Numerical Solution of Ordinary and Partial Differential Equation

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Lecture - 6
Error-Stability-Convergence of Single Step Method

Hello, so last class we have learnt higher order methods and also higher order equations. So, that is second order initial value problem has been solved by reducing it to a system of couple first order equations. Now, when we define a method how do we know that what is the error? Suppose somebody gives this is the method how to compute the error? And of course, we discuss in this lecture how a particular method is stable? And whether the particular method gives convergence solutions? And what is the reference equation with, which we analyze this process. So, let us discuss first how to compute error given approximation?

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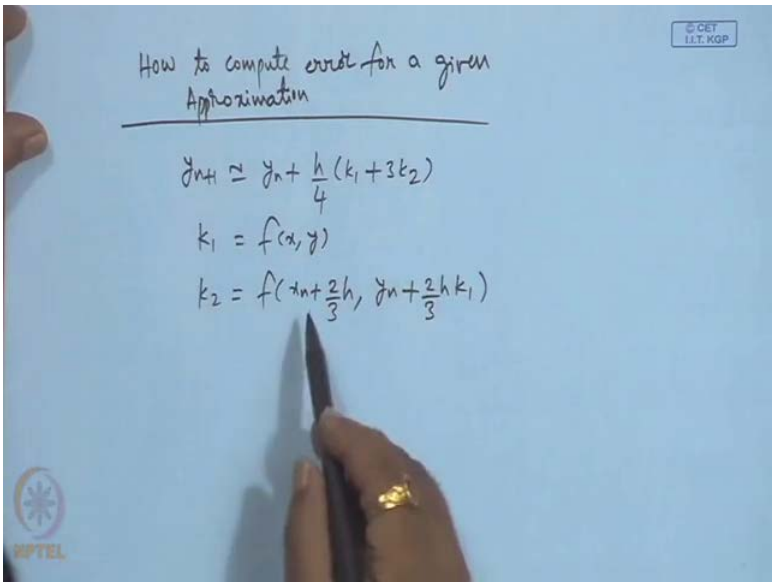

So, how to compute error or a given approximation? So, let us say we have the following approximation. So, I am not naming the method what is this method called etcetera. Suppose there is a sum given therefore so these are the weights k 1, k 2, w 1, w 2 are the weights 1 by 4 and 3 by 4 and the k 1, k 2 are defined like this. Now we would now like to know what if we approximate by this formula or a given I V P what should be the

error? So, how do we do it? As I mentioned we have to make the expansion terms of h and then we have to expand the Taylor series and try to compare. So, let us try to do that.

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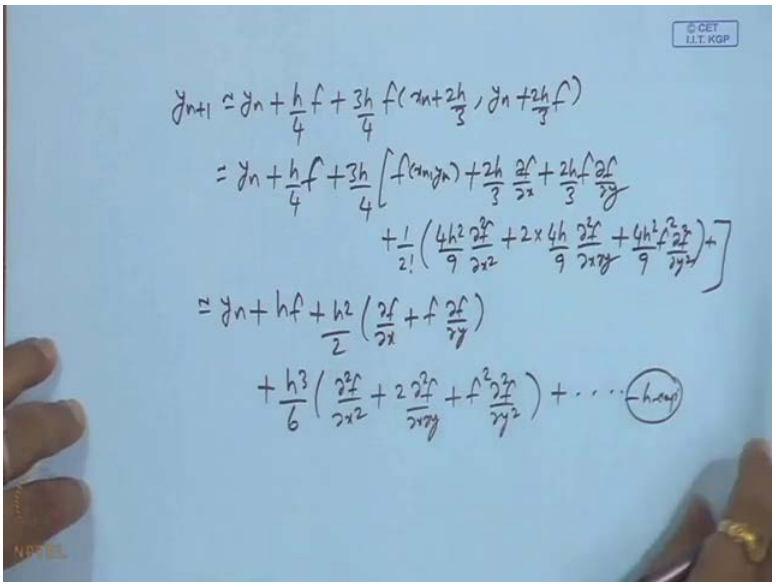

So, we have y n plus 1, so y n plus h by 4 f plus 3 h by 4. Now this has to be expanded in the powers of h. So this one f of x n y n plus the increment is this 2 h by 3 dou f by dou x plus increment with respect to this 2 h by 3 f. So, these are the first order terms plus second order terms square of that. So, two times plus. So this is our expansion.

So we have 3 by 4 and 1 by 4. So h f plus h square by. So, h square terms. So, one term h square term is 3, 3 gets cancelled and 2 by 4 is half. And we have. Then the next term from here we get plus h cube h cube by 6. So, this is our h expansion. Then we have to compare this with Taylor series expansion and you can see in Taylor series this term exists. So, first term approximates y of n and second term and third term. So than if you make equal coefficients of h power 0, h power 1 x square.

