Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 38
Method of Characteristics for Hyperbolic PDEs - II

Hello, good morning. So, in the last class we consider second order hyperbolic PDEs, and discussed method of characteristics. So, let us do some examples today and then also proceed to first order PDE. So, let us before we go for the examples, let us check quick recall on yesterday's thing.

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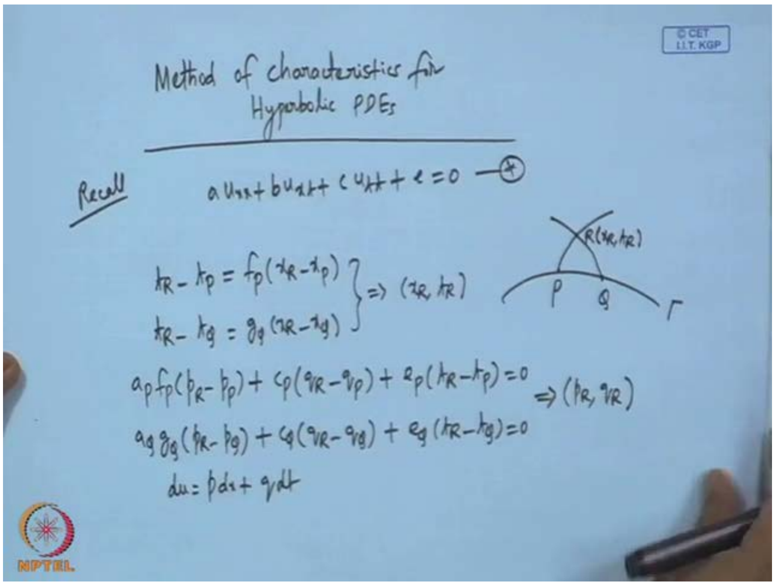

So, the recall is we consider PDE of the form, so then if gamma is any non characteristic curve, if we take two points. So, they intersect the point R, now we have obtained. So, these are the two equations and two unknowns. So, from here we get x R and t R then once we are done with this we go for, so these two would give p R and q R and then ultimately we use d u equals; of course the discussed version and obtained, right? So with this let us proceed.

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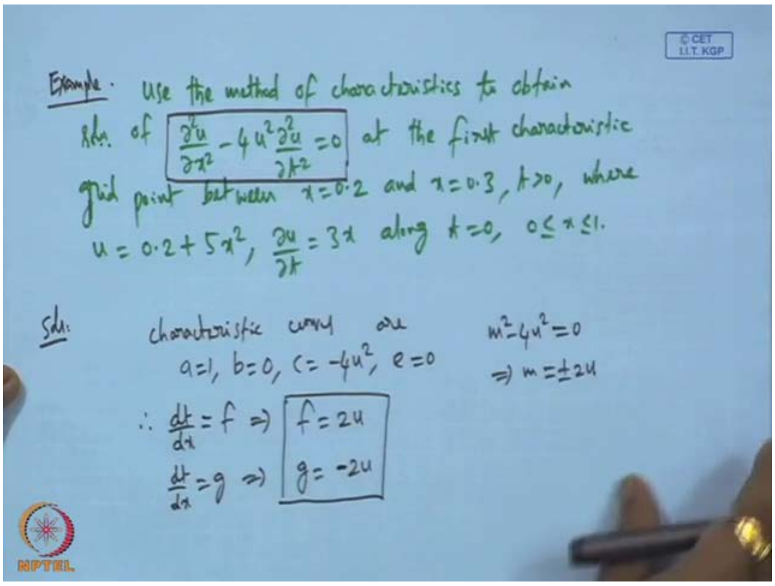

So, example so use method of characteristics to obtain solution of at the first characteristic the point between x equals to 0.2 and x equals to 0.3, where u is 3 x along p equals to 0. Suppose, this is a problem, so this is a given PDE and we are expect to get the solution between this two along the characteristic, so solution. So, what are the characteristic curves is so in this case a is 1, b is 0 c is minus 4 u square and e is 0 when we compare with the standard form. So, the characteristic curves are m is therefore, dt by d x equals to f implies f equals 2 u dt by d x equals to g implies g equal to minus 2 u. So, these are required for us.

