Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 40
Summary, Appendices, Remarks

Hello, welcome back so, we have come to the end of this course and today will be the last lecture. So, as such we started our journey from numerical solutions of ODEs and then we have moved to PDEs and we discussed various methods etcetera. So, I would like to mention few remarks and before that maybe, we just briefly summarise how this course has proceeded. And then some appendices, which are kind of useful tools in addressing these numerical solutions. So, we started with single step methods for initial value problems so, I will just put the summary quickly and then, we move to the appendices.

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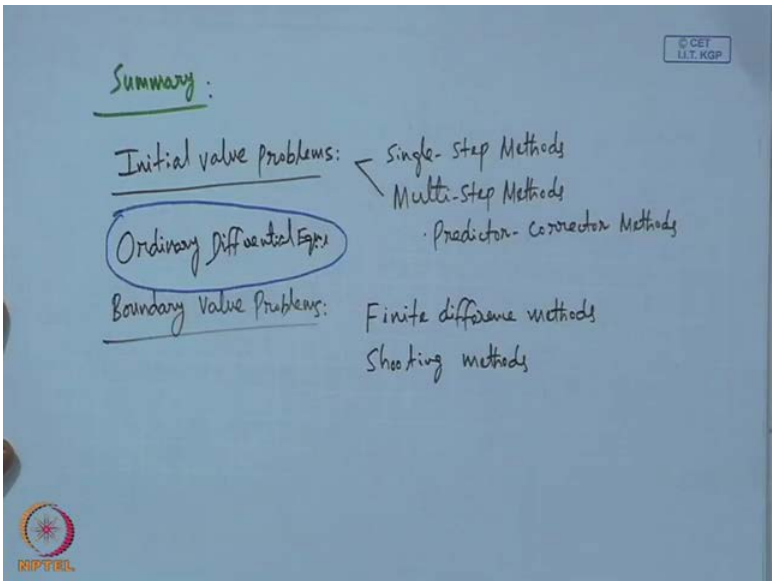

Summary:

Initial value problems: Single-Step Methods, Multi-Step Methods · Predictor-Corrector Methods

Ordinary Differential Eqns

Boundary Value Problems: Finite difference methods, Shooting methods

So, initial value problems so, here we have single step methods and multistep methods then, in multistep methods we have predictor corrector methods. So, then we move to boundary value problems, boundary value problems so, where we discussed finite difference methods and shooting methods. So, in initial value problems the single step methods and multistep methods, they are based on the principle of approximation by weighted average of slopes.

So, this is by and large the principle whereas, in boundary value problems then, finite difference the principle is approximating at various grid points. That means, to compute a past value we need, I mean to compute at a particular grid point, you need past several values. So, this is a boundary value problems so, this is with respect to ordinary differential equations. So, this is within the framework of ordinary differential equations.

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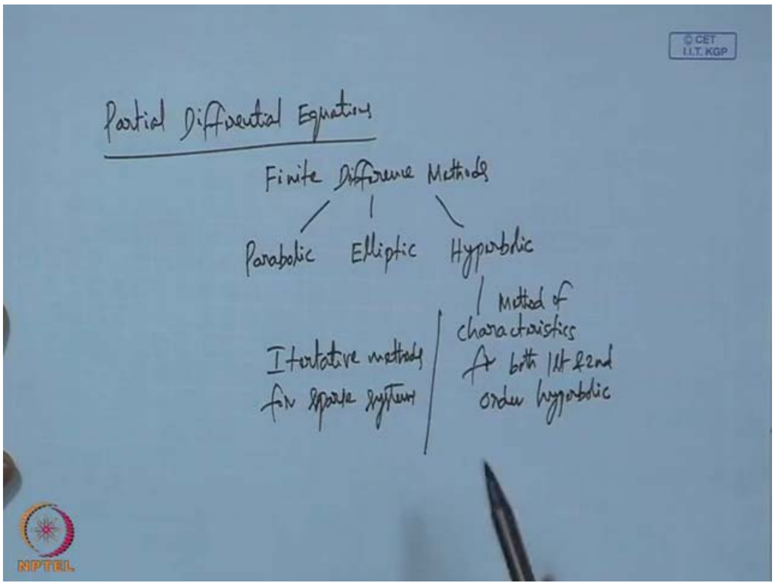

Partial Differential Equations

Finite Difference Methods

Parabolic Elliptic Hyperbolic

Method of characteristics for both 1st & 2nd order hyperbolic

Iterative methods for sparse systems

Then, we move to partial differential equations so, here we considered finite difference methods for parabolic, then elliptic, then hyperbolic. So, while we have done this kind of so, this is method of characteristics for both first and second order hyperbolic. So, this is how we proceeded then, we have to focus somewhere on iterative methods for sparse systems. So, this is typically when we deal with elliptic equations so, we have discussed how each approximation occurs and then, how the error comes into play and then, convergence stability aspects etcetera.

So, the aim of this course is that at the end of it, atleast you get a better idea of what are finite difference methods for both ODEs and then PDEs. And to get little advanced knowledge so that, you try to pick up and then maybe you concentrate on any advanced topic, you can do on your own. So, atleast you will be in such a position so, with this definitely you are at advantageous position. Because, once you are done on your own and practice some examples, definitely you can pick up any advanced book, covering advanced topics and numerical methods.

As I mentioned for example, how to solve problems on staggered grid, alternate direction and multi dimensions. And moreover, if you, if you come across problems dealing with fluid mechanics, as you know most of the studies these days are done numerically. So, you have to, definitely have a fundamental knowledge on various numerical techniques and then definitely the advanced techniques.

So, we come across some tools so, throughout the course as and when we met some technique and then, we had a brief discussion about them. And then proceeded further, but there are certain issues which I thought these topics should be put it as appendices so that, these are helpful.

