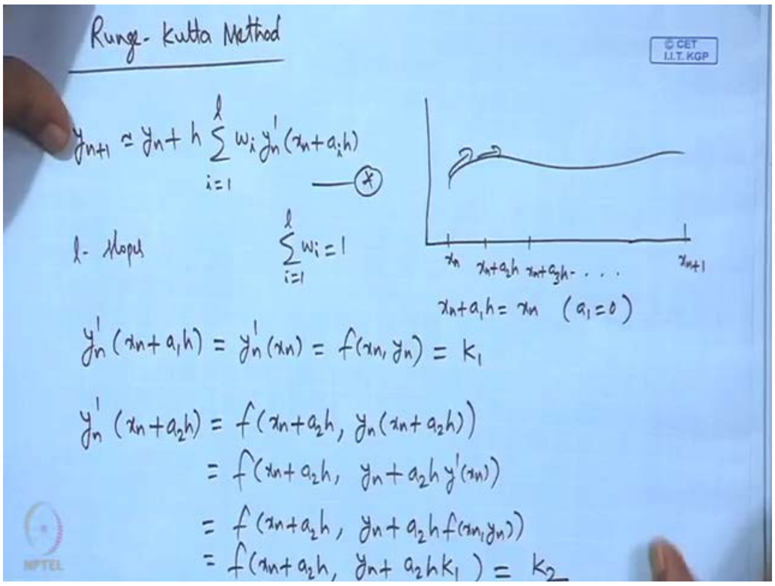
Numerical Solutions of Ordinary and Partial Differential Equations
Prof. G. P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 4
Runge - Kutta Methods for IVPs

Hello. So, we talk about Runge Kutta methods for initial value problems. So, in the last lectures we have talked about Euler method, then modified Euler method. So, we introduced the concept of slow power aging. So, the generalization is if you take an interval, you try to take some intermit points and then take the weighted average of the slopes. This is the more generalization which leads to Runge Kutta method. So, let us talk about the derivation is little lengthy, because its more technical based on the formula you have to compare with the with the Telesis expansion, and then try to determine the coefficients and weights and all that, so we have to be little patient while understanding the derivation.

(Refer Slide Time: 01:17)



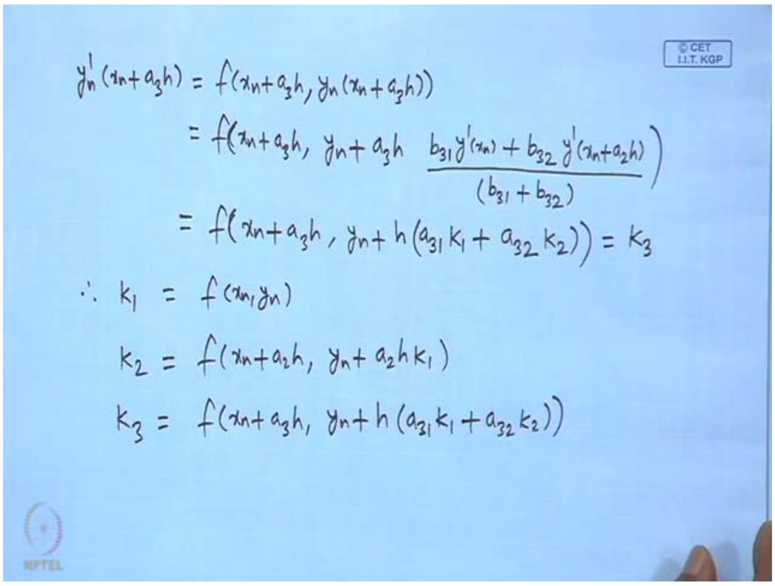
So, let us talk about this. So, as I mentioned the concept is within an interval, when you step from 1 x n you know the data and we try to compute the data at x n plus 1. So, that is what because this is single step method. So, we have been doing that now what is the idea use several intermediate points a 3 h. So, on and then take the slopes at each point.

So, that is weighted average that is what I said. So, your approximate solution is y n plus h i 1 to l, the weights and slope at these intermediated points.