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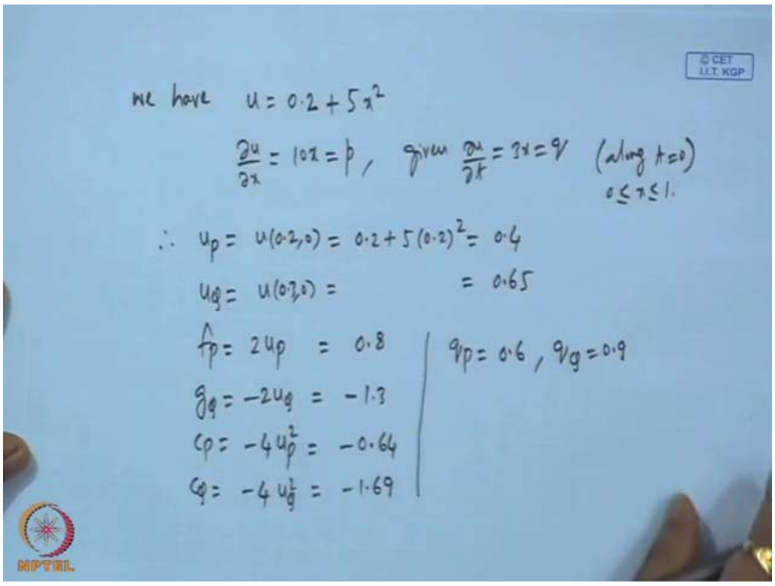

Now, we have u equals to 0.2 plus 5 x square therefore, d u by d x 10 x which is equals to p and by given d u b y d t equals to 3 x, which is q. Of course, these are along t equals to 0 and therefore, let us compute u p u p is u at 0.20. So, this is than u Q, so this is than f P is 2 u p. So, this is 0.8 than g q minus u Q, so this is minus 1.3 than c P is u P square. So, this is minus 0.64. So, this is so having obtained also we need the corresponding data q P. So, we have q which is 3 x q P is 0.6 and q Q is 0.9. Now, let us first compute we would like to compute first the point.

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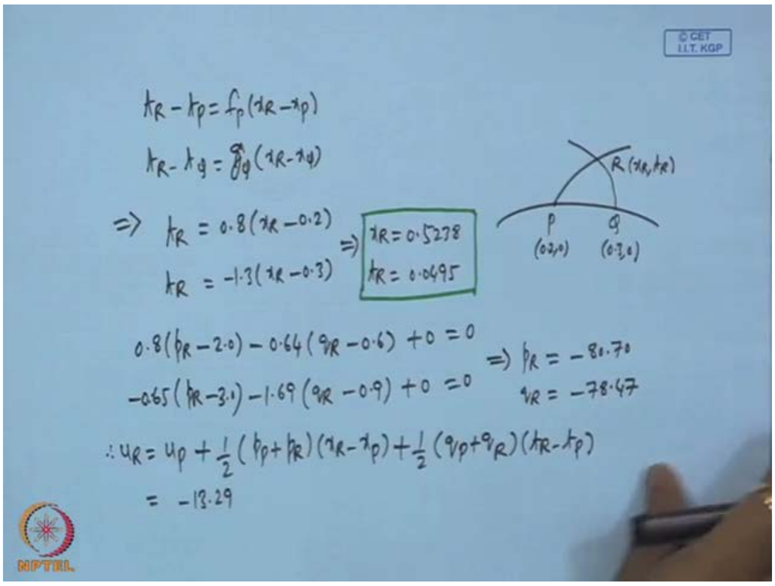

So this is P and this is Q now R, so we would like to compute this points therefore, we have to use. So, if we use this so t p and t q are 0, so t R is 0.8, so this implies x R so you can solve the system. So, this is the point, so once we get this point what is our next task to compute the first derivatives at R. However the corresponding equation so I am referring these equations if you recall so e p and e Q. So far the further equation that we have considered, so I have already mentioned e is 0, so what would happen if e is 0 the corresponding equation to compute p r and q r from this system this is 0.

So, e P is 0, e Q is 0. So, there is no contribution from this term so the equation is simplified to so by simplifying solving the system, we get and we have approximately for u. So, this we approximate by taking the average of p and average of q, so accordingly we get u R equals to u P plus half p P plus p R. So, this if we compute by substituting the corresponding values so this is the technique to compute. So, let us see one more example.