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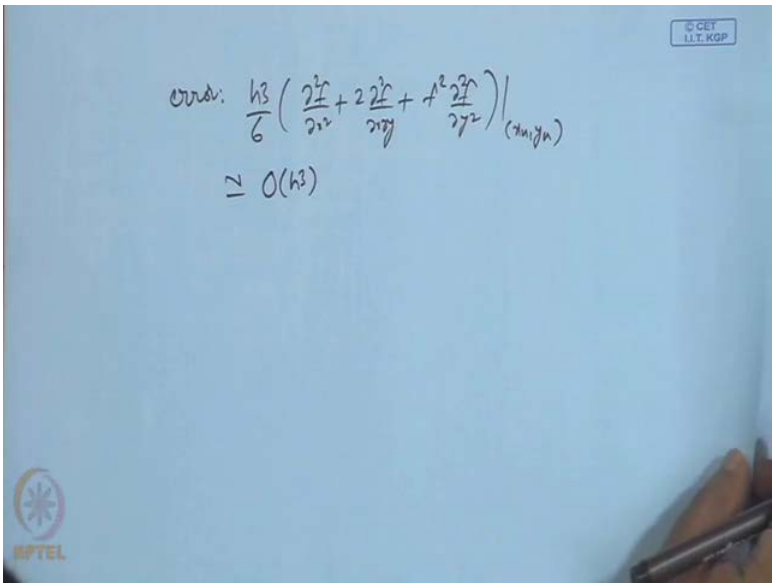

So, than we get error starts from. So, for any method we try to determine using this method, using this way. So, we know x and y n. So, this f we know. So, the error is a order h cube and hence the method is of order h square. So, that is second order. Now as I mentioned we have to discuss the stability and convergence. So, what do you mean by stability and convergence? So, let us try to do the understand this with respect to a reference equation.

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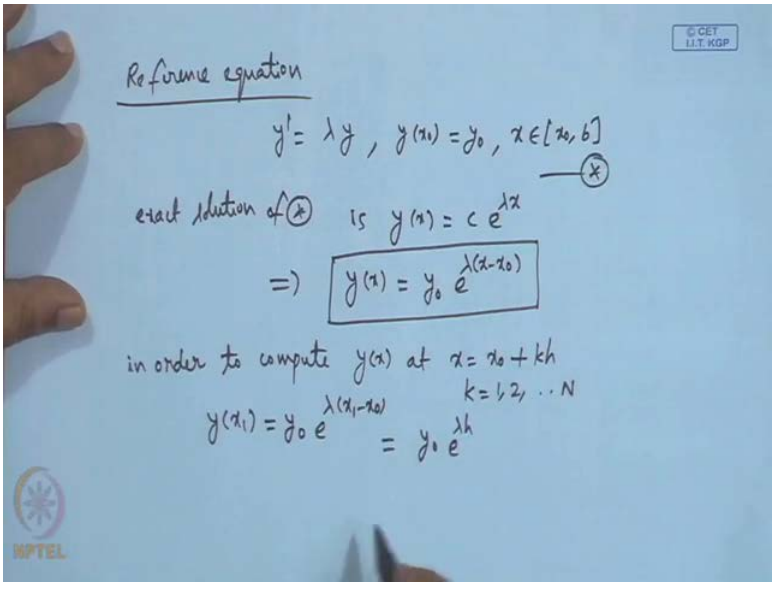

So, what is the reference equation, what is our reference equation? Y dash is lambda y, y of x 0 is y 0 x so this is our star. And exact solution of star is y of x equal to c times. So, if you use this initial condition we get y 0 e power lambda. So, this is our exact solution right? Now in order to compute y of x at... So, that is idea right? So, what is y of x 1, y 0 e power lambda x 1 minus x 0? So, that is y 0 e power lambda h because x 1 minus x 0 is…

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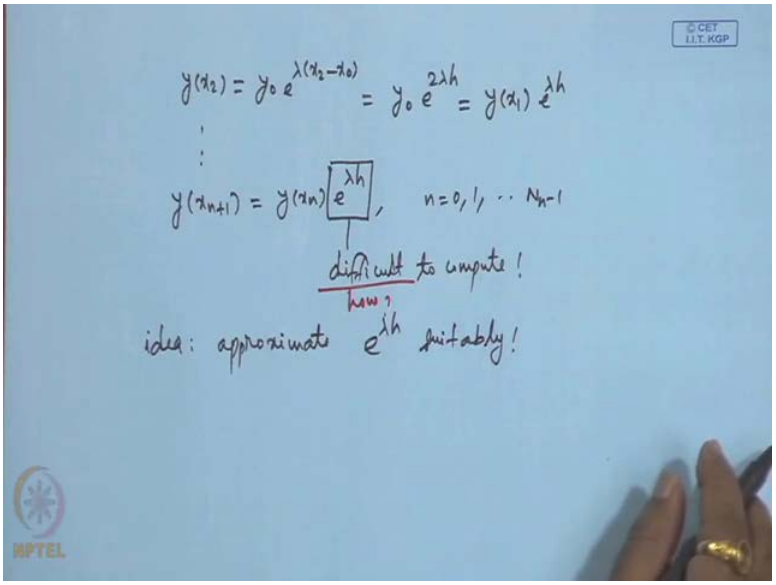

Then y of x 2 is y 0 e power lambda x 2 minus x 0, so y 0. So, this is y of x 1. So, we can generalize. Now in order to compute the solution we have to compute this, difficult to compute, but difficult to compute means how difficult, so how difficult? So that idea is approximate e power lambda h suitably, approximate suitably right? So, how do we approximate?