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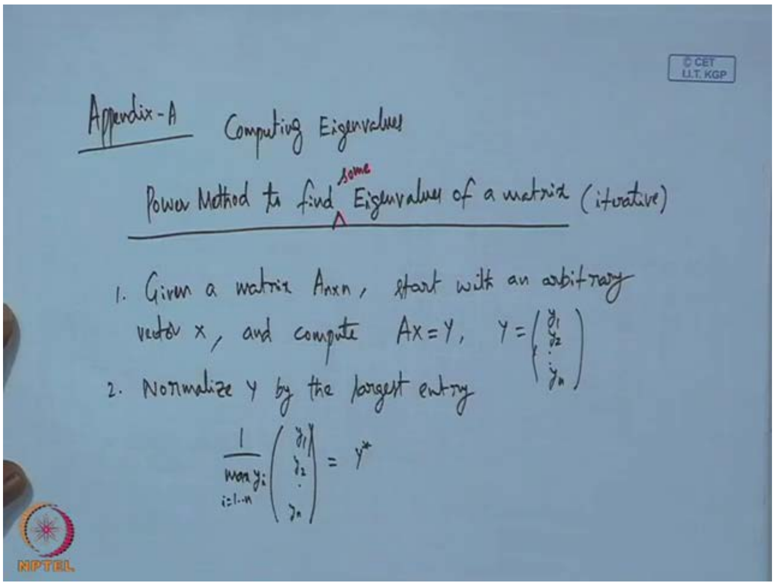

Appendix-A Computing Eigenvalues

Power Method to find $\overset{\text{some}}{\wedge}$ Eigenvalues of a matrix (iterative)

1. Given a matrix $A_{n \times n}$, start with an arbitrary vector x, and compute $Ax = Y$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$
2. Normalize Y by the largest entry

$$\frac{1}{\max\limits_{i=1..n} y_i} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = Y^*$$

So, the first one is computing say appendix A, computing Eigen values. So, computing Eigen values is general issue, which for stability aspects we have been doing. In fact I have given this matrix and then these are the Eigen values, but sometimes it is easy to compute. You can do it, but sometimes you have no clue, but there are methods, one of them which does iteratively is called power method.

Power method to find Eigen values of a matrix of course, this is iterative then, when I say Eigen values of matrix, we should put some because, we can get only some of them. So, what is the idea, the idea is given a matrix A, n cross n start with an arbitrary vector x and compute A x say, A x equals to Y. So, then what we do normalize Y by the largest entry say, here Y is y1, y2, yn. So, then you have to normalize by the largest entry, that means 1 over max of yi so, call this so, y1 max so, this is y1, y2 we normalize so, you get some y star.

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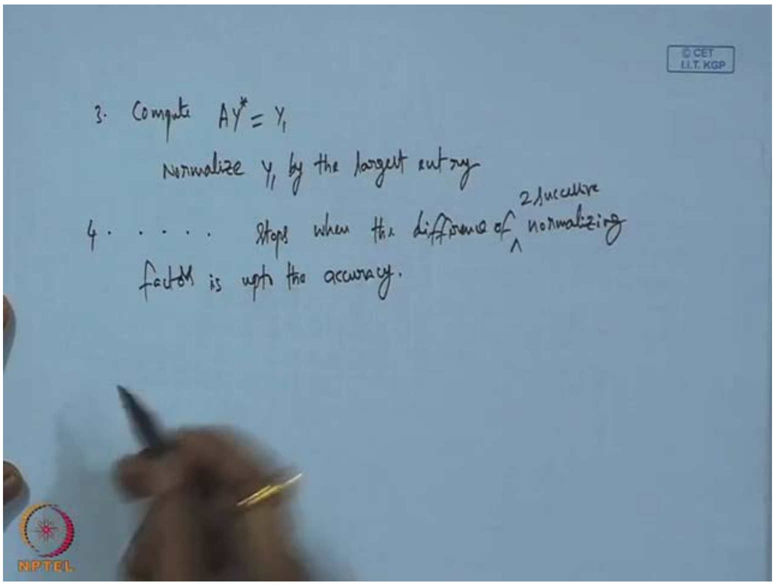

3. Compute $AY^* = Y_1$

Normalize Y_1 by the largest entry

4. . . . . . Stops when the difference of 2 successive normalizing factors is upto the accuracy.

So, then when you get y star compute A y star equals to say, y1 then normalize y1 by the largest entry. When I say largest entry, largest entry in y1 so, this we do and then, when do we stop, when. So, this process continues and then stops when, the difference of normalizing factors, the difference of two successive normalizing factors is up to the accuracy so, for example.

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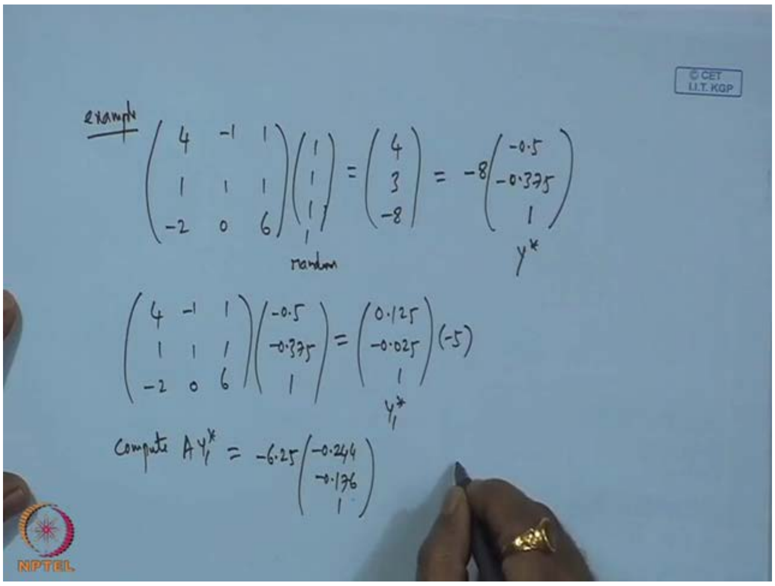

example

$$\begin{pmatrix} 4 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & 6 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{random}} = \begin{pmatrix} 4 \\ 3 \\ -8 \end{pmatrix} = -8 \underbrace{\begin{pmatrix} -0.5 \\ -0.375 \\ 1 \end{pmatrix}}_{y^*}$$

$$\begin{pmatrix} 4 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & 6 \end{pmatrix} \begin{pmatrix} -0.5 \\ -0.375 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0.125 \\ -0.025 \\ 1 \end{pmatrix}}_{y_1^*} (-5)$$

Compute $A y_1^* = -6.25 \begin{pmatrix} -0.244 \\ -0.176 \\ 1 \end{pmatrix}$

So, let us take an example so, consider so, this is a matrix then let us say random we start with this. So, this is arbitrary so, when we compute we get 4 3 minus 8, now the maximum entry in magnitude definitely is 8. So, we normalize so, we take it common minus 0.375 1 so, this is our y. What was the notation I have given, this is our y star because, we have normalized so, this is our y star. Then we compute so, we do this then we get 0.125 times so, this time the normalizing factor is minus 5. And this is our say, y1 star so, then again compute A y 1 star so, this would be something like. So, if we do we get after a few iterations.

