Numerical Solutions of Ordinary and Partial Differential Equation
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Lecture - 7
Tutorial -1

Hello good morning, so far we have learnt some single step methods to solve initial value problems. So, it would be better to work out some exercises in order to get the flavor of each of the methods and feel more confident.

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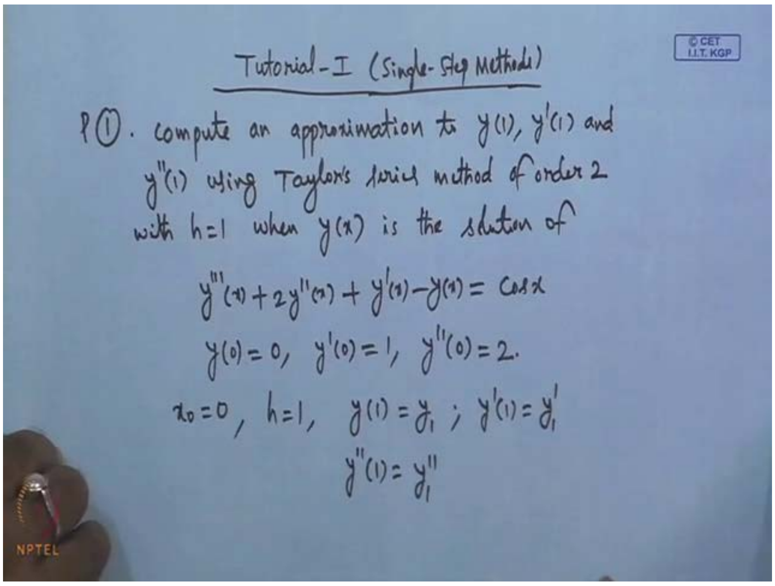

So, let us do some problems. This tutorial is on single step methods, so better to have your calculator with you and then start working with me. So, this is problem 1, compute an approximation to y of 1, y dash of 1 and y double dash of 1 using Taylor's series method of order 2 with h equals to 1, when y x is the solution of... So, let us read the problem carefully, compute an approximation to y of 1, y dash of 1 and y double dashed of 1 using Taylor's series method of order 2 with h equals to 1, when y axis the solution of the following initial value problem.

Look at that, this is the third order, so hence we have 1, 2, 3 initial conditions all at. So, we have x 0 is 0, h is 1, well I have taken h is 1 just for a computational e's. Now, we are asking to compute approximations for y of 1, which is y 1, y dashed of 1 is y 1 dashed, y double of 1, so these are to be computed. So, it is to be computed using Taylor series suppose, y of 1 that is y 1, so we know usual Taylor series expansion, but then how do we compute y dashed of 1, y double of 1 along similar lines as it will be done for y of 1.

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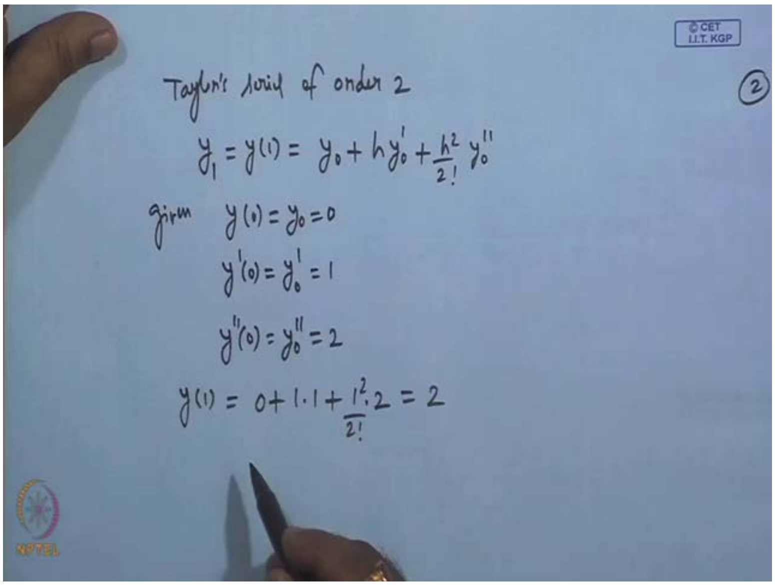

So, let us have a look at that. So, Taylor series of order 2, so this is y 1, this is y 0 y 0 prime h square by y 0 double. So, given data y of 0, y dash of 0, y double of 0 therefore, y 0, y 0 dashed, y 0 double we have, so hence y 1 can be easily computed. So, this is 0 h is 1, y 0 dashed is 1 plus h square by 2 factorial therefore, this is 1 there 1 there two. Now, what is next to being computed? So, y 1 we have computed, now we have to compute y 1 prime, so how do we do it? So, let us write down the Taylor series of order 2 for y 1 prime.