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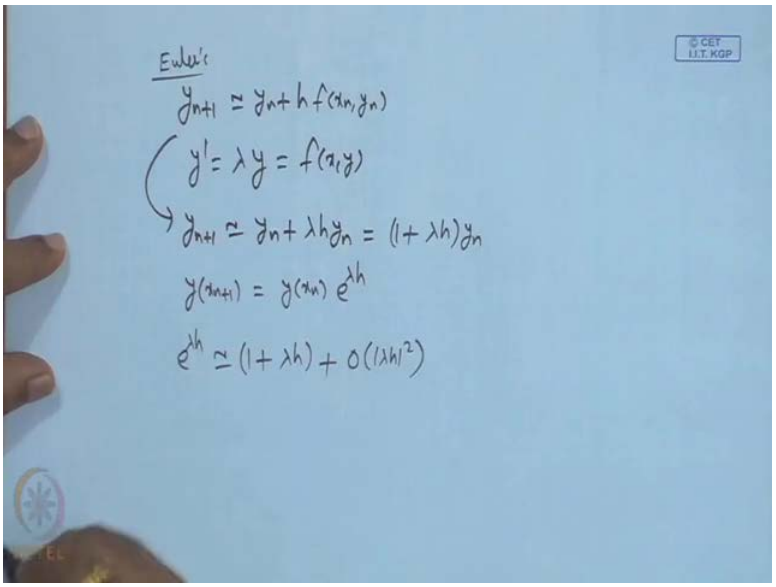

See, in case of Euler's method. Let us talk about Euler's y n plus h f of. And what was our reference equation? Lambda y. So, this is our f of x y. So, therefore, this reduces to y n plus lambda h y n. So, this is 1 plus lambda h y n. And what was our approximation? y of x n plus 1 is y of x n e power lambda h. So, y n is approximating this. That means e power lambda h. 1 plus lambda h is approximating this. So, therefore, this is Euler. So, for Euler method e power lambda h has been approximated by 1 plus lambda h plus order of right?

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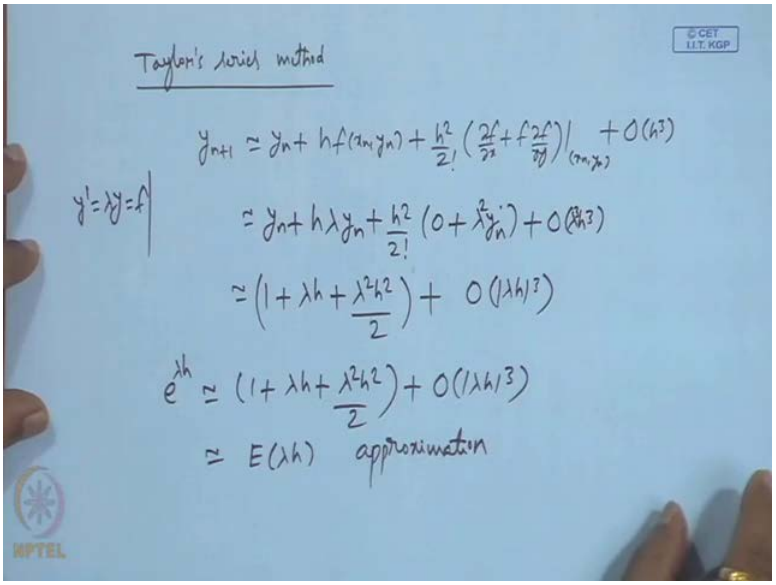

Suppose, let us take Taylor's series method. So, take y n plus 1 is y n plus h f of x m y n h square by 2 factorial so y double. So, that is 2 f dou x. Now this with our reference equation that is y dash equal to lambda y which were as f. This reduces to y n plus or f as lambda y n. Dou f by dou x f lambda y square plus order of h cube. So this is one plus lambda h plus lambda square plus. So, when I mention the there is a lambda sitting when we substitute this definitely there will be a lambda q. So, that means e power lambda h has been approximated by. So, this is the approximation right? So, let us call this approximation for.

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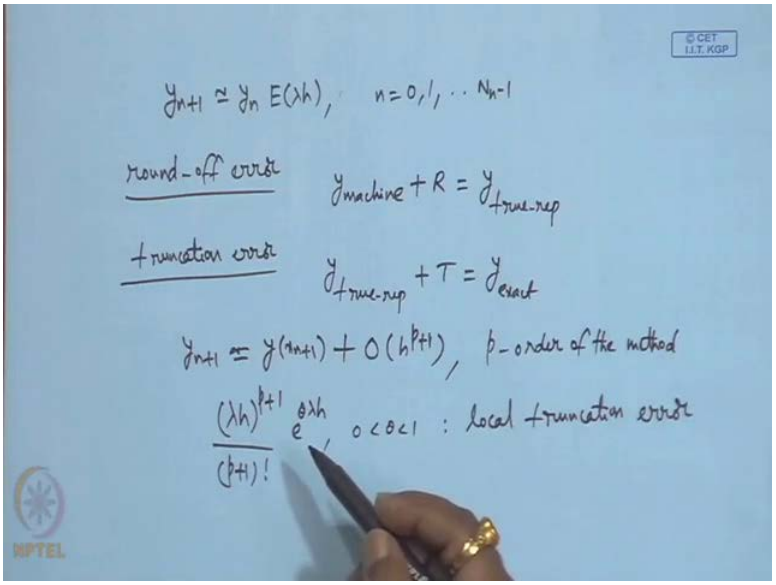

So, that means given y n approximation is as follows y n. So this is what is happening. Now we have to discuss a stability right? So, what is round of error? When you have an approximation there is a value, which is machine value plus R is y true representation. This is the round of because of the machine distractions we are rounding of under truncation error. So, y true representation plus T is y exact. So, that means we are expecting something exact, but y we are representing by a approximate formula and we are throwing some terms so that is the truncation error.