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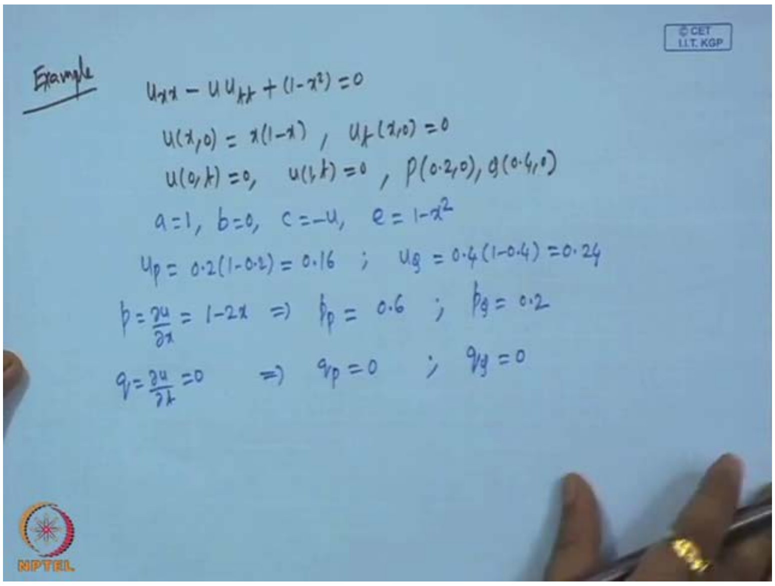

Where e in this case e is 0 therefore, there is no contribution literally even though we have obtained this point that was not used to compute this. However to compute this u r we use. So, the next example then u t and the points p is this and q is now comparing with the standard form a is 1, b is 0, c is minus u, e is 1 minus x square, u P and u Q then p d u d x. So, this will be 1 minus 2 x this implies p P is 0.6 and p Q is 0.2 just you substitute the point p and q then q 0, so this implies q P is 0 q Q is 0.

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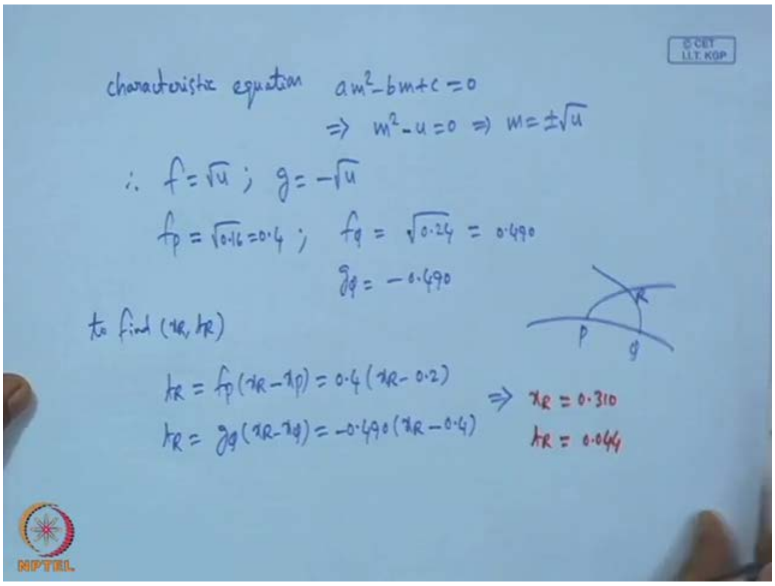

Now, let us go for the characteristics equation, so characteristic equation the equation we have so a is 1 b is 0 c is minus u. Therefore, so this implies m square minus u equals to 0. Therefore, f equals to root u and g is minus root u, now let us compute f P so this is then f Q, f Q is p is 0.2, q is 0.4. So, f Q is root then g Q is minus now we have to find x R t R because this is the next task to find R. So, t R is so this implies we get x R and t R so I have pre calculated by considering some random values so you must verify this.