So, as I mentioned we are taking weights such that and l points, l intermediate points this l denotes l slopes, and moreover I started with a 2. So, in x n plus a 1 h equals to x n. So, that I have taken a 1 equals 0. This is just for convenience. So, that there is a symmetry in that derivation that is all nothing serious about it you can take even this a 1 a 2. There is no harm now when you talk about star. So, what is the first 1 i starts from 1 therefore, x n is. So, the first entry is y n of x n plus a 1 h. So, this is x n this is, call this k 1. So, the next entry x n plus a 2 this is f of x n plus a 2 h y n of x n plus a 2 h. We use Euler so this is y n plus a 2 h y dash of x n. So, that means in Taylor series, we have neglected the second rod onwards that is what we have.

So, this is y n plus, this y dash f x n is f right. So, this is a 2 h, further this can be generalized in the sense this is equals to f of a 2 h this is k 1 right. So, I introduce k 1 there. So, this let us call k 2, see this for try to follow we started here and f of x n plus a 2 h is this. So, then we used Euler for this is just Euler then this is nothing but k 1 as defined. Now, we have defined something called k 2. So, that means each slope at each point were computing were giving a notation right now we move to the next right. So, what is the next point.

(Refer Slide Time: 06:02)



The next point is y n prime a 3, so this is f of. So, still we have not used the concept of averaging, now the first time we are trying to introduce here because where are we computing, we are computing y dashed at x n plus a 3 h. So, there is already x n 2 x n plus a 2 h and from here to here. So, there already 2 slopes involved prior to that right. Now, we try to use a averaging concept. So, that is this we have to, this is y n plus the increment is a 3 h, then we have y dash of x n right. So, this we tried to get it in weighted average sense.

So, we introduce some notation b 3 1 y dash of x n plus, b 3 2 widest of x n plus, a 2 h. See this is a 3 h therefore, we have used passed 2 slopes x n x n plus a 2 h . So, this is weighted average. So, this can be simplified this more generalization. So, we are not so much worried about particular constant because ultimately we end up with a generalization in terms of specified constants. So, this is plus, now I take h common here this h look at these constants orbited constants. So, this a 3 multiplied by b 3 1 divide by this. So, this we can call some new constant.

And y dash of x n is just k 1 plus again a 3 multiplied by b 3 2 divide by this, this I call another constant and y dashed of x n plus a 2 h is k 2. So, it is more of technical you should try to follow. First one this is at x n second slope is at x n plus a 2 h. So, that is k 2, then third slope is at x n plus a 3 h. So, this k 3, let me define k 1 is this k 2 is, k 1 is involved k 3 both k 1 and k 2.

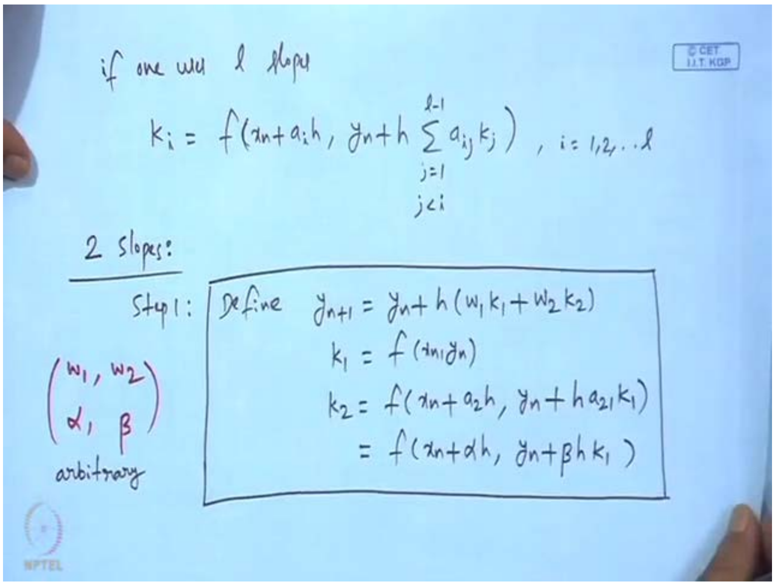
So, one can proceed further and try to get, but we do not want to do that because we fix finite number of intermediate points and take the slopes and try to do it. Now, an immediate concern is anybody would ask and who will decide this intermediate points and who will decide this weights and who will tell me how many points to be considered and all the stuff right.

So, we have to really consider this and since already I mentioned this is a bit laborious, may be somebody would start with less number of intermediate points. So, that your weighted average is done with less number of slopes, then you expect a little simpler method first and then straightly complicated. So, let us try to fix the number of intermediate points.

