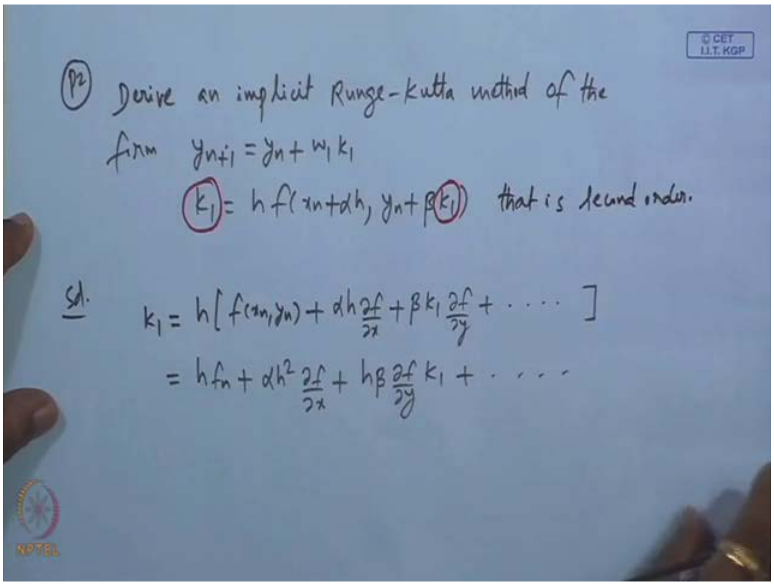

So, let us look at it is a kind of problem only, but in terms of implicit R K method derive an implicit Runge - Kutta method of the form. So, you may observe why we are calling this implicit. So, you have left hand side you have k 1, but right hand side as well we have. So, derive an implicit Runge - Kutta method of the form this that is second order.

So, how do you proceed? So, we in general in explicit we have learnt we have expanded this and tell us series and then we have expanded this in powers of h. And then we expanded the y n, y of x n plus 1 in tell a series. And then match the coefficient of equal parts of h.

Similar sought of thing we need to do here, but you may observe right hand side you have k 1. And left hand side we have k 1. So that means if you keep on substituting k 1 this same expression recursively k 1 will be sitting on the right hand side. So, let us see how do we proceed further? So, k 1 expanding alpha h dou f by dou x plus beta k 1. So, the higer term etcetera. So, this is h f n alpha h square plus h beta dou f by dou y k 1. So k 1 is sitting. So now again we need to substitute k 1.

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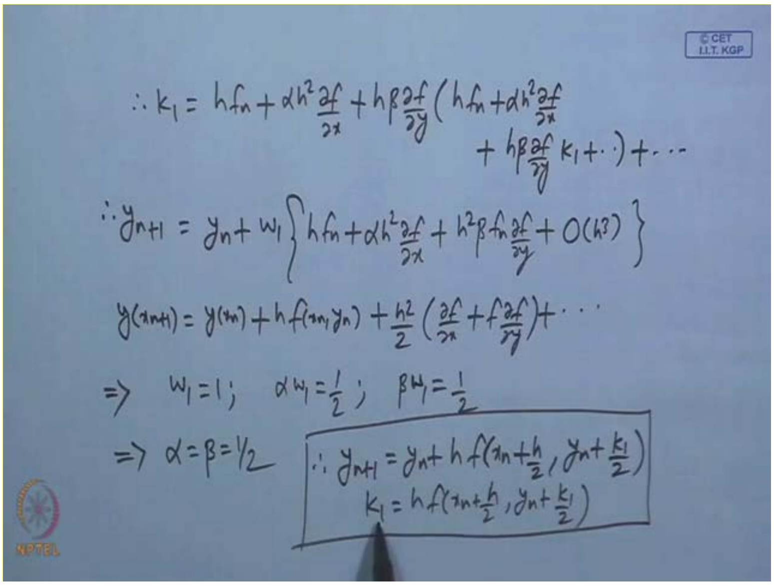

And the other terms. So, therefore y n plus 1 is y n plus w 1 h f n plus. If you multiply these h square beta, h square beta h f n dou y plus h q terms. So, plus so tell a series. Now comparing these two because the method is second order we can pair up to h square. So, this approximate, this and for f w 1 must be 1. Then for f x coefficient is half there and here alpha w 1 is half. Then for here w 1 is half. So, this implies alpha equals to beta equals to half.