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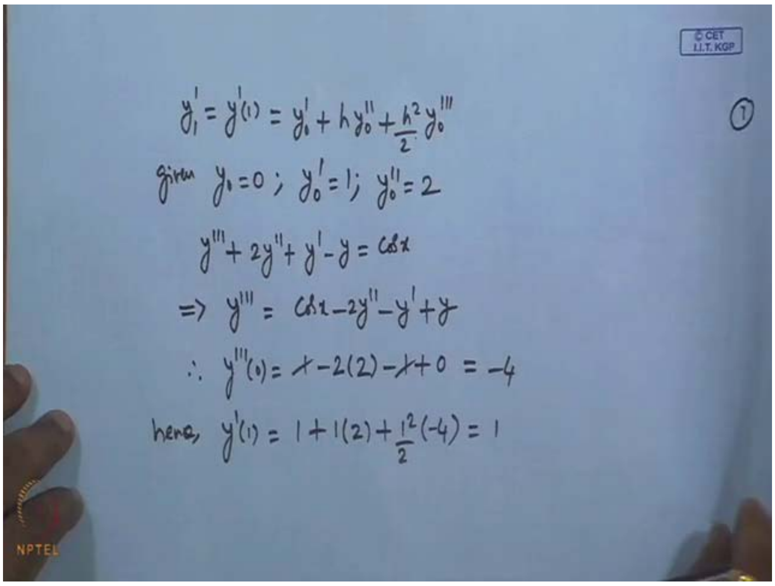

So, y 1 prime is equals y 0 prime plus so it is like y 1 prime, so let y 1 prime equals to z. So, then for z of 1 we write it and replace it, so same thing we are doing, so this is our expression, but this is the given data. So, in order to compute y 1 prime, we need y 0 triple, which we do not have, but we know that y is the solution of given ODE. So, this implies y 3 equals to minus 2 y double minus y prime plus y.

Therefore, y 3 of 0 1 minus 2 y double is minus y 1 y 0 will be 0, so this is minus 4. Now, we are in a position to compute this one, hence this is y 0 prime is 1. So, h y 0 double prime is 2 1 square by 2 y 0 triple prime, so this is y 3 y dash of 1 this is minus 2 and this is 2 get cancelled and this will be 1. So, we got approximation y dash of 1.

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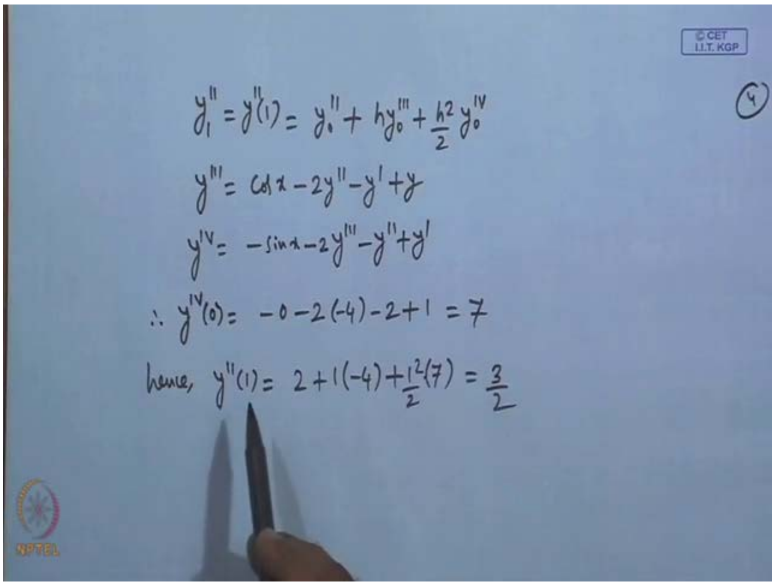

So, Taylor series for this, now we do not know y 4, but given from the ODE, we have computed. So, from here y 4 therefore, y 4 of 0 is y 3 of 0 just now we have computed y 3 of 0 minus 4 y 2 of 0 y dashed, so this is 7. Hence, y 0 double 2 this is y threes minus 4 y 4 is 7, so this will be using second order Taylor series we have obtained approximations to Y of 1, y dash, y of double dashed of 1. So, all that we have done is we made used of the given having known that y x is the solution of ODE, we have computed the higher order derivatives. Suppose you have to compute y 4 of 1 then we differentiate this further and try to obtain the higher order of derivatives.

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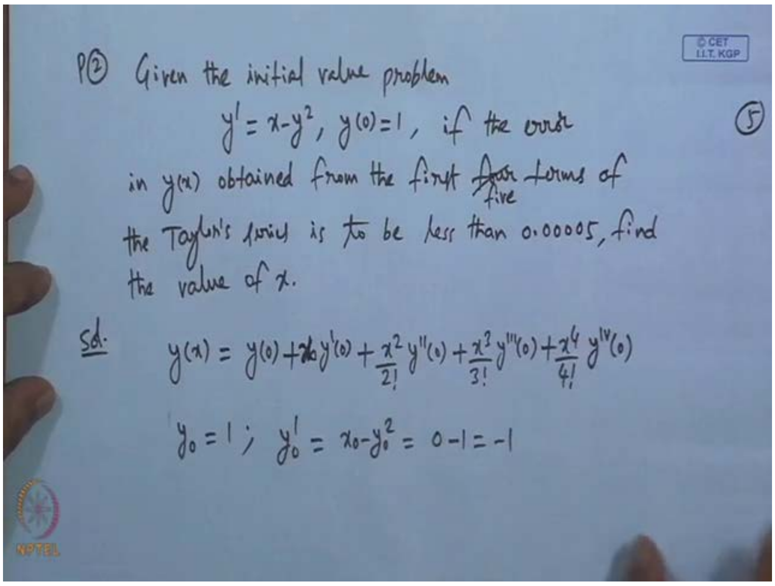

So, let us proceed further, given the initial value problem y dash equals to x minus y square, y of 0 is 1, If the error in y of x obtained from the first five terms of the Taylor's series is to be less than 0.00005. So, let us try to understand given the initial value problem, if the error of y of x obtained from the first 4 terms of the Taylor series is to be less than, that means you obtain y of x and if the error has to be less than this then what should be the condition on x. So, first 4 terms means we have to compute up to y 4 that means y of x obtained from first 4 terms.