So, if one uses l slopes, we generate k i of the form x n plus a h y n plus h. We have taken common then we get this kind of pattern where i is. So, it is one can identify easily, see this k 2 is x n plus a 2 h, h is common so a 2 k 1 k 3 is xn plus a 3 h, then h comes out then a 3 1 k 1 plus a 3 2 plus k 2. So, the generalization is a 4 1 k 1, a 4 2 k 2, a 4 3 k 3, like that for k 4 so this is the generalization right. So, this gets reduced to simpler forms depending on the number of slopes.

So, as I mentioned let us start with just 2 slopes. So, if u consider 2 slopes right, then step one you define the method. So, we are defining the method, y n plus 1 is y n plus just 2 slopes. So, what will we get, h w 1 k 1 plus w 2 k 2 i mean instead of the arbitrary constants, earlier notation we can just generalize it. So, this is these are the weights standard I have used. Now, k 1 k 2 contain arbitrary coefficient.

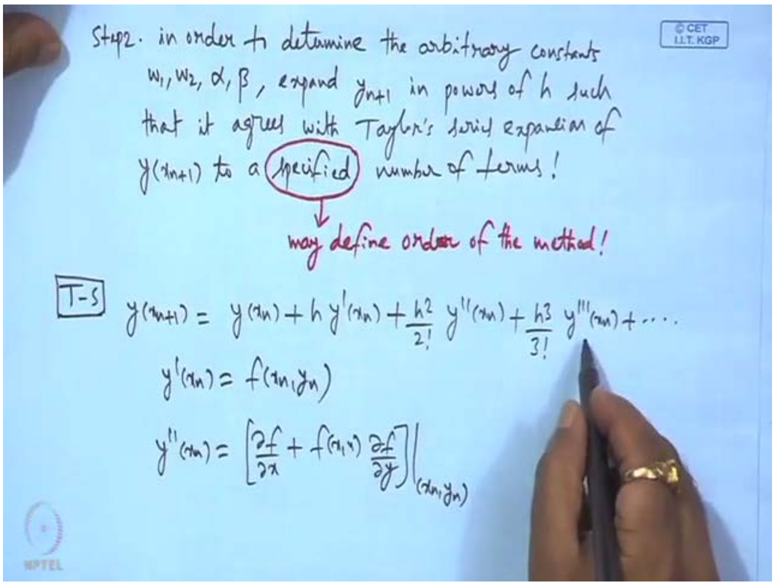
(Refer Slide Time: 13:19)



So, what are they k 1 is f of and k 2 is f of x n plus a 2 h y n plus h a 2 1 k 1. Now, at this stage what is left, yes we have to determine the arbitrary coefficient and the weights. So, to do that I would like to switch to some other notation, just for a convenience because using such indices will be little nasty in the derivation, x n plus say alpha h y n plus beta h k 1. So, I hope you follow right.

So, I am not going with these notations just for convenience head 1 is replaced by beta. Now, what is a next task. So, step 1 we have defined this that means this is going to be our method, k method with 2 slopes provided one determines what are the arbitrary coefficient left, the arbiter coefficient are w 1 w 2 alpha beta. So, these are the orb try coefficient. Now, one has to determine right. So, what is step 2?

(Refer Slide Time: 16:36)