So, therefore, y n plus 1 is y n plus h x n plus h by 2 y n plus k 1 by 2, where k 1 is. So it is a recursive because if you substitute we keep on getting k 1 recursively k 1 there k 1. Again if you substitute again you get k 1, but now the less up to second order by comparing by we get the coefficient and then this is explicit. Why it is explicit? You cannot recall. K 1 we are expecting to compute, but it is given the right hand side. So, this is a very important observation in implicit R K method.

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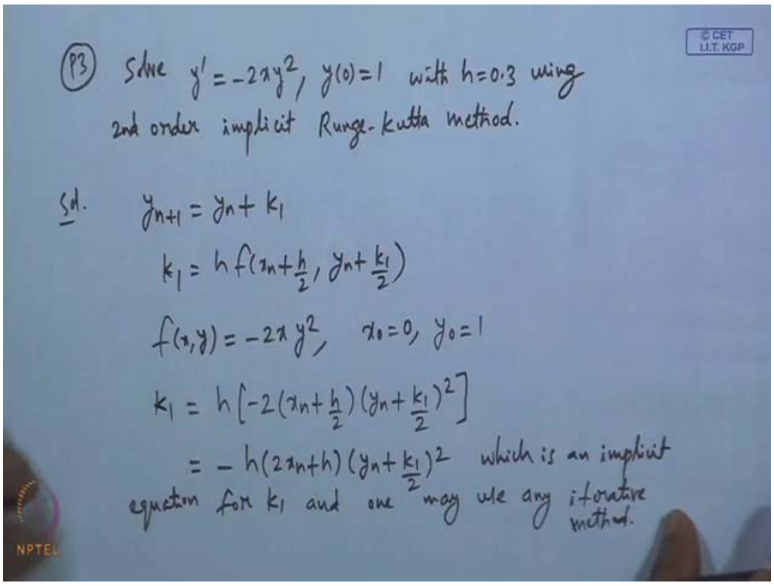

So, let us solve some problems, but one remark before we proceed is generally expanding in a tell a series is very difficult because recursively how long we do it. So it is. So, let us see through this example slightly. Different story with h equals to point 3 using second order implicit Runge - Kutta method right. So just now we have derived by expansion now we would like to solve a problem.

So, this was our implicit R K method. For the given problem f of x y is this x 0 0 y 0 is 1 right. Now let us try to compute k 1 h f of this so therefore minus 2 x n plus h by 2 y n plus square. So, minus h 2 x n plus h, which is an implicit equation for k 1. And one may use any iterative method because we have to solve for k 1. See this we have to solve the problem if you keep on substituting the cursively so that is not end. The method has been obtained in that fashion, but now we would like to compute the solution. So this is an implicit equation. So we have to solved using any itorative method.

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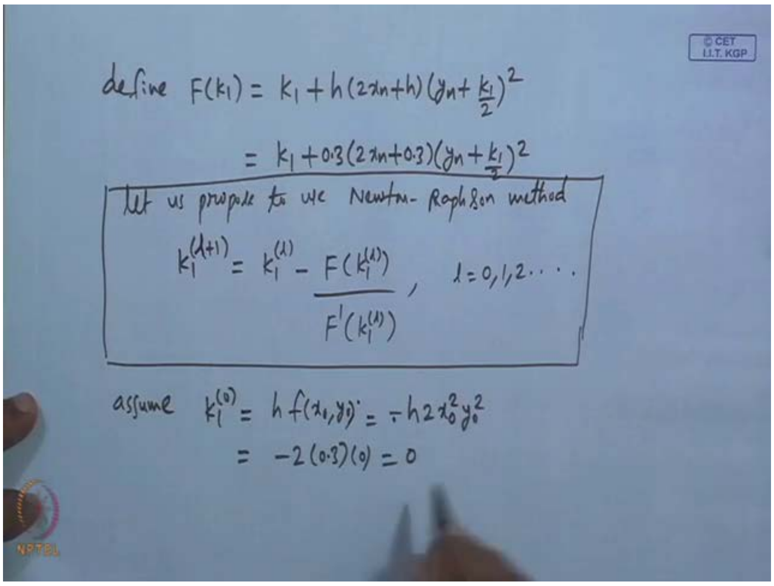

So to this extent let us define f of k 1 as k 1 plus h. So, this is k 1, which is point 3. Now since it is any itorative method we try to use Newton - Rophson method we propose to use Newton - Rophson method. So, therefore k 1 s plus 1 say l plus 1 minus. So let us propose to have Newton - Rophson method. So, accordingly let us compute. Now we need an initial guess. Assume k 1 0 equals h. So, this is f or f s minus 2 x 0 square y 0 square this is minus 2 point 3 x 0 square 0. So, k 1 is 0.