So, this is y of 0 plus x plus x square by 2 factorial plus Taylor series method first 1 2 3 4 first 5 letters make it 5, so that is plus x 4 by 4 factorial 1 2 3 4 5, first 5 terms. So, we need to compute the higher order derivatives, so y 0 equals to 1, y 0 dash equals to x 0 minus y 0 square so that is 0 minus 1.

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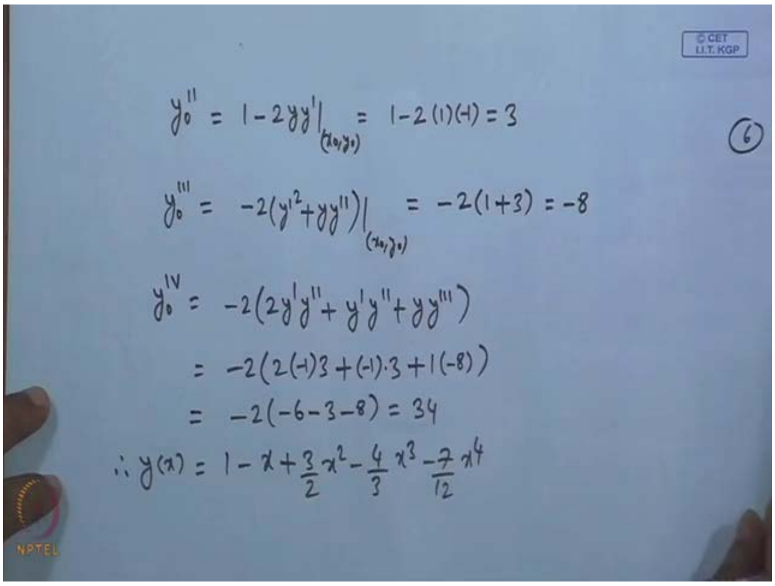

Then y 0 double so that is 1 minus evaluated at so this will be 1 minus 2, y 0 is 1, y dash is minus 1 then y 0 3 is minus 2, y dash square evaluated at, so this will be 1 y 0 is 1 and y 2 is 3 then y 0 4. So, this will be 2 y dash is minus 1, y 2 is 3, plus y das is minus 1 3, so this will be therefore, y of x is we have written already y 0 plus y dash x 0. So, by substituting we get 1 minus x plus 3 by 2 x square minus 4 by 3 x cube minus 7 by 12 x 4, so this is using first 5 terms. Now, if this is the case then what will be the error?

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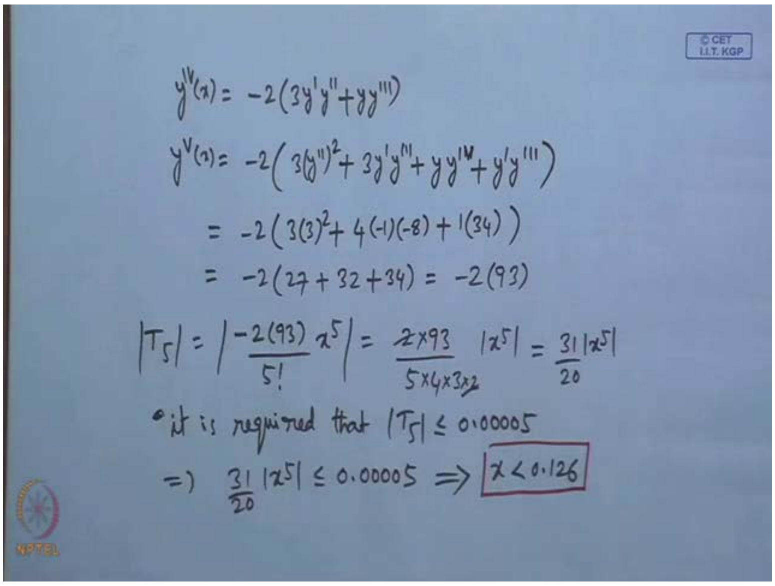

So, the error y 5 x is minus 2, so this is y 4, so y 5 this will be 3 there, so 3 y 2 plus y dashed 3 plus y 3 plus y dashed y. So, this will be minus 2 3 plus this is 3 and 4 times y dashed is minus 1, y 3 is minus 8, this is 1 y 4 is 34. Therefore, the error this is minus 2 times 93 by 5 factorial, so this is equals to 3 31 by 20 and it is required that mod T 5 must be less than equals to point this much. This implies 31 by 20 must be less than or equals to it is lightly difficult to solve, but then one obtains the estimate, so this is the range of x.