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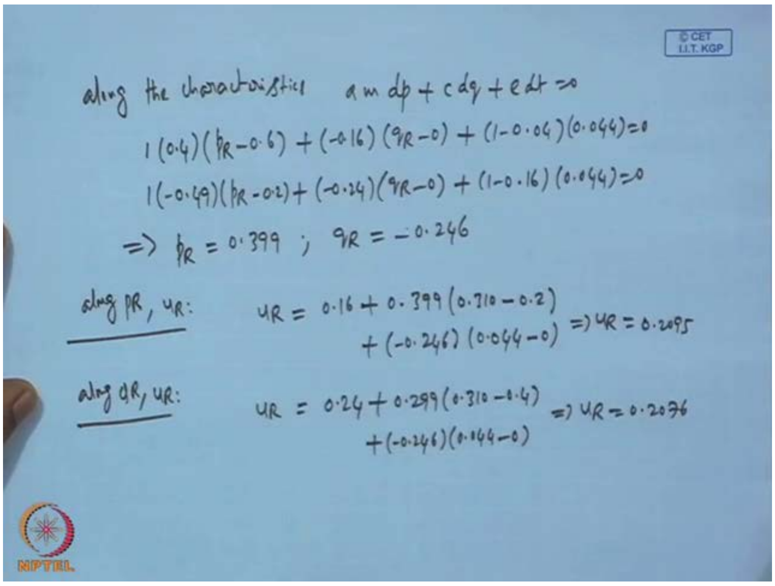

Now, along the characteristics we have the relations a m d p. So, the corresponding discretized versions if we use then we get a is 1. I am just substituting the corresponding values nothing else so we get p r is q r is, now the next task is to obtain along P R, u r. So, this will be u R equals to 0.16 this is u P plus so this implies u R equals to zero point along q R u R so along q R so their close because u R must be same they have to improvise small error. So, this is the method of the characteristics for the second order PDE. Now, let us make use of the characteristics properties and then try to see the numerical solution of the first order PDE. So, the story is small as same.

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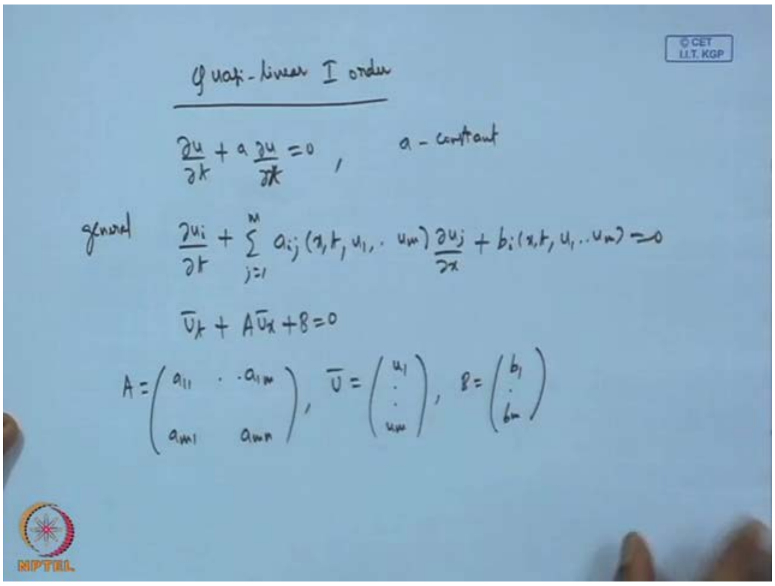

So, before we go to the numerical so these are nothing but Quasi-linear first order. So, we have to reduce a bit. So, this is the first order we considered yesterday as an example of conduction of with particle kind of, however there the scenario is different a is not constant. So, u is dependent an x x and e are independent variables. Now, more general is something like this. So, this can be put it in a matrix form right where A is so we revisit this, and we saw discuss numerical method of system as well now in this.

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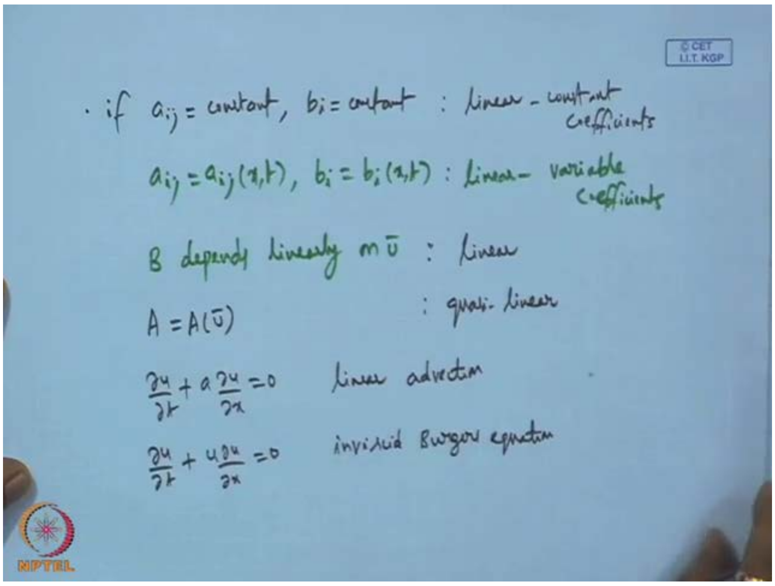