So, y n plus 1, so y n plus 1 is y of x n plus 1 plus where p is order of the method because this approximation is up to something and that is the truncation error. So if the method is pth order some theta. So, this is local truncation error more generalization right? Now if you say this is up to some approximation some truncation. Now let us say we compute y

1. So we have thrown T corresponding to y 1. Now compute y 2. So, we use y 1 that means whatever your thrown T at computing y 1. So, that will be still sitting in the system and then we have to understand what kind of a impact it is creating.

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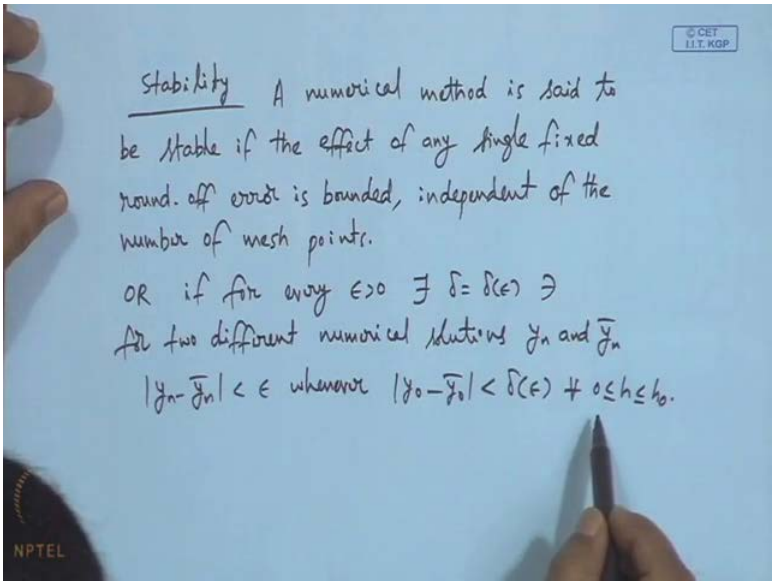

So, let us talk about stability of a method. So, formal definition nothing but how the error is playing a roll? A numerical method is said to be stable if the effect of any single fixed round off error is bounded, independent of the number of mesh points. A numerical methods is said to be stable if the effect of any single fixed round of error is bounded. That means the effect of round of error is bounded since it is not really growing. And its independent of the number of mesh points. This is another important parameter.

So or if for every epsilon is greater than 0 there exist delta of epsilon such that for two different numerical solutions. y n and say y n or the difference is less than epsilon whenever the corresponding initial conditions for every h. So this is very important for every epsilon there exist a data. It depends on epsilon such that see the scheme is same we are talking about two different solutions with two different initial conditions. So now as long as the difference is delta of epsilon the approximations should not be differed by large quantity. And this is, this must hold independent of the mesh size.

So when we ate approximating with what your, so for example in Euler method we have seen, in Euler method we have expanded then now we first considered two terms. So with two terms we have got 2 plus lambda h then Taylor's dependents on suppose your

using three terms suppose we get 1 plus lambda h lambda h square. Then if we use more terms that means your exact solution, which is of e power lambda h we are approximating that right? So a comparison between these two must be made. So that means compare to what your approximating e power lambda h. It should have been e power lambda h, but now your approximating by some something. So also there should be a comparison.

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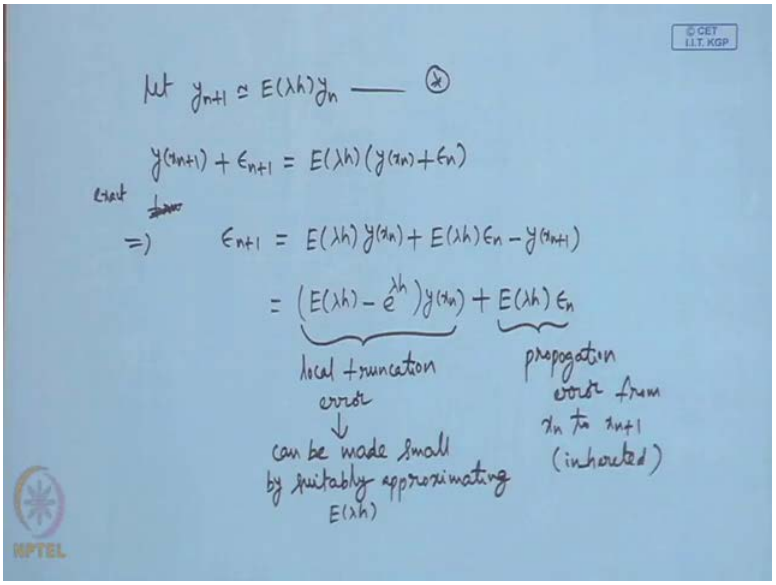