So, let us try to understand what we have done, we have given initial value problem if we use the first 5 terms, so first 5 terms of the Taylor series is to be is used and if the error are on using first 5 terms is to be less then find the range of x. So, what did we do? So, first 5 terms goes up here hence, the error come from T 5 that is from the fifth term as a sixth term and that needs to be less than this, so on solving we obtain the corresponding estimate. So, this is how we find out the corresponding estimates.

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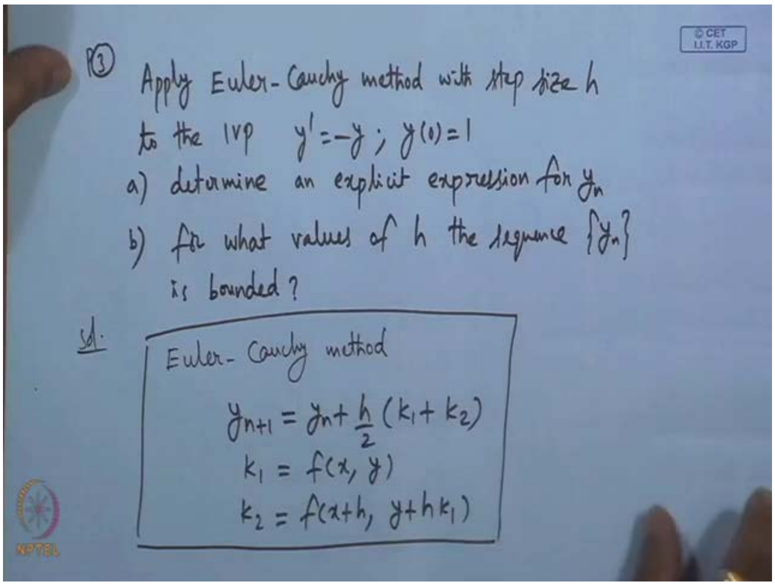

Now, problem 3 apply Euler-Cauchy method with step size h to the I V P and determine and express it expression for y n, for what values of h the sequence y n is bounded. So, what the method says apply Euler-Cauchy method with step size h to this initial value problem. Determine explicit for determination y n and for what values of the sequence of h y n is bounded. So, when we determine expression then b part is triple, so how do we proceed? So, first let us write down Euler-Cauchy method, so this is Euler-Cauchy method. Now, we have to use this method and determine the solution y n.

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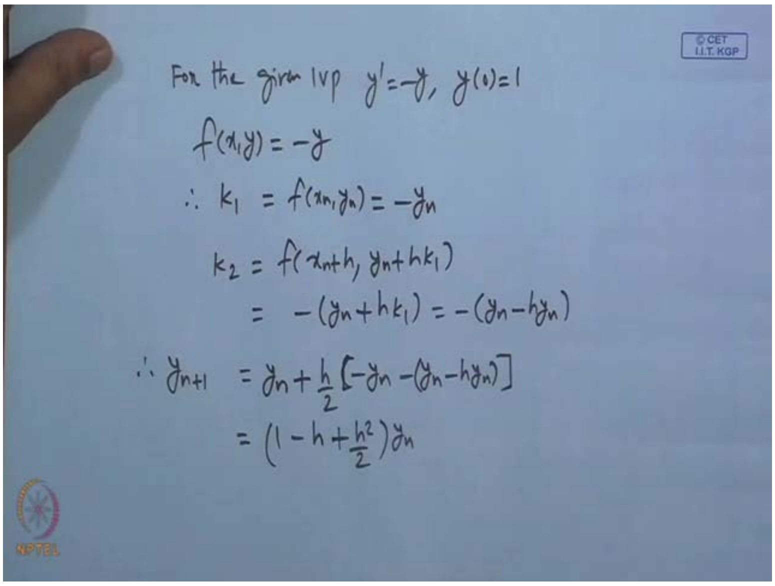

So, for the I V P f of x y is minus y therefore, k 1 equals this, k 2 is so this is nothing but minus y n plus h k 1, this is minus y n, k 1 is minus y n so minus h y n. Therefore, y n plus 1 is y n plus h by 2, k 1 is this plus k 2, so minus y n minus h. So, this is one coefficient here and so 1 and minus 2 get cancelled, so minus h then this is plus h square by 2 y n.

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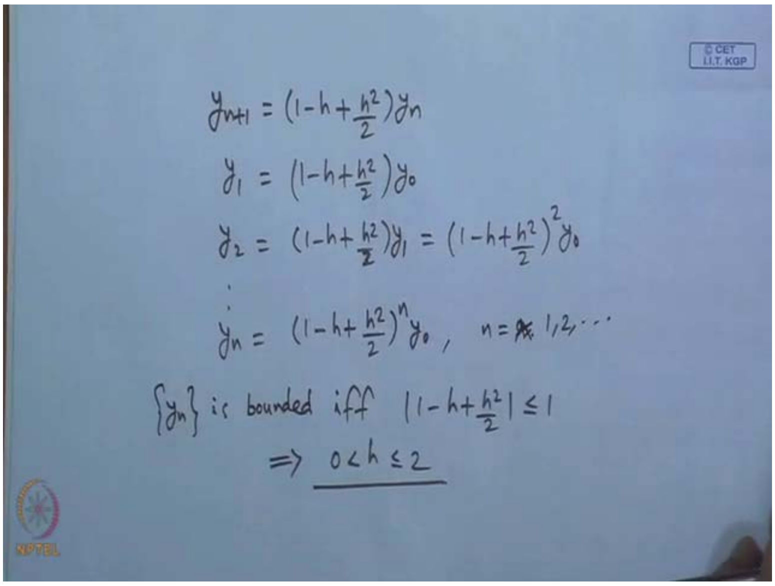