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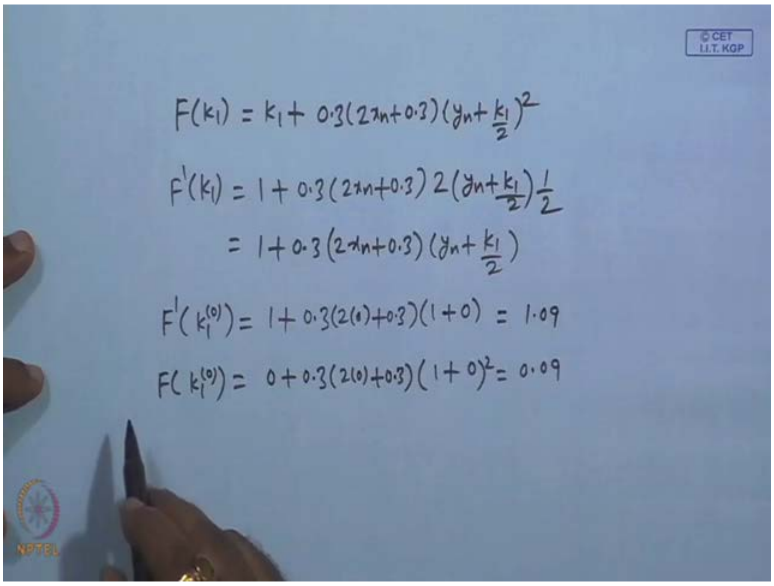

Now let us compute f dashed. Say f dashed k 1 is 1 plus 2 times half. So, this is 1 plus 0.3 2 x n 2, 2 get cancelled, Y n plus this. Now so this will be x 0, y 0 so 1 plus k 1 0, 0. So, this equals, then f of k 1 0. This will be… so this will be.

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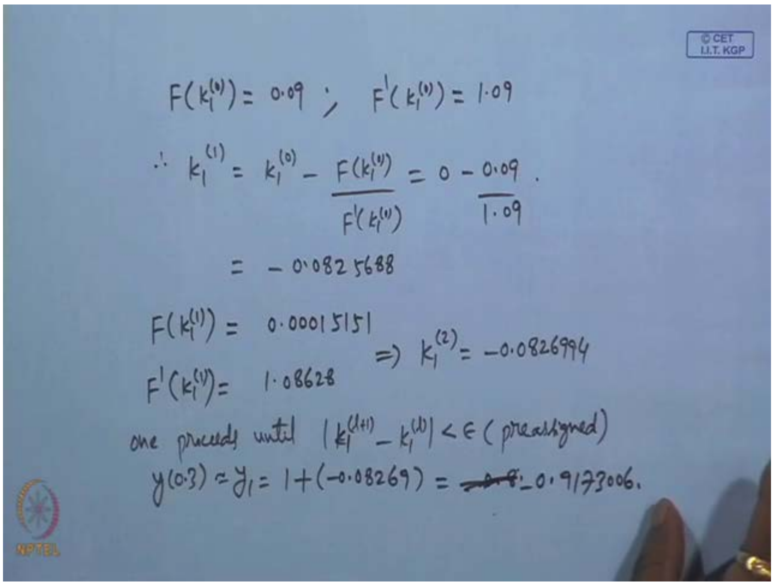

So, therefore... So, we have f of this is f dashed of is. So, therefore k 1 1 is. So, this is minus. So, this k 1 prime now we have to compute f of k 1 sorry not k 1 prime k 1 1. Now we have to compute this so we may do using calculator and also we need this. And hence k 1 2 we get. So, one may stop here or proceed until. So, we proceed until two consecutive. The difference the difference was the two consecutive values is less then x large some pre assigned we will proceed. So, let us say here we stop. So, one we stop we get. So this is y 1 y 0 plus k 1. So, this is the answer. So, this is how to sole using implicit R K method.