So, let us do that. So, let us call our approximation. So, which means. So this is a true. Exact plus epsilon n plus 1 epsilon. Plus 1 is e power lambda h y of x n plus. Y n 1 plus 1 is exact plus epsilon n plus 1 at one stage. Then we have this approximation then y n is y of x n plus 1. So, this implies lambda h y of x n epsilon n. Now this is minus. How did we write this? Y of x n plus 1 is actual solution is e power lambda h into y x n. So, I am writing y of x n is replaced by e power lambda h e of x n. So we take common plus. Now what is this tells? This is exact, but this approximation right? So this is nothing but local truncation error.

Now how can you minimize this? You can minimize this by approximating e power lambda h more closely with e power lambda h. So, that means this can be made small by suitably approximating that means suitably choosing e of lambda h. We choose better e of lambda h its close to. So, you can see Euler method 1 plus lambda h where as Taylor's series is with more number of terms more close to e power lambda h so in that sense.

Then what about this? See this is approximation and this is the error attain stage and that is contributing to error at n plus stage. Therefore, this must be the propagation error.

So what is the propagation error? This is the propagation error from x n to x n plus 1 and this is inherited. So error attain at a 1 place consist of two parts. One is local truncation error, which is due to the approximation. The better you approximate close to e power lambda h the minimum the local truncation error other is the propagation error. The error at nth stage has been magnified by this factor.

Now one can guess easily how do you minimize this? Yeah, as long your approximation is not magnifying because this is the factor method it is magnified by this? So, when do you say that the propagation error will diminish, if e of lambda h is not magnifying it to a large extent. So we discuss, this is the stabilizing factor because what is the stability? Stability means I mentioned. See your computing your defined an a process it is a single step process to compute a value at same stage you require value at the past one stage. Now you it is a recurrence processor.

So you compute y 2 and use it to compute y 3 than compute y 4 using y 3 and so on so fourth. So, whatever the errors existing at one step ahead that has been incorporated into the higher steps right? So know you have to control this. So, this can be controlled only when approximation is playing some role. So, we define that now.

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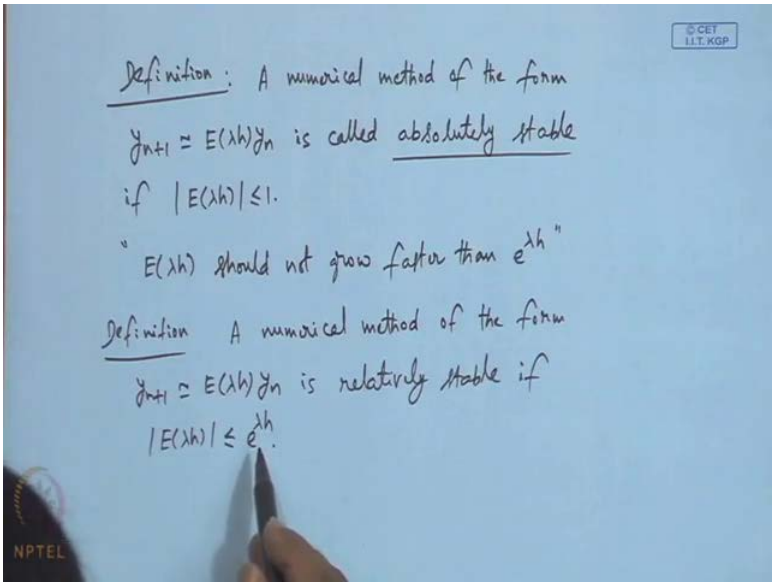

So, the definition. Say a numerical method of the form is called absolutely stable. If what here their your. This is the magnifying factor. So this should not magnify to the large extent that means here the model must be equal to 1 right? So, that means E of lambda h should not grow faster than. Because your approximation is making it magnified the error definitely is not stable. So, the comparison is between this and this approximation. So hence a numerical method of the form is relatively stable. If see, if you are, you can ensure to that this is always less than equal to 1 than its absolutely stable.

That means whatever the errors, which are been carried at nth stage. So they are not magnified beyond magnitude of less than 1. So, the error is going to be bounded. One can ensure that at next level the error is bounded. So this is absolute stability, but another is compared to what? This approximation is compared to E power lambda h. So, therefore, this is, if this is less than equal to E power lambda h. So then it is called relatively stable, right?