So, what did we get y n plus 1 is 1 minus h. So, y 1 is y 0, recursively if you apply y 0 hence, y n is y 0, n is equal to 1 2. So, this is an expression so a part is done, now for b part. So, for what values of h is the sequence y n is bounded, so this is the sequence is bounded f and f mod 1 minus h is less than or equals to 1, this implies this is a range for which the sequence is bounded. So, such methods will give an idea how to compute the boundedness of the solution and the range and the step size. So, we have simply applied on this method and then tried to determine the solution y n explicitly and then put the condition for the boundedness.

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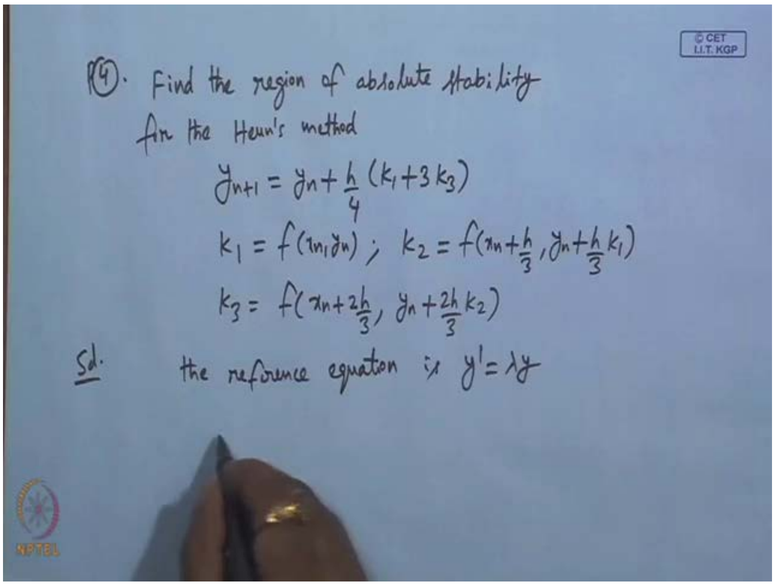

So, let us move further, find the region of absolute stability of the Heun’s method given by where k 1 is this and find the region of absolute stability of Heun’s method. Now, what do we do for absolute stability, there is a reference equation so we have to consider only the corresponding reference equation. So, what will be that the reference equation is y dash equals to lambda y so the reference equation is y dash equals to lambda y.

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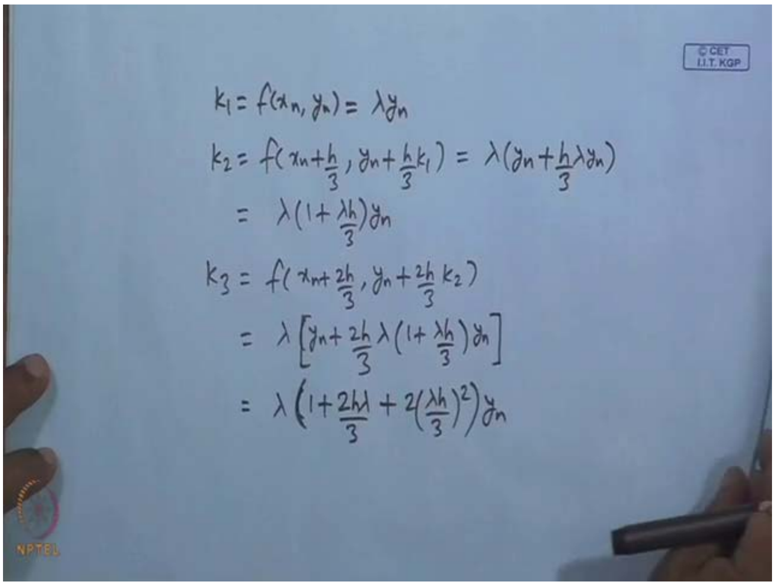

So, let us work out k 1 is f of x n y n, which is lambda y n, k 2 will be lambda times y n plus h by 3 k 1, k 1 is so this is lambda 1 plus then k 3. So, this will be lambda k 2, k 2 will be lambda 1 plus 2 by 3 h lambda plus 2 square y n.