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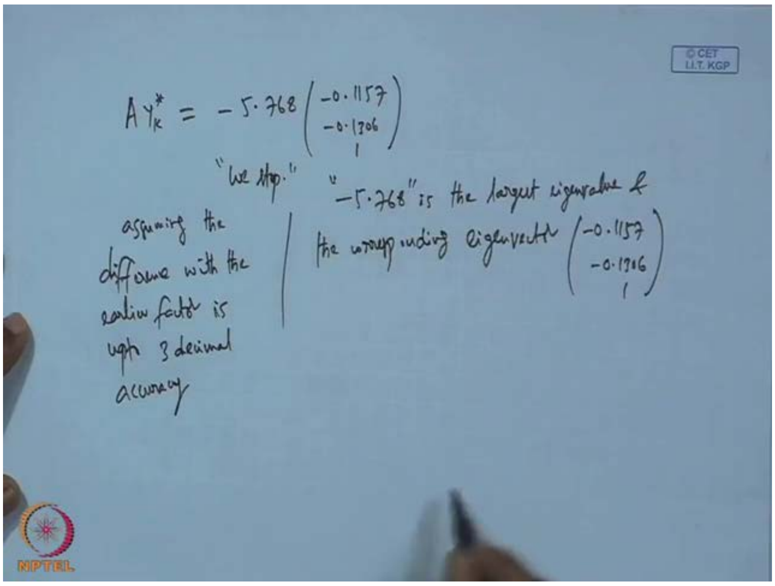

$$A\,y_k^* = -5.768\begin{pmatrix}-0.1157\\-0.1306\\1\end{pmatrix}$$

"We stop."

assuming the difference with the earlier factor is upto 3 decimal accuracy

"-5.768" is the largest eigenvalue & the corresponding eigenvector $\begin{pmatrix}-0.1157\\-0.1306\\1\end{pmatrix}$

So, after a few iterations A on some y k star equals so, this is differing with the previous one so, we stop here. So, that means we declare that is the largest Eigen value and corresponding Eigen vector is when we stop, when I say we stop, assuming the difference with the earlier factor is up to 3 decimal accuracy. You can compute and check it so, we declare this is the Eigen value so, this is an interesting method. Of course, there are proof, why this works out and all that, but this is an iterative so, this gives maximum because, every time you are normalizing by maximum and then we get that. So, now similar method can be used to get smaller because you get the larger one and then inward.

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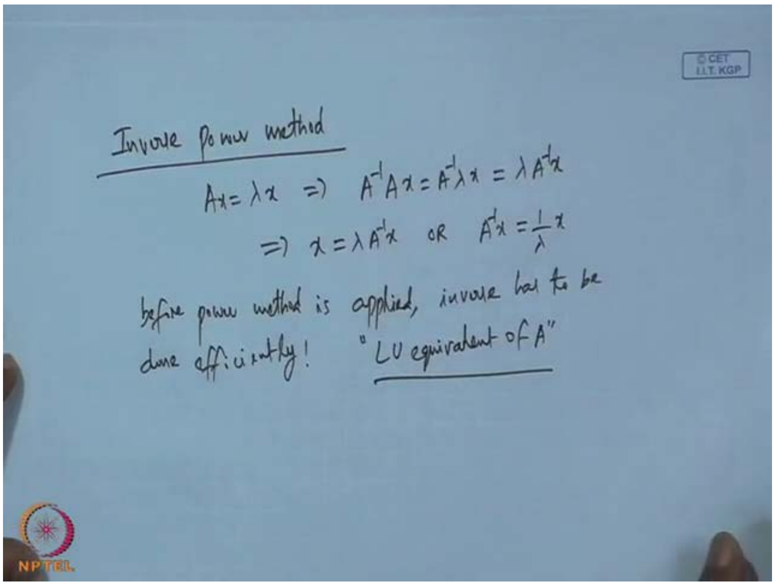

Inverse Power method

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}\lambda x = \lambda A^{-1}x$$

$$\Rightarrow x = \lambda A^{-1}x \quad \text{OR} \quad A^{-1}x = \frac{1}{\lambda}x$$

before power method is applied, inverse has to be done efficiently! "LU equivalent of A"

So, it is slightly in a different way so, this is called inverse power method so, our aim is this. So, this implies A inverse A x is A inverse lambda x, this is lambda A inverse x so, this implies x is lambda or 1 over lambda x. Now this is, see you are you are normalizing every time, but inverse computing, inverse could be expensive. So, before power method is applied, before power method is applied, inverse has to be done efficiently. For example, maybe LU equivalent of A so, this is one suggestion. So, this is one appendix I want to just give a brief idea on power method.