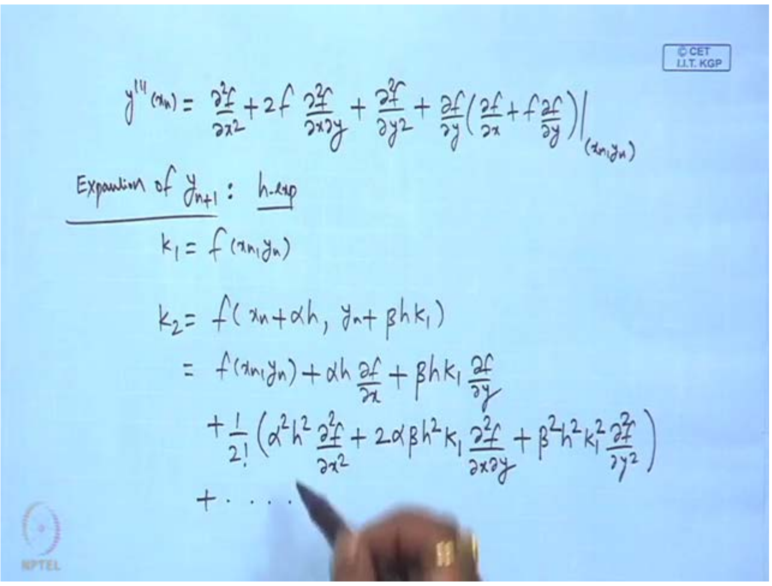
Step 2 is in order to determine the arbiter constants of coefficient w 1, w 2, alpha, beta what we do. Already I mentioned see what is this is an approximate solution for the given initial problem. Now, approximating what approximating the true solution therefore, who is fellow with whom we have to compare, we have to compare this fellow with the exact solution, but then exact solution is not known to us. So, what we do is your exact solution is if it exist I mean under the sums of existence and uniqueness. Your exact solution will be expanded in terms of l series.

Now once you expand your Taylor Series, compare your Taylor Series with your approximation here. So, let us do that, in order to determine the arbiter constants w 1, w 2, alpha, beta, expand y n plus 1 that is the approximation in powers of h, such that it agrees with Taylor series expansion of true. That is this and how do you compare how long to a specified number of terms.

Now, when you say specified number of terms, may be this is specified number of terms. This may define may define order of the method. So, comparing up to this order that means that will dictate the order of the method. So, let us do that as I mentioned it is going to be very, very technical so we have to follow the steps. So, let us expand Taylor series first.

So, I am writing the t s that is Taylor series expansion first, y of x n plus 1 is y of x n plus so on. We have this terms, y dashed is f of x n y n, y double is dou f dou x plus f x y dou f dou y evaluated x n y n, I mean it is better to compute the next time as well before we proceed with comparison. So, let us it is straightly complicated, but then we need this term y 3.

(Refer Slide Time: 21:56)



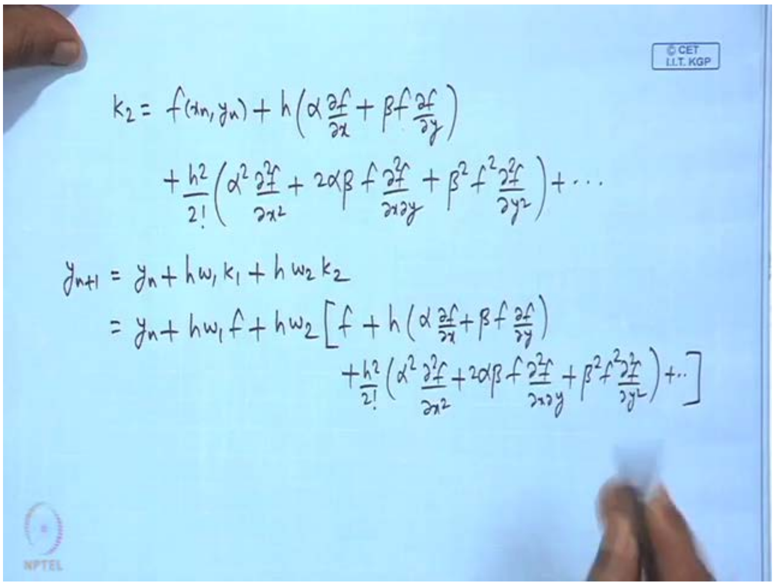
So, y 3 one can do it 2 f of course, all this validated at. So, this as I mentioned this is lightly complicated. So, we have t s, this is supposed to be the exact. Now, we have the approximation this is approximation. So, what is our aim, our aim is to expand this in powers of h so that we compare with this. So, its slightly complicated, let us try to do that. Now, our immediate concern is to expand this in powers of h that means see for example, k 1 there is nothing to expand, but when you take k 2, there is you have to expand this right. So, let us do that, y 3 is done. Now, we proceed to expansion of y n plus 1. So, this I call h exp.

So, k 1 there is nothing, k 2 it is better to write the form of k 2, f of x n plus alpha x alpha h y n plus beta h k1. So, this is f of x n y n so it is a 2 variable expansion Taylor series. So, we have to expand this plus alpha h with respect to this variable. So, I am not writing the evaluation point. So, it is understood at each step I have to write, but one can use generalized notation just plus first order terms I am writing first f expanded with respect to this, then the first order term.