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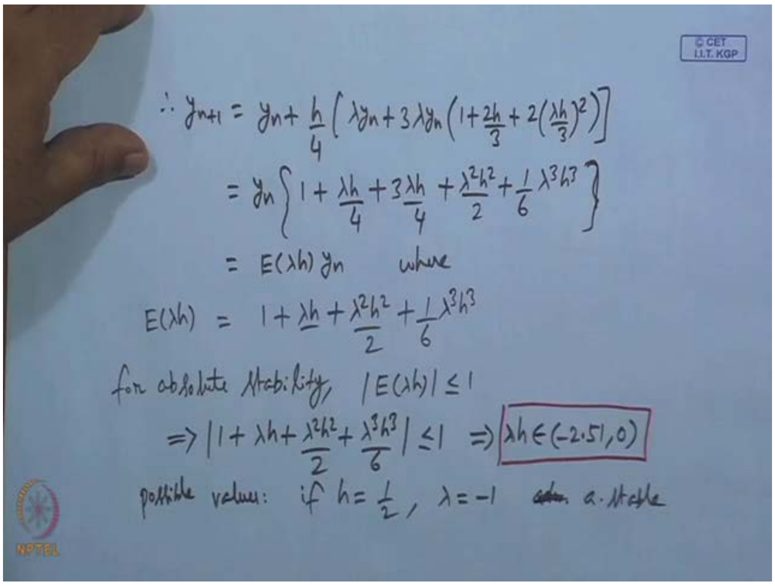

So therefore, y n plus 1 is y n plus h by 4, this method is k 1 plus 3 k 3. So, k 1 is lambda y n 3 k 3, k 3 is this much, so 3 lambda y n 1 plus so this is simplified 1 plus lambda h by 4. Now, this is exactly E of lambda h y n where, E of lambda h equals 1 plus lambda h plus this. Now, for absolute stability, this implies lambda h belongs to minus 2.51. So, it is pity tough, but one can solve it and this is the internal of absolute stability.

So, for example, if we have some possible values so for example, if h is of course, always positive so if lambda is positive, so whatever may be the value so if it is beyond this then it is not administrative values. So, lambda equals to say minus 1 so this gives absolute stable, so one can obtain various values.

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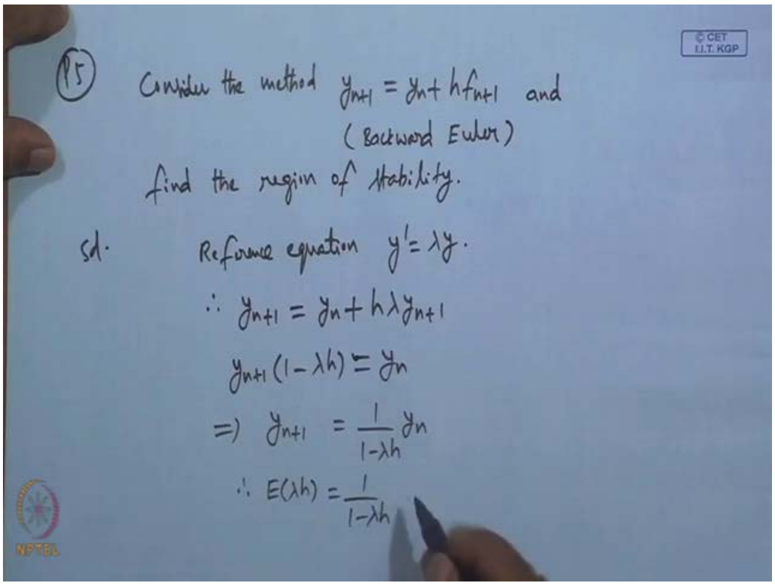

So, let us consider this is p 5 so consider the method y n plus 1 equals to y n plus h f n plus 1. So, this is almost looking like Euler, but this is called backward Euler. Consider the backward Euler method and find the region of stability. So, the method is again state forward, so the reference equation is therefore, the reference equation is this therefore, y n plus 1 is plus h lambda y n plus 1. So, we need to get a form, so how do we get it 1 minus this equals y n, therefore, E of lambda h is this.

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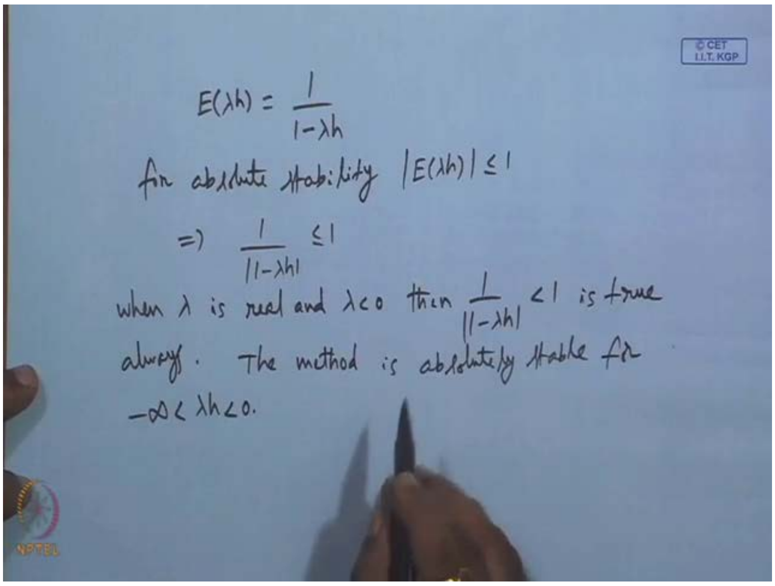

Now, for absolute stability we had E of lambda h is 1 minus lambda h for absolute stability. So, we need this so this condition we have to come up with the range, so before error at the range we can argue lambda is positive what happens and negative what happens like that? When lambda is real and lambda is negative then see if lambda is negative so h is positive, if h is negative so this becomes 1 plus lambda h. So, this 1 hour of that is then is true always therefore, the method is absolutely stable for minus infinity lambda h 0.