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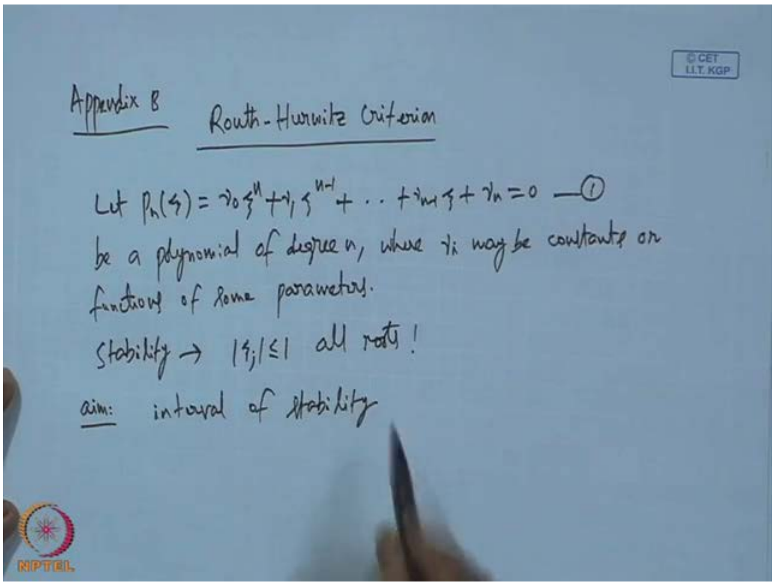

Appendix B Routh-Hurwitz Criterion

Let $P_n(\xi) = \gamma_0\xi^n + \gamma_1\xi^{n-1} + \cdots + \gamma_{n-1}\xi + \gamma_n = 0$ —(1)

be a polynomial of degree n, where γ_i may be constants or functions of some parameters.

Stability → $|\xi_j| \leq 1$ all roots!

aim: interval of stability

So, then there is other one appendix B so, this is Routh Hurwitz criterion so, many times in order to get the stability we are ending up with a polynomial. And then, we have to find the roots of this polynomial or sometimes the range so, many times finding all the routes is quite complicated. So, atleast if we can get the region where the roots lie then, atleast we can conclude the corresponding stability. So, Routh Hurwitz criterion is very popular and then it indicates how to compute the region of stability so, let us have a look at it.

Let Pn zeta is n minus 1 equals to 0 be a polynomial of degree n where, mu I maybe constants or functions of some parameters. Now our aim is, for stability we have to compute all roots so, this is difficult. Now what we are looking for, aim, interval of stability or values of the parameters for which it is stable.

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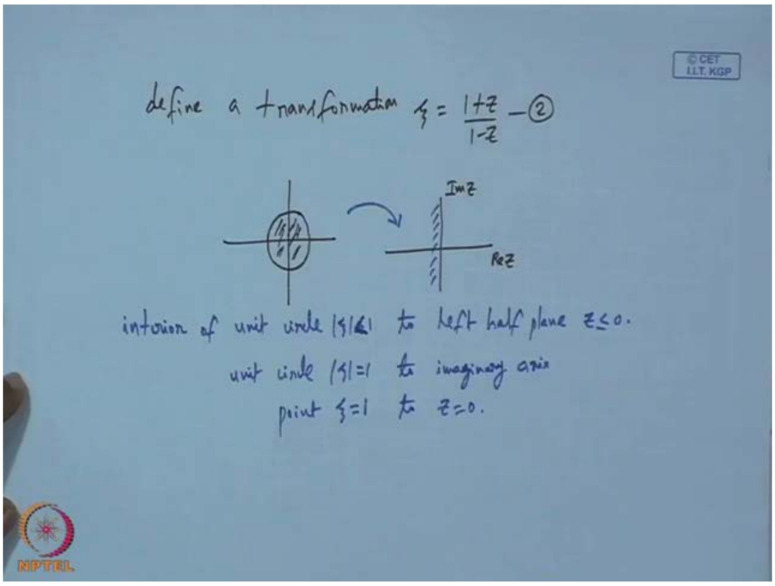

define a transformation $\zeta = \frac{1+z}{1-z}$ — ②

interior of unit circle $|\zeta|<1$ to left half plane $z \leq 0$.

unit circle $|\zeta|=1$ to imaginary axis

point $\zeta=1$ to $z=0$.

So, now this is done via transformation, define transformation so, if you define this transformation what this does so, real z and imaginary z. So, this part that means interior of the unit circle onto left half of so, we can write so, interior of unit circle to left half plane then, this is ticked. So, the boundary, this is to imaginary axis then point eta equals to 1 to z equals to 0.

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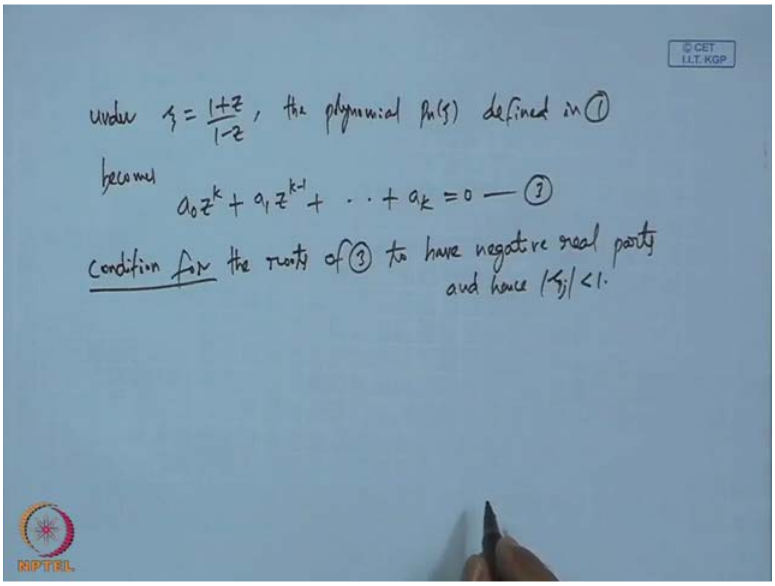

under $\zeta = \frac{1+z}{1-z}$, the polynomial $P_n(\zeta)$ defined in ①

becomes

$$a_0 z^k + a_1 z^{k-1} + \cdot\cdot + a_k = 0 \quad — ③$$

Condition for the roots of ③ to have negative real parts and hence $|\zeta_j| < 1$.

So, now under this transformation under the polynomial Pn zeta defined in 1, becomes so, it becomes this. Now, what we need is condition for the roots of 3 to have negative real parts and hence. So, this is we are looking for so, Routh Hurwitz criterion ensures this.