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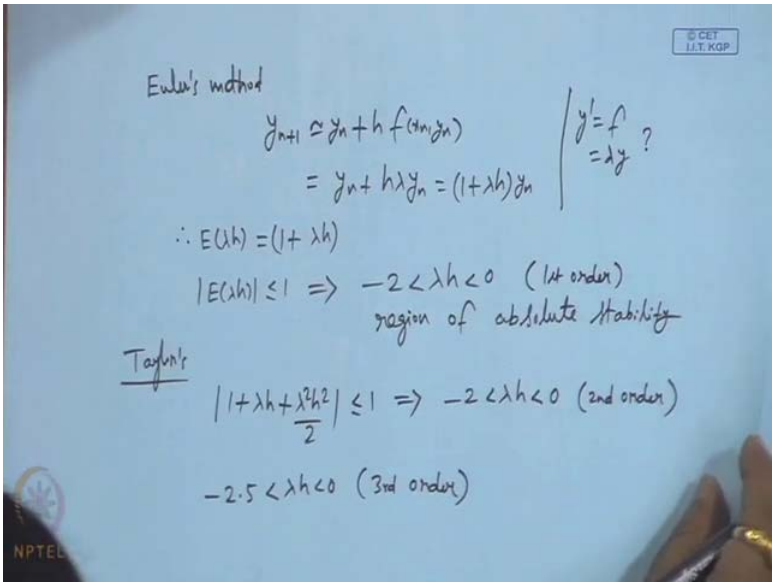

So, take Euler method. y n plus 1 is y n plus h, which is equal to y n plus h lambda y n. So, therefore E f lambda h is 1 plus lambda h. Hence more E of lambda h is less on 1 implies so that means this is the region of absolute. That means the method is going to be absolutely stable as long as it chooses step size such that lambda h for holes in this range. So, what is lambda? So lambda is coming from our reference equation, but we have more general form that equals to f. So, that means at a lighter stage y dash equals to f.

So why are we talking y dash equals to lambda y as a reference equation were, as we need for a more general case, so will do it little later. What is the connection between this? Why do we take such a reference equation when we need a more general case of f? So, then Taylor's method, suppose up to two terms, three terms, so this implies. So, this is first order second order. Then one can also obtain third order right? So, one is propagation error, another is local truncation error. Then we talk about, we talk about absolute stability with respect to this factor. And we talk about relative stability with respect the distance between these two. So, that is what we have done right? Now we go back to this question. Why did we consider y dash equals lambda y when we need y dash equals to f?

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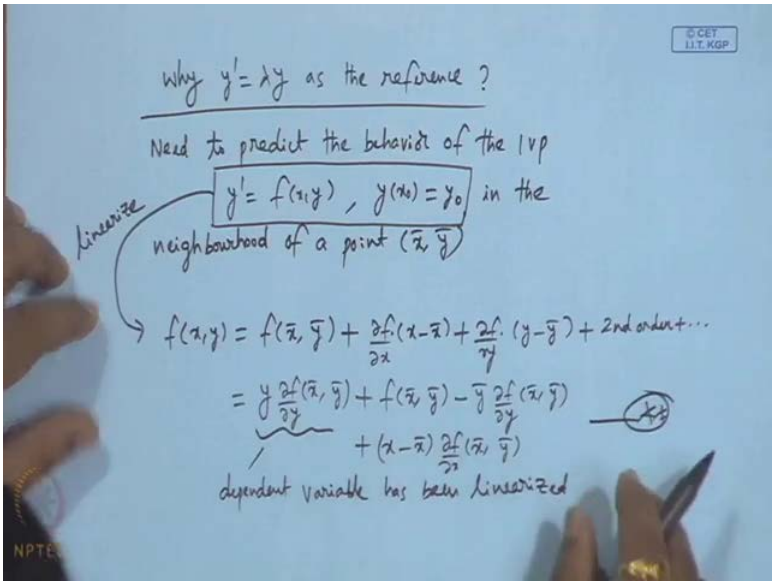

So, why y dash equals to lambda y as the reference right? So, what we are doing? We are considering a particular point x n y n. And we are trying to analyze for that value what happens to the approximation? So, that means we need to predict the behavior of the I V P y 0 in the neighborhood of a point. So, this is what we are trying to analyze, but with respect to the general case. Then what we do is linearize. So, how do we do it? f of x, y around this point plus this is multiplication plus. So, this equals y dou f by dou y plus f minus y bar dou f by dou y plus. Now this is an Euler function, but we have line arise it with respective the dependent variable. So, that means with the respective to the dependent variable the question has been linearized.

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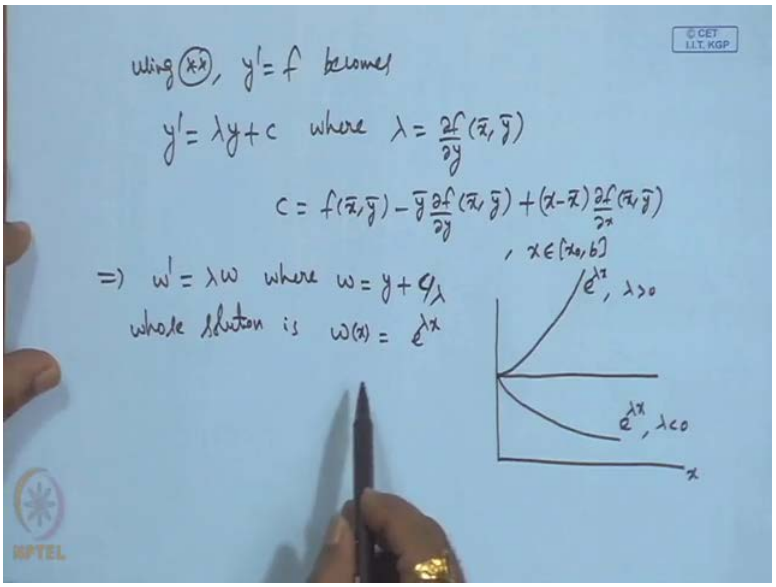

Now with this f your y dash equals to f can be written as this becomes y dash the coastal the lambda y plus c were lambda equals to lambda y. So, lambda y is dou f dou y and c. So, were of course, were x is in. So, this can be transformed using c by lambda whose solution is. So, that means. So, this is. So, we tried to analyze the behavior of the solution and it depends certainly on the lambda. And were lambda in turn depends on f.