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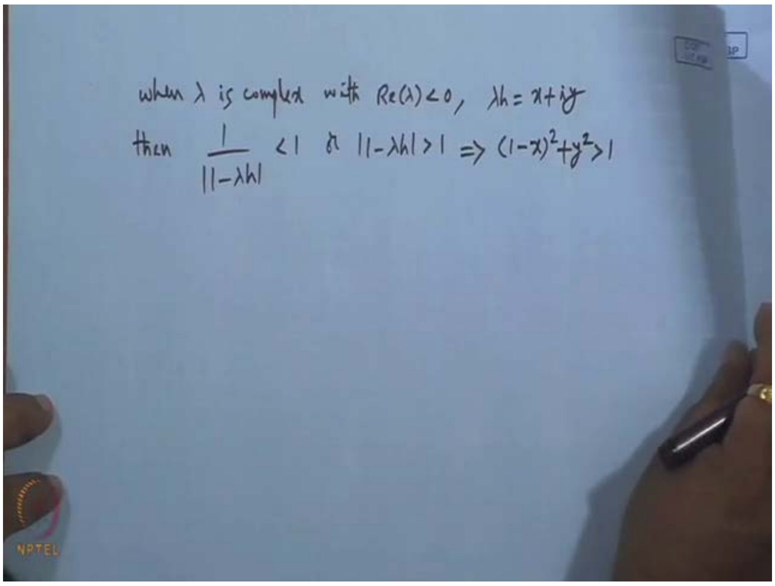

In the other case the lambda can be complex as well, when lambda is complex with real part of lambda is less than 0, so lambda is complex real part is this. So, this quantity is less than 1, we need or 1 minus lambda h greater than 1, so this gives 1 minus x square greater than 1, so what is this region? This is the region outside the circle, it is a interact 1 0, so this is region and radius 1. So, depending on the problem we error it different stability conditions, but for a given problem how do you come across a lambda. See f is non leaner here so f can be any non-liner quantity, but then the linearization says you can linearism it and then figure out what is f?

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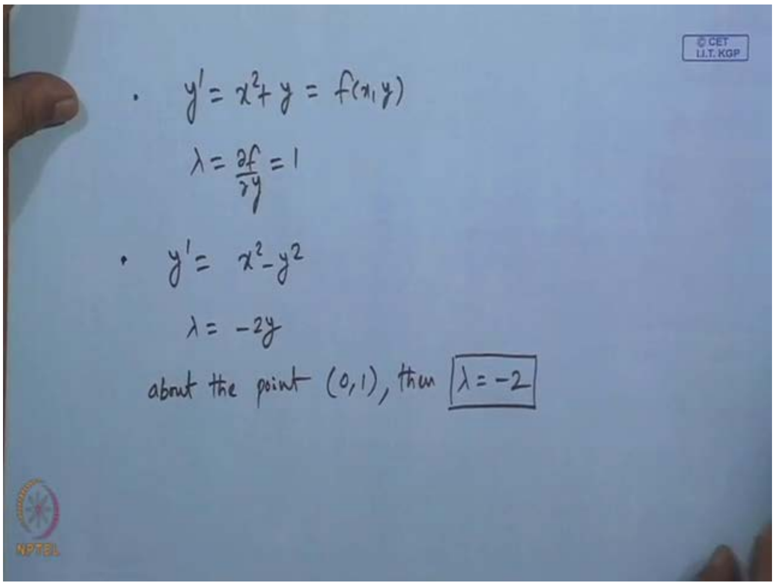

So for example, if y dashed is say x square plus y so our f of x y then our lambda is duo f by duo y, so in this case there is just 1. Suppose, y dashed is x square minus y square so then lambda is minus 2 y. Now, suppose we would like to analyze about the point 0, 1 then lambda is minus 2. So for example, when you using Euler method what kind of h will give us, so that the method is absolutely stable so keeping this lambda in you, for this initial value problem we have to choose h such that the range of the absolute stability interval is met. So, these are some observations we have to be careful about. Now, let us look at another problem, this is R K second order.

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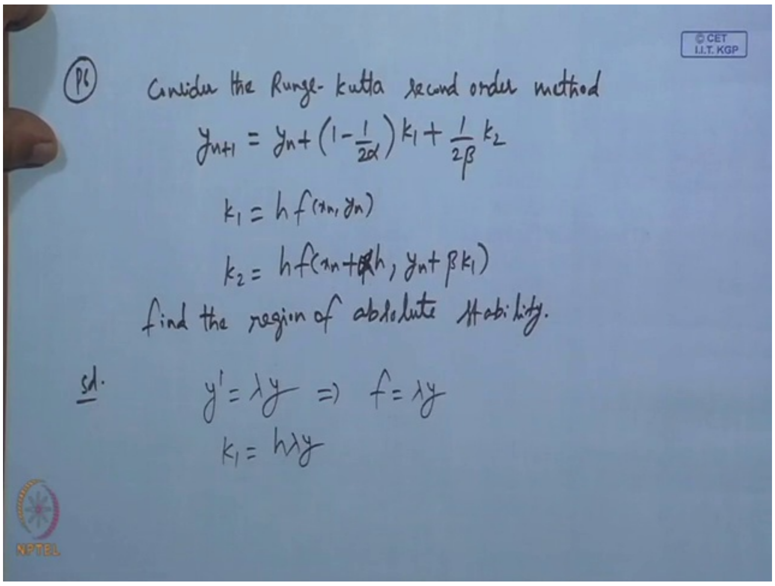