Now I will expand with respect to the first order term with respect to this. So, that will be beta h k 1 dou f by dou y because with respect to this variable. So, this the first order terms, plus second order terms, this is the increment. So, alpha square h square, second order derivative plus the mix term 2 alpha h beta h.

So, that will be 2 alpha beta h square k 1 this is the mix term plus second order term of with respect to y beta square h square k 1 square plus well look at this. So, we have to expand with respect x and y. So, this function about the point x n y n, then first order terms alpha x so f by so x plus this is increment for y therefore, the increment times dou f by dou y. Now, the second order terms, 1 naught 2 factorial. So, this is second order term of this, this is second order term of this and this is the next term. Now, this can be simplified as follows.

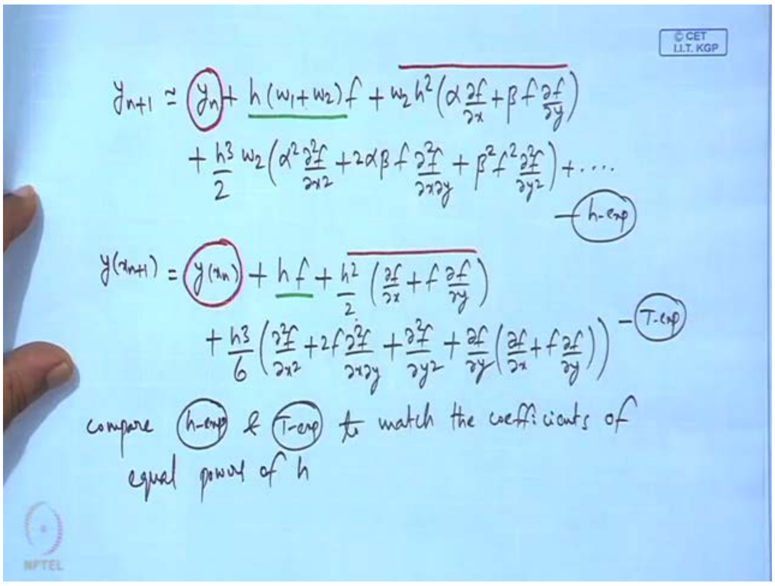
(Refer Slide Time: 27:23)



So, k 2 is given by plus h so beta, we have k 1 there. So, that k 1 is nothing but f so that becomes an f there plus because what is our aim, our aim is to compare with the Taylor series expansion. So, our Taylor series expansion has exactly this kind of form h h square by 2 factorial. So, we are trying to put it in that form . So, this is k 2, what did we do, we are trying to get the h expansion of that approximation.

So, k 1 is this k 2 we have expanded, now what we have to do, we have to substitute the expansions in y n plus 1. So, let us try to do that therefore, your y n plus 1 is y n plus h w 1 k 1 plus h, this is our approximation. So, this is given by y n plus h w 1 k 1 is just f plus h w 2 k 2, this entire expression we have to write down f plus h. This is first term then let me put it here.

(Refer Slide Time: 30:46)



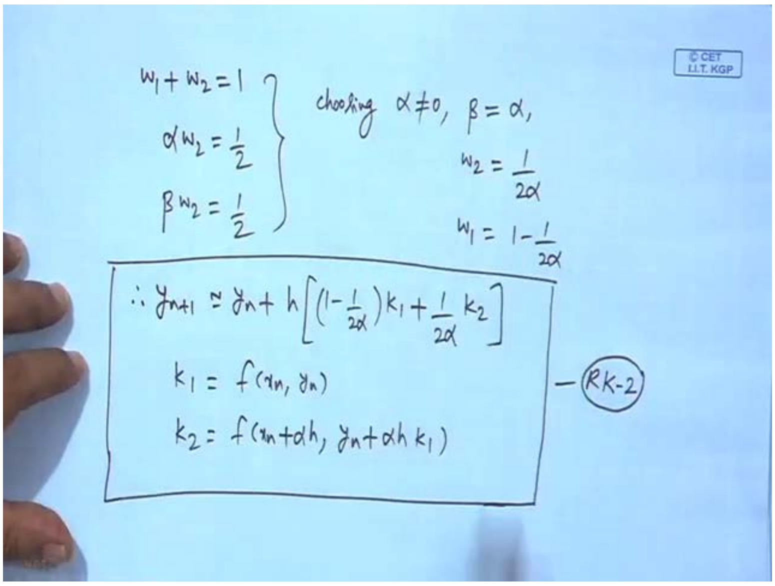
So, this can be further simplified as follows, y n plus h. Let me explain the we get this look our aim is to put it in a Taylor series form, so coefficient of h coefficient of h square. So, look at this you have one term here and with h w 2 f. So, if you take common we get exactly this term. So, similarly, we get this is the next term. So, h square, how do we get it, look h square w 2 times this now h cube from this term. So, plus 2 alpha beta is a mix term.