If a i j is constant then b i is constant then the equation is linear constant coefficients, or if a i j is a i j x t, b i is b i x t then linear variable coefficients. If B depends linearly on u, so than linear depending on other coefficients variable, but it is linear if A is depending on u. So, then it is Quasi-linear, so examples. So, we discussed numerical solution of this little later, so this linear advection then so this is inviscid burgers equation.

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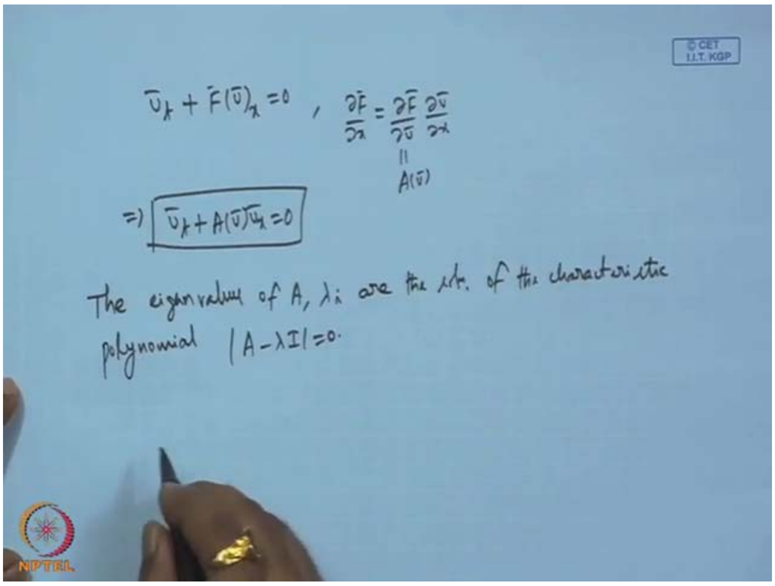

On the other hand more generally, so these are partial derivatives this notation, so where dou f dou x and this we call a of u the system reduces to, so this is more general. Now, the characteristic concept before we proceed, first order can also be classified depending on the characteristics, so we must discuss that so the Eigen values of A, lambda i are solution of the characteristic polynomial. Now, let us consider an example.

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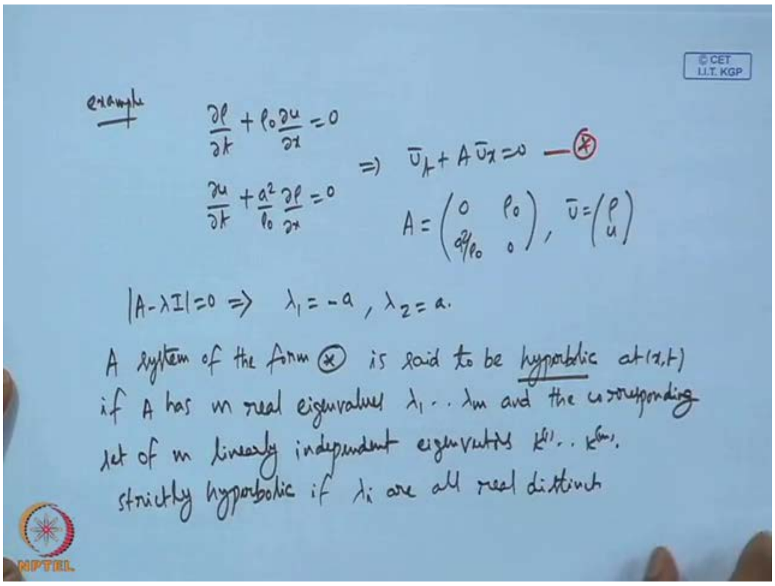

Suppose, the system is given by this and it must be coupled u naught is constant A is constant. So then this can be written as u t plus A u x equals to 0, where A is given by 0, rho naught A square rho naught then u is now, if you consider the Eigen values of this matrix, we get lambda 1 equals to minus a lambda 2 equals to a now similar to second order one can define depending on the Eigen values.