Consider the Rangakutta second order method given by equals to y n plus 1. Well if you take h there we need not otherwise we have to take h here this alpha h beta k 1. So, if this is the case find the region of absolute stability, so the story is the same. Now, consider the reference equation y dash equals to lambda y this implies this, so k 1 is h lambda y then k 2 is h.

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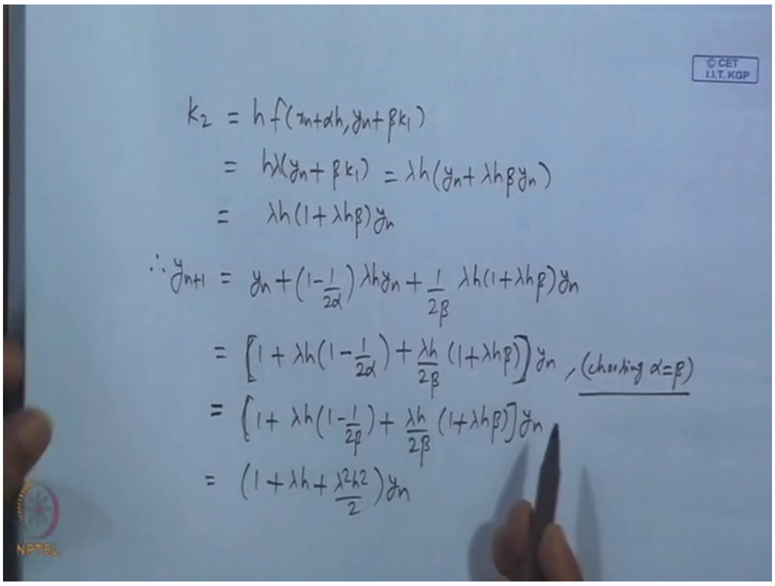

So, this is h into lambda y n plus, so this is lambda h y n plus beta k 1, that will be lambda h beta y n, so this will be 1 plus lambda h beta y n. Therefore, this was our y n plus 1 so y n plus 1 minus k 1, so k 1 plus lambda h y n plus 1 by dou beta k dou lambda h 1 plus lambda h beta y n. So, this can be written as 1 plus lambda h, it take common 1 minus plus lambda h by dou beta 1 plus h lambda beta.

So, from here we have obtain what is E of lambda h. So, consider a special case choosing alpha equals to beta 1 plus at least in this case we get 1 plus lambda h plus into y n. This is the special case otherwise you get a very complicated expressions, so we have to get the range of the interval will consist of alpha and lambda h.

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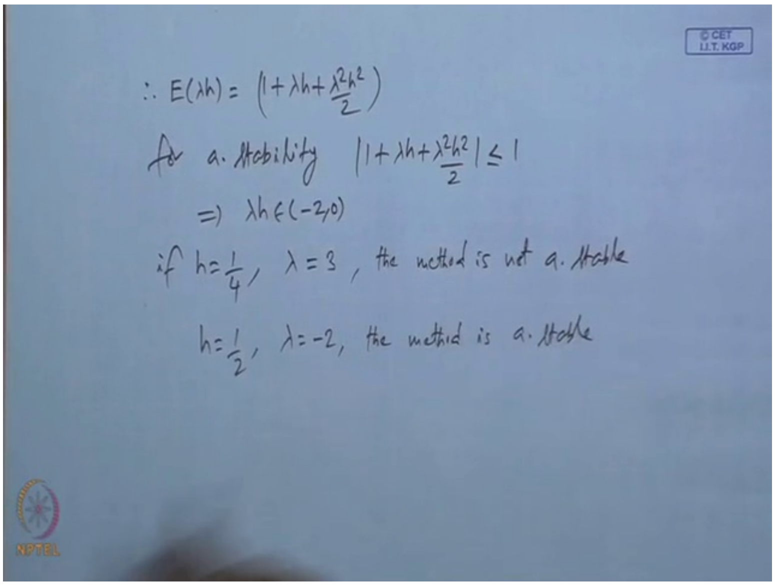

So, at least in this case it gets simplified therefore, E of lambda h is 1 plus lambda h plus lambda h square. So for absolute stability mod, this must be less than equal to 1, so this is satisfied and lambda h belongs to minus 2, 0. So, that means if say h is one-fourth so lambda equals to say 3, the method is not absolutely stable because it is minus 2. Suppose h is half and lambda equals to minus 2 then the method is absolutely stable.

So, estimating these stability intervals is very much useful, so that there is a trade of between the lambda and then step size and the internal lambda depends on the given non leaner function. So, may be next class we may do some more problems until then.

Thank you, bye.