So, this is our h expansion and what was our t expansion. So, this is Taylor series expansion this is our Taylor series with these 2 terms. So, may be its better to write down y of is y of x n plus h f plus h square by 2 dou f by dou x plus f dou f by dou y plus h cube by 6 h cube by 6. So, this our Taylor series expansion. Now, what is our aim, aim is to compare as I mentioned the next task is, to compare the Taylor series expansion and the h expansion up to desired terms.

So, that means you have coefficient of h coefficient of h square so we compare. So, that we get some we except a system of equations let us see. So, again I would like to this is an approximation which were x expand in powers of h and this is our Taylor series expansion, now we try to compare. So, what we do is compare h expansion and t expansion to match the coefficient of equal powers of h. So, let us look at it y n must be approximating y of x and then look you have h f. So, h f, w and plus w 2 must be 1.

So, this is approximating this, then this you have h f in this term. So, if you try to compare this two, what happen w 1 plus w 2 must be 1 right. So, let us try to do that. So, if we do that we get from first case, then so let me use again red. So, this is red these 2 then the green, now h square upper I am using. So, you have h square h square by 2. So, there is a half a team for dou f by dou x, if you remove h square there is just a factor of half a team for dou f by dou x whereas, her you have alpha w 2 sitting. Therefore, alpha w 2 must be half there and here beta w 2 is sitting for f dou f by dou y and there is another half sitting for f dou f by dou y therefore, beta w 2 must be half there.

(Refer Slide Time: 37:38)

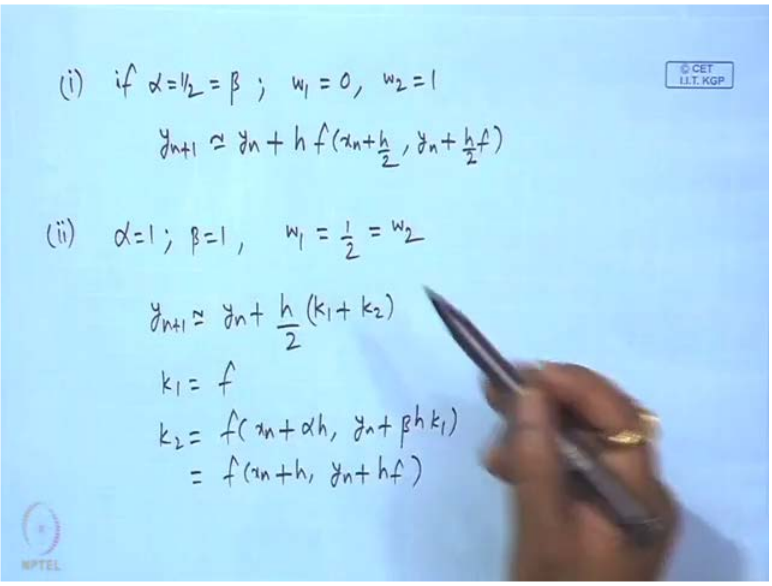


We get this is one equation, then alpha w 2 is half, beta w 2 is half. So, this I have done it only up to how many terms up to order of h square. Now, compared so y n is approximate of y f x n and by comparing by this we get, w 1 plus w 2 is 1 and comparing this h square terms we get alpha w 2 is half beta w 2 is half. So, I am not comparing these three terms. So, that means I have compared up to terms up to h square right now from this system. If you stop then we should be able to determine the coefficient, but suppose let us stop at this stage. That means you have compared up to h square.