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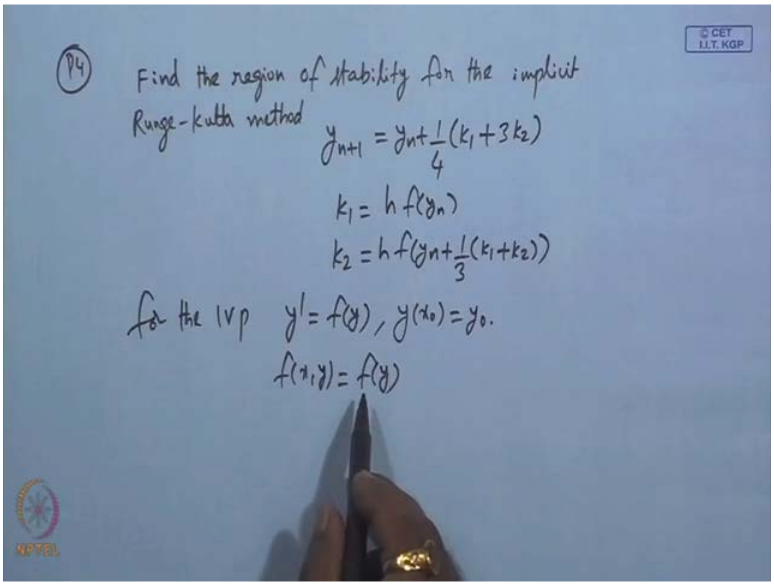

Now we have solve lot of stability problems for the express it so let us do it for implicit. Find the reason of stability for the implicit Runge - Kutta method given by this for the I V P? Please make a note. So, this is a case of f of x y equals. So, that is no explicit dependency on x. So, this is a kind of a special case. Now for this we would like to have stability interval, stability analysis. So, now let us proceed with our reference equation.

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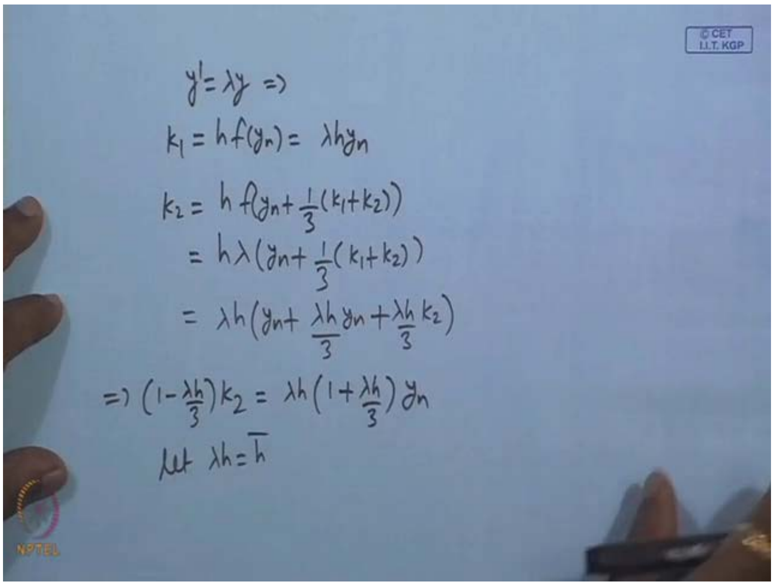

y dashed equals to lambda y this implies k 1 is h and k 2 is h. so you can see k 2 is implicit. So, k 1 lambda h y n, so since k 2 is explicit we try to collect the co efficient and solve as an algebraic equation. So this implies. So, this is kind of algebraic equation let.

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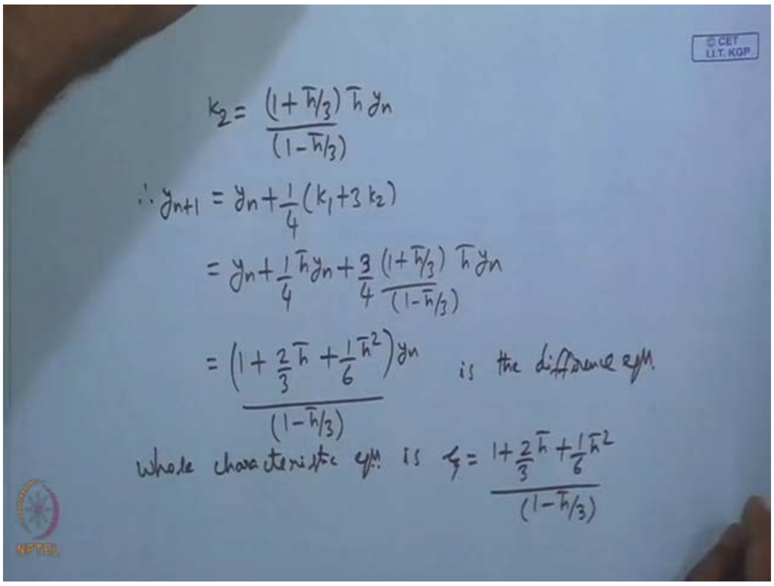