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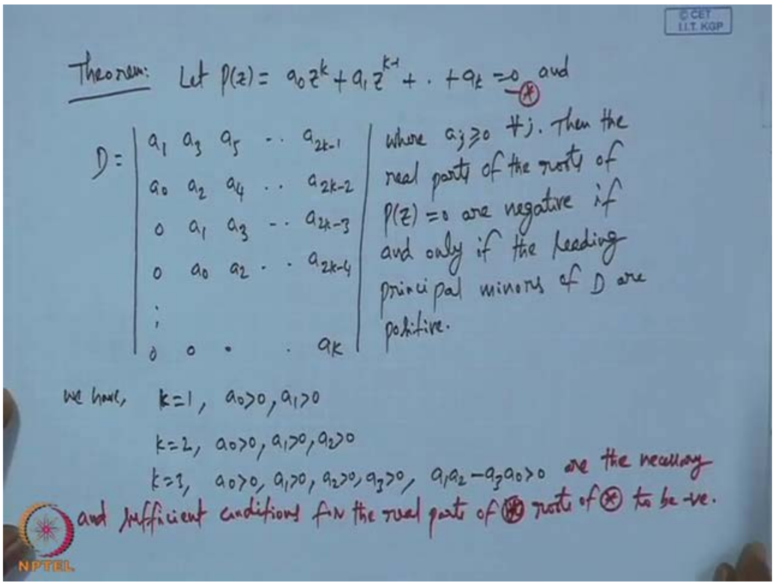

Theorem: Let $P(z) = a_0 z^k + a_1 z^{k-1} + \cdot + a_k = 0$ — ⊛ and

$$D = \begin{vmatrix} a_1 & a_3 & a_5 & \cdot\cdot & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdot\cdot & a_{2k-2} \\ 0 & a_1 & a_3 & \cdot\cdot & a_{2k-3} \\ 0 & a_0 & a_2 & \cdot\cdot & a_{2k-4} \\ \vdots & & & & \\ 0 & 0 & \cdot & \cdot & a_k \end{vmatrix}$$

where $a_j \geq 0 \;\forall j$. Then the real parts of the roots of $P(z) = 0$ are negative if and only if the leading principal minors of D are positive.

we have, $k=1$, $a_0 > 0, a_1 > 0$

$k=2$, $a_0 > 0, a_1 > 0, a_2 > 0$

$k=3$, $a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0$, $a_1 a_2 - a_3 a_0 > 0$ are the necessary and sufficient conditions for the real parts of ⊛ roots of ⊛ to be -ve.

So, what is Routh Hurwitz criterion, let P z is and so, determinant where a j greater than or equals to 0 for all j. Then the real parts, all the roots of P z equals to 0 are negative if and only if, the leading principal minors of D are positive. So, this is the Routh Hurwitz criterion so, then accordingly we have suppose, k is 1 a 0 greater than 0, a 1 greater than 0, then k is 2, then k is 3.

So, these are the necessary and sufficient conditions for the real parts of, star real parts of, I mean roots of star real parts of roots of star, to be negative. And hence, for stability as well so, this is a general concept, which is very popular for checking the stability regions, one uses Routh Hurwitz criterion it is very standard practice. So, now the next is on bit on errors so, because this is very important sometimes what we do, an approximation could be a function and sometimes the approximation could be value grid points. So then maybe one expects a kind of a correlation between, a correlation with the behaviour of errors, with the function notation, as well as the grid notation.

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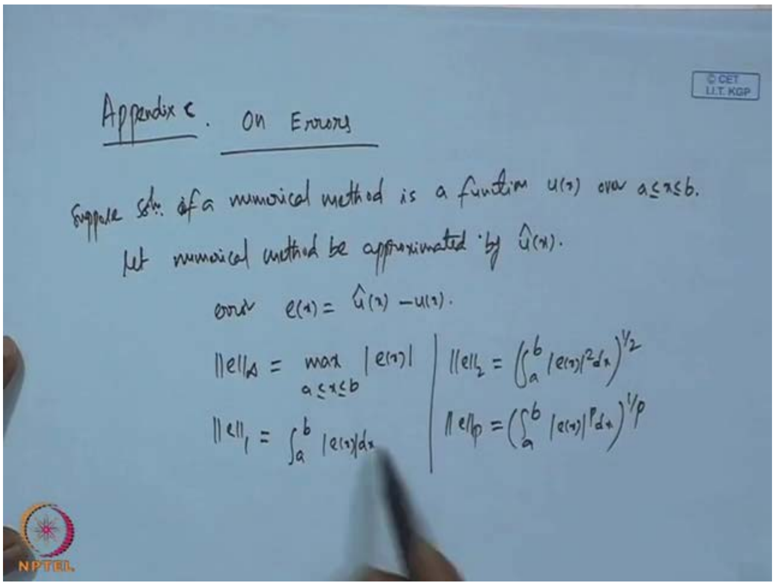

Appendix C. On Errors

Suppose soln. of a numerical method is a function $u(x)$ over $a \le x \le b$.

let numerical method be approximated by $\hat{u}(x)$.

error $e(x) = \hat{u}(x) - u(x)$.

$\|e\|_\infty = \max_{a \le x \le b} |e(x)|$

$\|e\|_1 = \int_a^b |e(x)| dx$

$\|e\|_2 = \left(\int_a^b |e(x)|^2 dx\right)^{1/2}$

$\|e\|_p = \left(\int_a^b |e(x)|^p dx\right)^{1/p}$

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So, let us have a look at this so, now consider suppose, we have a solution of a numerical method is function u of x over some interval. So, this is suppose, suppose solution of a numerical method is function then, let numerical method be approximated by this. Then, we define error function, then we define max norm, then we define one norm, then we have square norm or ((Refer time: 31:55)) norm and p norm. Now these are errors with respect to functions so, solution of, solution of the equation, when I said numerical method, then numerical method is approximated by this. So, these are the errors with respect to functions.