You should be able to solve, but how many unknowns you have, w 1, w 2, alpha and beta and you have only three unknowns right. So, that means you have to choose one of them arbitrary right. Let us say choosing alpha non zero, we get beta equals to alpha w 2 is 1 over 2 alpha w 1 is 1 minus 1 over 2 alpha. So, with this our approximation becomes y n plus h, what was our approximation, these are our approximation h into w 1 k 1 plus w 2 k 2. So, we have w 1, h into our w 1 is 1 minus 1 over 2 alpha, k 1 plus w 2 1 over 2 alpha.

So, this is our approximation where k 1 is given by f of x n plus alpha h y n plus alpha h k 1. So, this is our R K method, still alpha is arbitrary. Now, you can talk about specific values, so let me write this as R K 2 because I have compared only up to h square therefore, maybe we are optimistic just by comparing up to h square. We can call this method second order probably right, any way we can discuss that. Now, since alpha is arbiter let us take some specific value and try to see right.

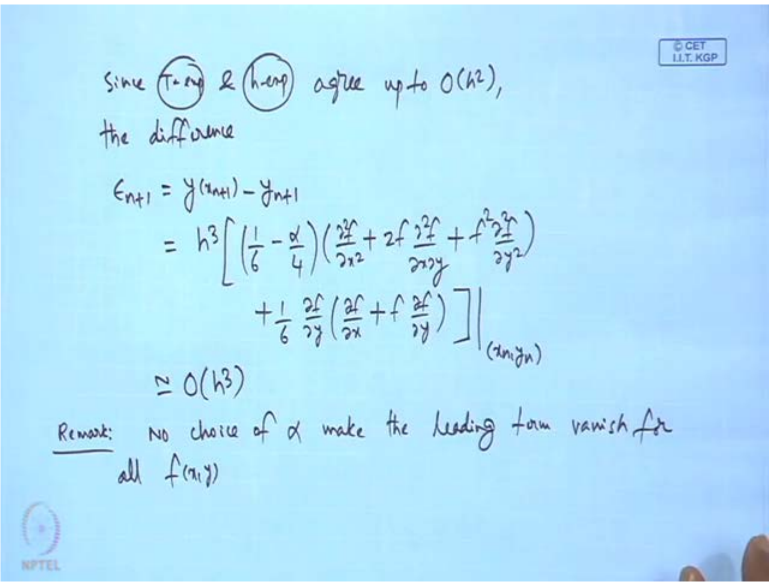
(Refer Slide Time: 41:21)



If alpha is half, then beta will also half then what will happen to your method alpha is half right. So, this fellow is zero and this fellow is 1 right, if alpha is half. So, let us if alpha is half, beta is half then we get w 1 0, w 2 is 1 and you method reduces to y n plus 1 is y n plus y n plus w 2 is 1, so h f of x n plus h by 2 y n plus h by 2 f. So, this is a method we get. So, y n plus this is a method alpha is half. So, this coefficient vanishes and alpha is half. So, this we get w 1 0 w 2 is 1 and beta equal to alpha equal to half. Therefore, the method reduces to this one. Suppose, alpha equals to one then beta equals to 1 then w 1 equals to half and which is also w 2.

So, in this case y n plus 1 y n plus, h times w 1 plus w 1 k 1 plus w 2 k 2. So, this reduces to h times k 1 plus k 2 and what is our k 1 k 1 is f and k 2 is k 2 is f of x n plus alpha h y n plus beta h k 1 this reduces to f of x n plus h y n plus h k 1 k 1 is f. So, this is the particular case when alpha is 1 alpha is. So, these are general choices now the story is not left we have compared up to 2 terms and then shown as system of this form and then one is arbitrary. So, of course, taking non zero, one can have different choice right. So, general choices are alpha is half and alpha is 1. Now, what happens to the terms beyond h square, obviously they should contribute to the error.

(Refer Slide Time: 45:57)



So, if you consider the difference, since t expansion and h expansion agree up to h square, what will happen to the difference. The difference must be the residual that is error. So, naturally that should from h cube. So, if you can do the long calculation, y 3 we have computed any way this is in our hand. So, this is the residual which is supposed to be the error of course, at a given. Since, we know f and we have determine alpha. So, what will happen, this entire thing is known to us and hence this is order of h cube. So, the residual is of order h cube and hence the method is of order h square.