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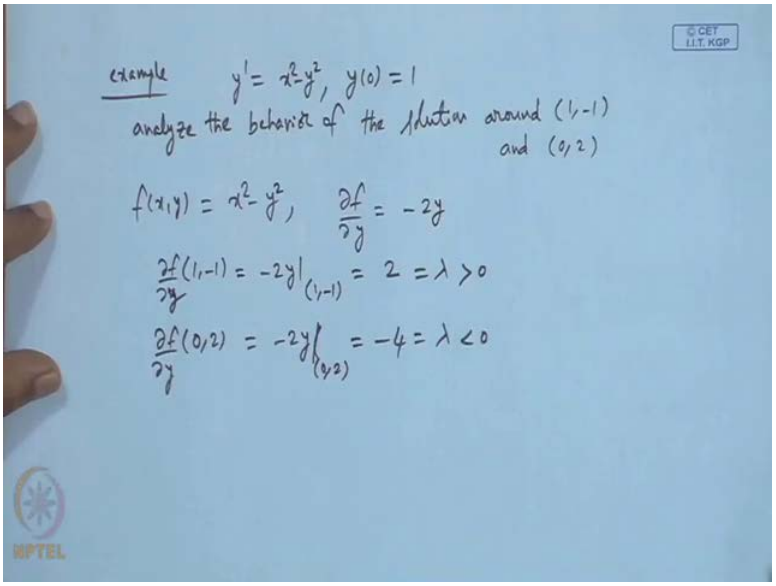

So, let us see. For example, say y dash is x square minus y 1. Then analyze the behaviors of the solution around say 1 minus 1 and say 0, 2. Now what is our f? Therefore, suppose

let us have. So this will be minus 2 y. Now dou f dou y at 1 minus 1 will be minus 2 y at 1 minus 1. So that will be 2, which is lambda positive dou f by dou y. So around 0, 2 will be minus 2. So that will be minus 4, which is lambda.

So that means this f around this point the solution behaves like this. And were as around 0 to the solution behaves decays and here it grows. So our question of why reference equation. That means the solution, overall the behavior of the solution of the non leaner case can be interpreted by linearizing f and estimating the behavior of the linearized k is in this sense, because this is in more general form.

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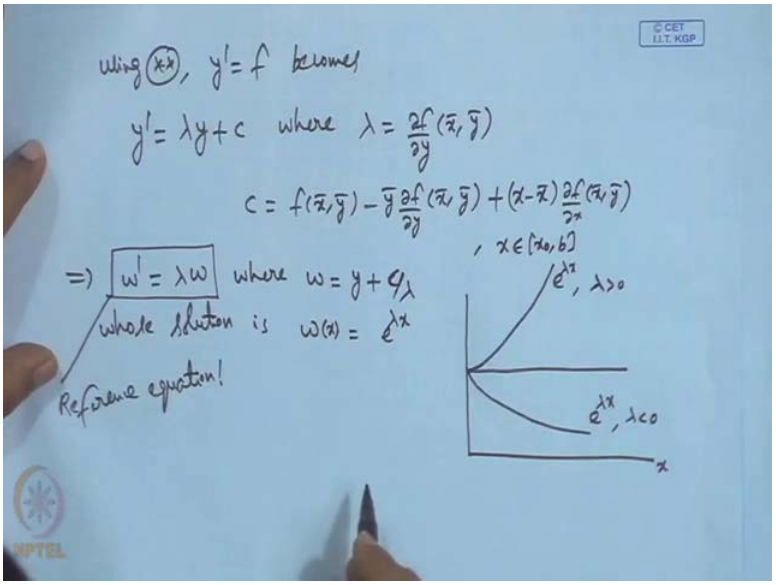

Hence this is our reference equation, this is our reference equation. So we started with stability. So, numerical method is said to be stable if the effect of any single fixed round of error is bounded, independent of the number of mesh points. So that means the approximation should be bounded by epsilon whenever the initial data is within the range of delta, which depends on epsilon, and then we have approximated given solution of the given I V P.

Then we introduced the error at n plus one stage. So, that will come from two different parts. One is the inherited, which is due to the error at the nth stage. And the other one is due to the approximation. And the better you approximate E power lambda h you get the better minimum local truncation error. Now once we have these two parts. This will define absolute stability. Why absolute stability? Whatever error at n stage it is

magnified by this factor. So as long as you are controlling this there would not be much inherited error.

Therefore, you can assure that absolutely stable were as this compared to what? How far you are from actual E power lambda h? This is your approximation how far you are. So, that will tell relative stability. So, therefore, numerical of the form this is called absolutely stable if this is the condition and relatively. So, it should not grow faster than this. So that will introduce numerical method of the form is relatively stable if this happens. So, for a particular method single step method these are important criterion absolute and relative. So, let us take the one of the R K methods and try to estimates the condition for absolute stability. So, for example, we take hewn method so the hewn method.