So, then we have k 2 equals. So, this our k 2. So, therefore y n plus 1 is y n plus. So, what was our method? Y n plus 1 by 4, k 1 plus. So, this is y n plus 1 by 4. So, this is our difference equation, is the difference equation whose characteristic equation is. So, this is a characteristic equation. So, now once we have the characterise equation. So, the procedure is for absolute stability we have to put the condition of on the roots and then try to determine. So, let us do that.

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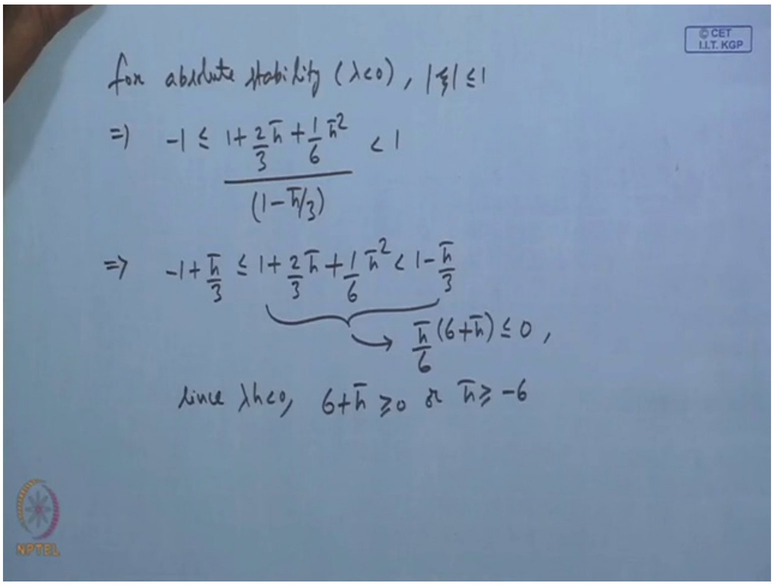

For absolute stability, where lamda is negative. So, this implies minus 1, so this implies right. So, from from this we get, we get look at it so if you bring it this side so we get h bar. So, h bar by 6 h bar 6 plus h bar less than or equals to 0 and since lamda h s d 0 h or h bar is equal to the minus 6. So, this is one side. So, then we have to see while the left. So, this is the right inequality now we have to see the left.

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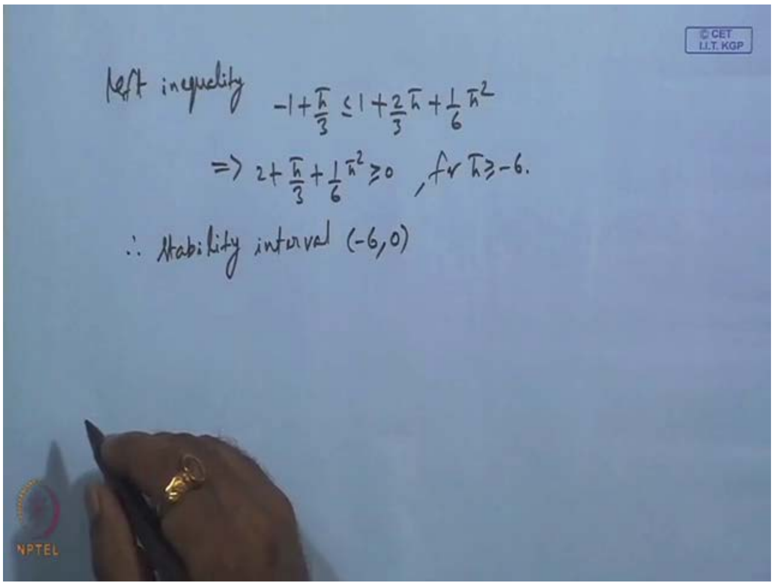

So, the left inequality. So this this gives so minus 2. So, this gives and which is true for. So, this holds and true for h bar greater than 2 minus 6, so therefore stability interval.