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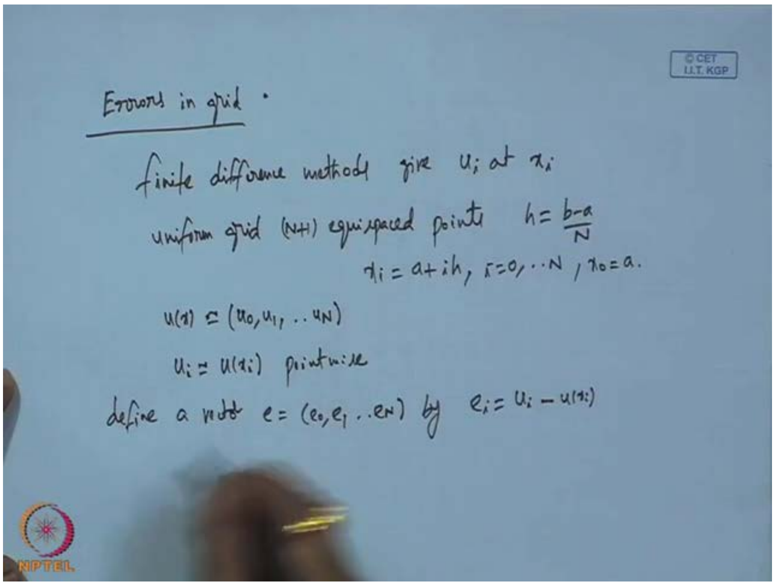

Errors in grid .

finite difference methods give u_i at x_i

uniform grid $(N+1)$ equispaced points $h = \frac{b-a}{N}$

$x_i = a + ih, \ i = 0, \ldots N, \ x_0 = a.$

$u(x) \simeq (u_0, u_1, \ldots u_N)$

$u_i \simeq u(x_i)$ pointwise

define a vector $e = (e_0, e_1 \ldots e_N)$ by $e_i = u_i - u(x_i)$

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Now errors in grid, errors in grid so, for example say finite difference, finite difference methods give u i at x i. So, for example if you have a uniform grid, uniform grid with N plus 1 equi spaced points with h equals to b minus a by N and x i equals to a plus i h, where x 0 is a. So, we have u x approximately u0, u1, un now, this we need to compare with a set of discrete values, with the function.

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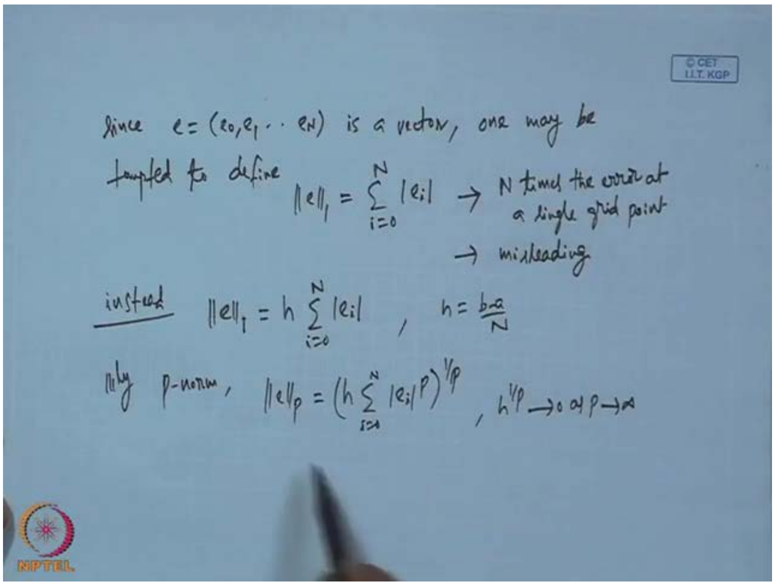

Since $e = (e_0, e_1 \cdots e_N)$ is a vector, one may be tempted to define $\|e\|_1 = \sum_{i=0}^{N} |e_i| \rightarrow$ N times the error at a single grid point

$\rightarrow$ misleading

instead $\|e\|_1 = h \sum_{i=0}^{N} |e_i|$, $h = \frac{b-a}{N}$

lly p-norm, $\|e\|_p = \left(h \sum_{i=1}^{N} |e_i|^p\right)^{1/p}$, $h^{1/p} \rightarrow 0$ as $p \rightarrow \infty$

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So, that means these uis are nothing but so, these are approximating point wise. Now define a vector e to be e0, e1, en by ei equals to ui minus, I could have used capital so that, no confusion, I am using small ones please understand the. So, if you define this then, how do we talk about error so, sometimes maybe averaging so, u i is just point wise, but sometimes averaging.

So, now since this is a vector, since e is e0, e1, en is vector one may be tempted to define the error function like this. But however, what will happen at each grid point this one, ei is the error so, this will spoil the whole calculation that means, if you consider this way. Because, this is N times the error at single grid point in some sense therefore, this is misleading.

So, what is the remedy, we have to normalize, we have not done rather, we are magnifying, multiplying it with N. So, this, the remedy is because h, the way it is defined so, in some sense we are normalizing by N. So, then similarly, for p norm because, we can see h power 1 by p this goes to 0 as p goes to infinity so, this normalization works. So, that so, what I am trying to say is, the functions we have defined this kind of norms for functions, but when it comes to grid so, similar concept may not be extended.

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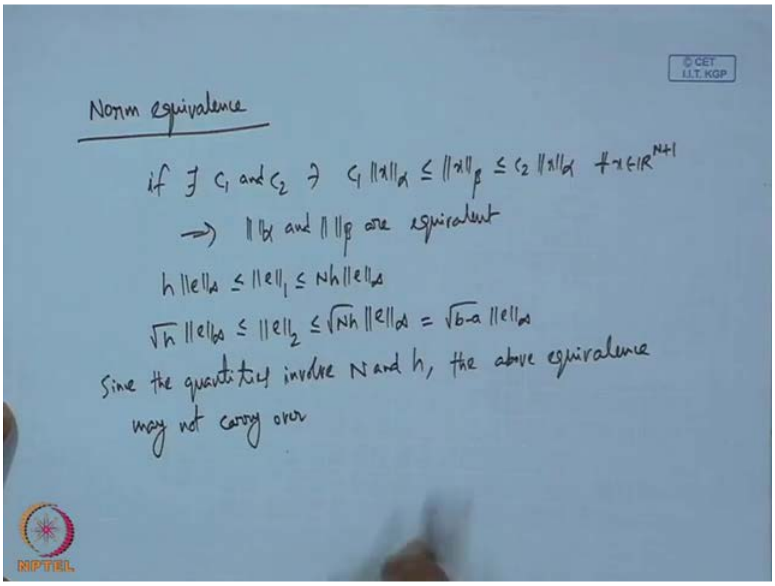