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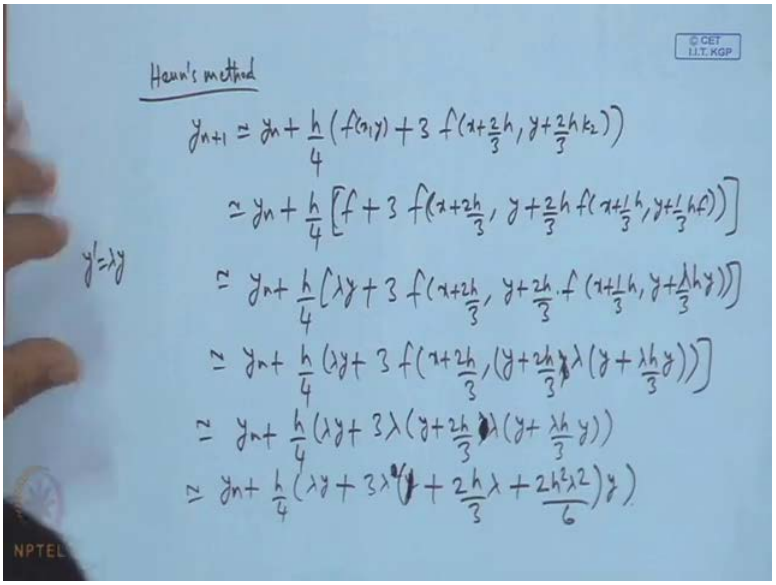

So, we have the hewn method. y n plus 1 is y n plus h by 4 n for this. So this is, so Hewn method h by 4 so k 1 is f of x y plus 3 k 3, so 3 k 3 that is f of x plus 2 by 3 h k 2. So, y n plus f plus 3, k 2 is... Now with our reference equation y dash equals to lambda y this becomes y n plus h by 4 f is lambda y and this is f of. So, y plus 1 by 3 h f is lambda y.

So, we have to f of y plus lambda h y. So, this is y n plus h by 4 lambda y plus 3 f of y plus 2 h by 3. Now this is our y. So, what is f lambda y. Therefore, lambda y plus. So, this is y n plus h by 4 lambda y. And this our y. So, lambda y must be lambda times. So, this is all y n plus 3 lambda square 1 y comes out right? This multiplies only here. I am

sorry this multiplies only here. So, lambda y plus 3 lambda y plus 2 h by 3 lambda plus 2 h square lambda square by 6 and this is 1 y.

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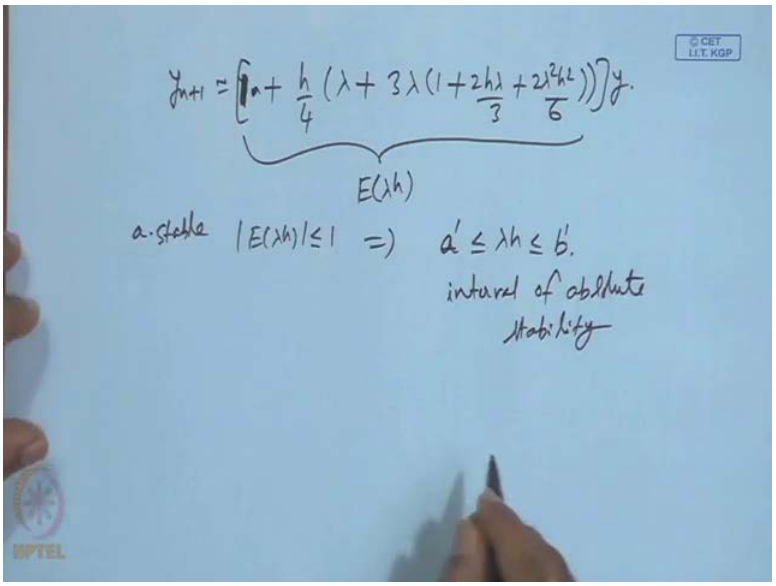

So, this is for hewn method. So, this is y plus h by 4, h by 4 lambda plus 3 lambda 1 plus. So, if you make this 1. So, this is which is slightly complicated. So, we can simplify a bit and for absolutely stable we must be able to get a, some a prime and b prime. So, this will give the interval of absolute stability. So, this looks quite complicated, but nevertheless attempting for at least higher order methods you get an algebraic expression for the range to compute their interval, which the approximate method is absolutely stable.

So, from that algebraic expression we can try to estimate the bones. In some cases it is straightforward and some cases it is tedious. But nevertheless given approximation that means the approximate method we have learnt how to compute the corresponding error. And further we have learnt how to compute the absolute stability interval. And also with reference to the reference equation how far it is, how closely it is approximating that is the relative the region of relative stability. So, this will give sensible idea about single step methods, and how the stability and error are related.

So, in particular the, for the absolute stability the approximation what you are making so that will play a vital role. So, we have learnt R K method this are called expletive

because to compute n plus 1, we have past one value. So, there are some implicit methods, but these are quite complicated. So, will see how we proceed. Have a good day.