From the left we, from the right way got h bar greater than equal to 6. And the left inequality this and this holds for this, so the stability interval minus 6 to 0. So, for implicit, the problem with implicit is for deriving a method we have use to tell a series. But then for a specific method we have to solve an algebraic equation. Sometimes it is nor linear and that we have solved using using Newton - Rephson method. Now let us have some exercises for your for benefit of practice.

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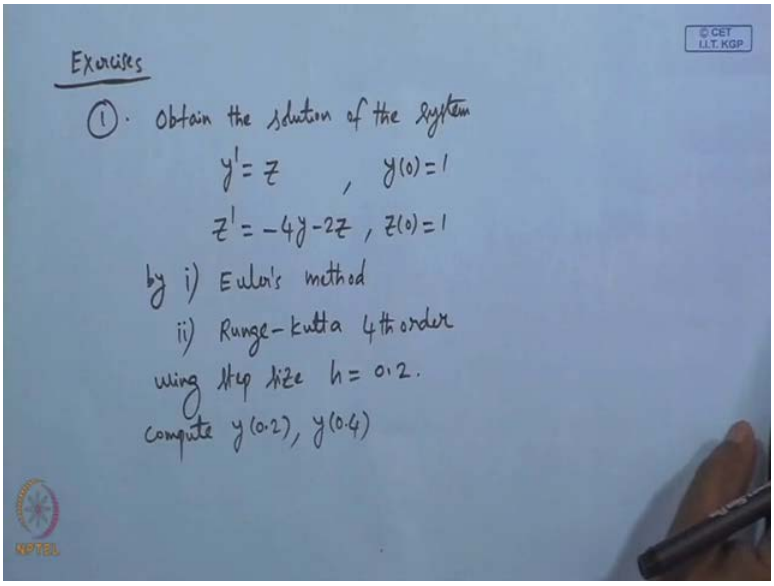

Obtain solution of the system y dashed equals to z. z dashed equals to minus 4 y minus 2 z y of 0 is 1 z of 0 is 1 by Euler's method. Two Runge - Kutta fourth order using step size h equals to 0.2. Obtain the solution of the system using step size h is equal 0.2 compute y of 0.2, y of 0.4.

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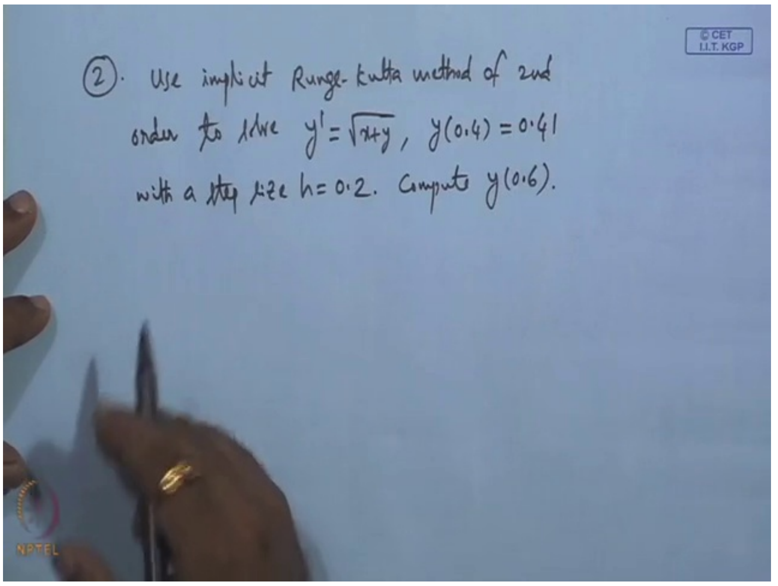

Right two use implicit Runge - Kutta method of second order to solve y dashed equals to x plus y y of point 4 is this with a step size h is equals to 0.2. Compute y of 0.6. So, These are some exercises for you you can try and then feel more confident on both explicit and implicit R K methods. However as I mention we discuss in detail what is an explicit method, when we proceed to multi step methods. So, let us wait for the lectures on multi step methods to hear more on implicit methods.

Thank you until then bye.