Norm equivalence

if $\exists$ c_1 and c_2 $\ni$ $c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha \quad \forall x \in \mathbb{R}^{N+1}$

$\Rightarrow$ $\|\,\|_\alpha$ and $\|\,\|_\beta$ are equivalent

$h\|e\|_\infty \leq \|e\|_1 \leq Nh\|e\|_\infty$

$\sqrt{h}\,\|e\|_\infty \leq \|e\|_2 \leq \sqrt{Nh}\,\|e\|_\infty = \sqrt{b-a}\,\|e\|_\infty$

Since the quantities involve N and h, the above equivalence may not carry over

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So, this is one issue and the other issue is norm equivalence so, the standard definition is two norms are equal, two norms are equal in the sense, if there exists constants. If there exists c1 and c2 such that, c1 times with respect to one norm, other norm alpha for every x belongs to the corresponding space. So, for example so, if this is the case then we say are equivalent so, for example. So, we can, we take c1 to be this, c2 to be this so, then if somebody concludes.

So, this is nothing but root b minus a so, what will happen if somebody, this is equivalent therefore, in this norm it is equivalent like this. So, then what would happen, this equivalence may not be carried because, these things depend on N and h. So, the remark is since the quantities involve N and h, the above equivalence may not carry over, when we discuss about error.

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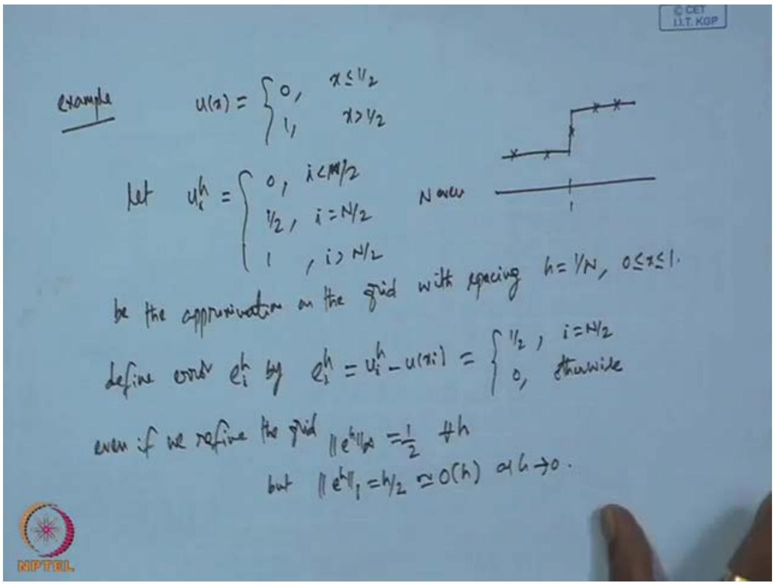

<u>example</u> $u(x) = \begin{cases} 0, & x \le 1/2 \\ 1, & x > 1/2 \end{cases}$

let $u_i^h = \begin{cases} 0, & i < N/2 \\ 1/2, & i = N/2 \\ 1, & i > N/2 \end{cases}$ N even

be the approximation on the grid with spacing $h = 1/N$, $0 \le x \le 1$.

define error e_i^h by $e_i^h = u_i^h - u(x_i) = \begin{cases} 1/2, & i = N/2 \\ 0, & \text{otherwise} \end{cases}$

even if we refine the grid $\|e^h\|_\infty = \frac{1}{2} \; \forall h$

but $\|e^h\|_1 = h/2 \simeq O(h)$ as $h \to 0$.

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So, this can be displayed by an example so, consider and let the approximated is say N even. So, if this be the discrete approximation so, this be the approximation on the grid with spacing h is 1 over N, on this let us say. Then, define error ei, h by so, this would be half i is N by 2, 0 otherwise so, basically the function is like this so, this is approximating so, this is.

So, the approximated is behaving like this, the stars and u of x is behaving like this, but even if we refine, even if we refine the grid, we have nor e infinity is half for every h. But one is h by 2 and this is big of 2 as h goes to 0 so, which means this converges to 0 whereas, this is not. So, the norm equivalence one has to be careful so, this converges to 0 but this does not converge. So, this is a general appendix I thought I would mention because, in general we use power methods to compute Eigen values and then, also the error etcetera are beneficial to discuss, what kind of norms should be used.

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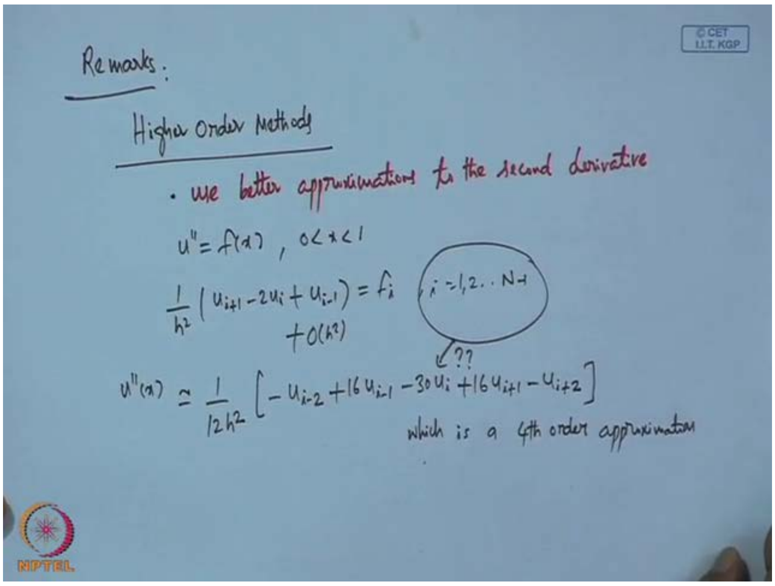

Remarks:

Higher Order Methods

- we better approximations to the second derivative

$u'' = f(x)$, $0 < x < 1$

$\frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) = f_i + O(h^2)$ $\boxed{i = 1, 2 .. N-1}$??