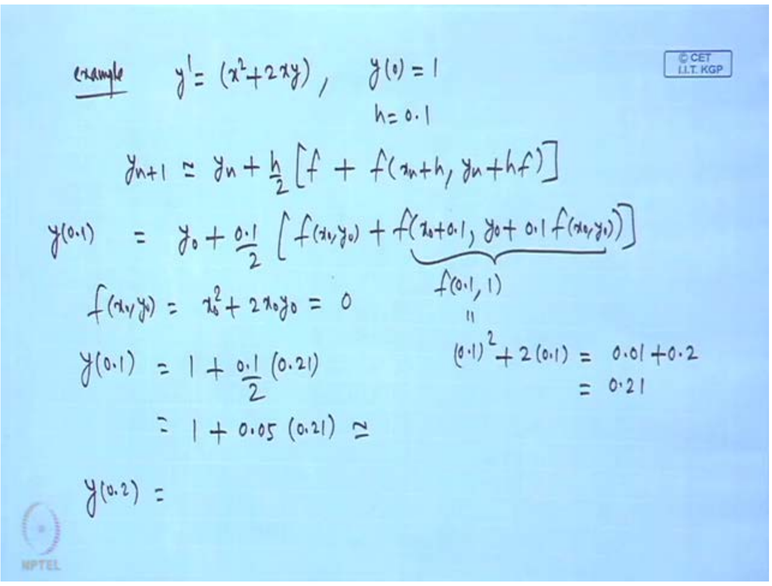
So, one remark see for example, here for all f means not for a particular f this is becoming 0 and for all of no choice of alpha will make the leading term vanish what is the leading term in this is the leading term. So, further you have h 4, no choice of alpha make the leading term vanish for all f of x y. So, this is an important mark to conclude that really the error is coming from the order of h cube. So, we have approximated with two slopes in generalized formula, but we have considered only two slopes.

Then we have taken the weighted average and expanded the approximate formula expand the Taylor series formula. Then determined the arbitrary coefficient up to h square right and then conclude that the residual comes from h cube. Hence this is the second order method so this is RK 2 that is what I have written. Now, suppose one would like to define a higher order method R K method what one has to do, one has to compare a the terms between the Taylor series expansion and the h series expansion of the approximate and try to determine the arbitrary coefficient.

So, this will determine the method because once you determine the coefficient the method is determined and then you get a new method. So, how long we try to determine of course, this is algebraically you try to do you get very complex system. So, the standard methods are first one you arrive at second order method. There is a choice of one arbiter coefficient and more general choice are half and 1.

So, these are R K second order, but they slightly vary. Now, if you really go for more number of terms, then you get let us say you go to 4 terms. Then you get four in the sense you got to compare h optical h power 4, you get fourth order R K method. So, that will be slightly complicated.

(Refer Slide Time: 51:31)



So, before for that let us just see some example. So, this is y dashed is x square plus 2 x y and y of 0 is 1 and say h is 0.1. So, then whatever the method we have determined, let us say alpha 1 case. So, alpha 1 case we have y n plus 1 is y n plus h by 2 k 1 is just f, there plus k 2 is f of x n plus h y n plus h f. So, this is the method we have. Suppose, we try to determine y f 0.1. So, this will be y 0 plus f of x 0 y 0 plus f of x 0 plus 0.1 y0 plus 0.1 into f of. So, we had compute each term, what is f of, this is x 0 square. So, x 0 is 0, this is 0. Therefore, so y 0 is 1 h by 2. This is 0 and f of we need this term right and this is 0. So, this reduces to this is 0, this is gone. So, this reduces to f of 0.1 and 1. So, this is 0.1 square.

So, this will be where we are. So, point h by 2 this is 0 and this is reduced to. So, we get the value. Next we compute a good exercise for you is for the same example you try to compute the values at least 2-3 values, y point 1 point 2 point 3, using Euler method and then R K method. Since, they form of f is very simple one can compare with the exact solution to know which gives slightly at the rate. So, in the next class we try to talk about R K fourth order and may be once we define the order of the method, then we have to talk slightly more about the error.

For example, you are computing y f 0.1, then there is some error introduced at that stage then to compute y f 0.2 you are using the value of y f point 1, so whatever the error introduced at 0.1 that will propagate. So, we have to really think of we have to really think of how these errors will be propagating and try to control. So, will talk on R K fourth order and then how in general error estimates are computed.

Thank you.