$u''(x) \simeq \frac{1}{12h^2}\left[-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}\right]$

which is a 4th order approximation

So, now some remarks so, the remarks so, we discussed finite difference methods so, that is the highest when we started single step, multi step and then we come to finite difference. So, there are higher order methods as well, higher order methods what are these so, these are typically, they use better approximations to the second derivative. So, in what sense better so, for example if we consider u double equals to f of x then, usual central what we have been using. Now when we see, this is nothing but the second order however, u double x can be approximated by the following, u double can be approximated by the following, which is a fourth order approximation.

So, accordingly if we use this in place of this and then get it so, you can see, we expect more accuracy because, it is spread over i minus 1, i plus 2, i minus 2. However, there could be difficulties because, we may not be able to run the equation, it is a question mark. Because, see you have these points, end points so, they will increase the fictitious values. So, one has to take care of the corresponding issues, but however such higher order methods are possible.

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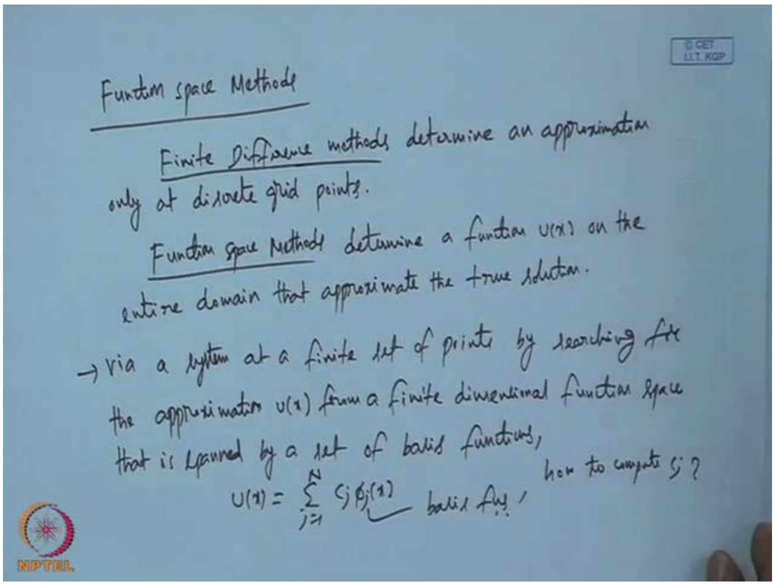

Function space Methods

Finite Difference methods determine an approximation only at discrete grid points.

Function space Methods determine a function u(x) on the entire domain that approximate the true solution.

→ via a system at a finite set of points by searching for the approximation u(x) from a finite dimensional function space that is spanned by a set of basis functions,

$$u(x) = \sum_{j=1}^{N} c_j \underbrace{\phi_j(x)}\ \text{basis fn's}$$

how to compute c_j ?

So, then function space methods so, typically we have the finite difference methods so, these methods determine an approximation only at discrete grid points. However, these function space methods, function space methods they determine a function u of x on the entire domain, that approximates the true solution.

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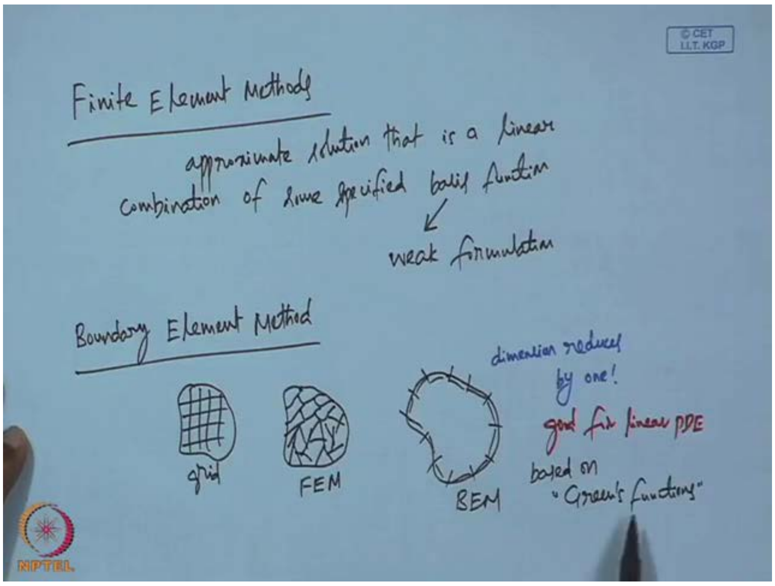

Finite Element Methods

approximate solution that is a linear combination of some specified basis function

↙

weak formulation

Boundary Element Method

dimension reduces by one!

good for linear PDE

based on "Green's functions"

However, how this is possible, this is via so, this is via a system at a finite set of points by searching for the approximation u of x from, a finite dimensional function space, that is spanned by a set of basis functions. That means so, these basis functions are such that so, it is approximated like this. And these are the base basis functions however, the task is how to compute cj so, this is the general task.

So, then we have finite element methods so, even finite element methods, these are nothing but approximate solution, that is a linear combination of some specified basis function. However, there is a slight variation compared to function space methods, this basis function is obtained something called weak formulation. And also, there are finite volume methods, but another method I would like to mention, before we conclude is boundary element method.

So, for example finite difference, these are based on grid so, these are based on some triangulations. So, this can be thought of some triangle elements, in case of finite element so, this is boundary element method. However, what would happen compared to these two so, the dimension would reduce dimension of the computation. Here for example, it is 2d you have to completely handle the 2d domain even here whereas, here the 2d means a line.

Suppose it is 3d, you have to completely handle the entire volume, but whereas, here a surface however, these are good for linear PDE. What is the reason, the reason is these are based on Green’s functions so, if you have Green’s function so, then only this boundary element method can be used. Because, one would end up with the boundary integral formulation based on the Green’s functions.

So, then you end up with integral equation on for example, if the problem is 3 dimensions, you get an integral equation on the surface. If the problem is 2 dimensions, you get an integral equation on the line so, then you just discretize using any general colocation methods and get the solution. However, there are restrictions of this as well. So, this is general idea of little more advanced methods. So, with this we come to end of this course. I hope you enjoyed hearing some of this and your valuable feedback would definitely help.

Thank you, bye.