So, the definition is as follows a system of the form star is said to be hyperbolic at x t, if A has m real Eigen values and the corresponding set of m linearly independent Eigen vectors we can call k m. So, this is said to be hyperbolic x t if a has m real Eigen values. Now, the system is said to be strictly hyperbolic if lambda i yes you are right lambda i are all real distinct. So, with respect to first order also one can classify than suppose you consider another example.

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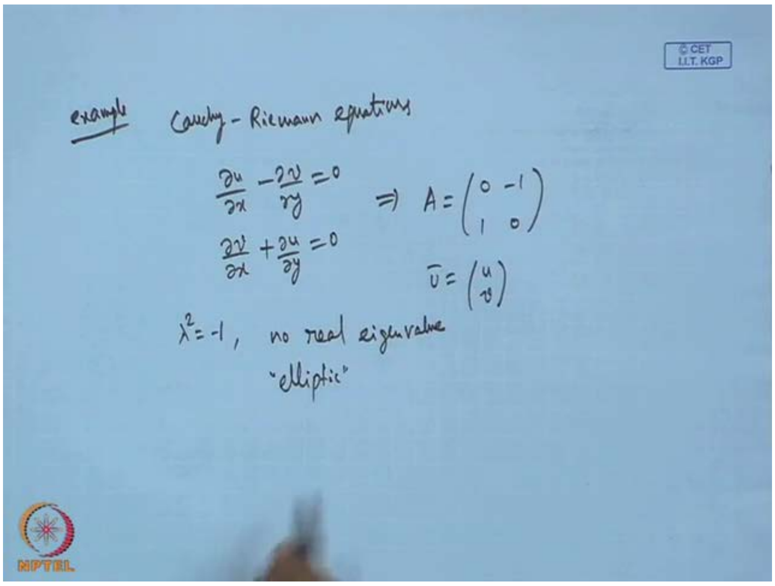

Which is quite popular, suppose you consider C R equations Cauchy Reimann then u x equals to v y, u y equals to minus v x. So, this implies A is and u r is u v in this case lambda square is minus 1. So, no real Eigen value hence this is elliptic, so we have extended the concept to the first order equation as well, now the characteristics of first order as I mentioned.

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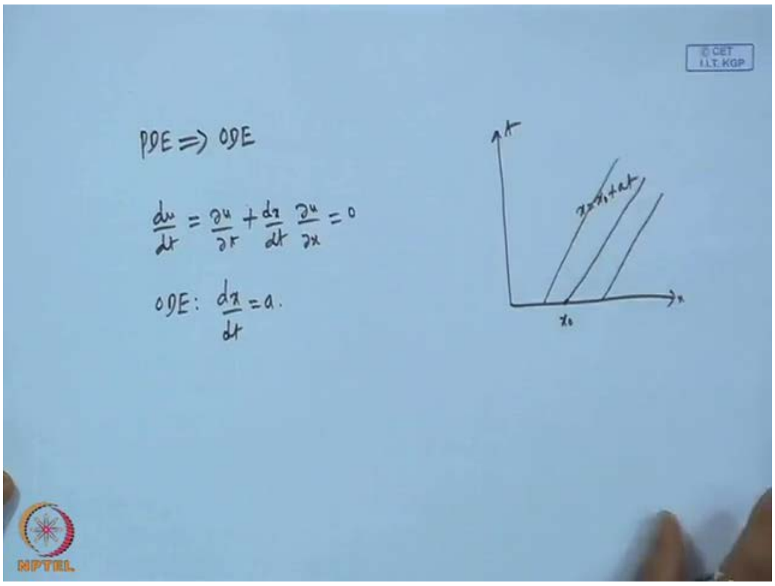

What is the property of characteristic, so if you are taken an initial point. So characteristic curve is given by this so this is x naught this is x equals to x naught plus a t where a is slope. So, these are the characteristic curves right, so what will happen? Characteristic curves in some sense they reduce PDE to Ode, so and then one can obtain along the characteristic curve. So, this is 0 and what is the ODE, the ODE is you see here a. So, the ODE is d x by d t equals to a now similar to second order, we would like to discuss briefly how do we proceed to compute solution along characteristics.

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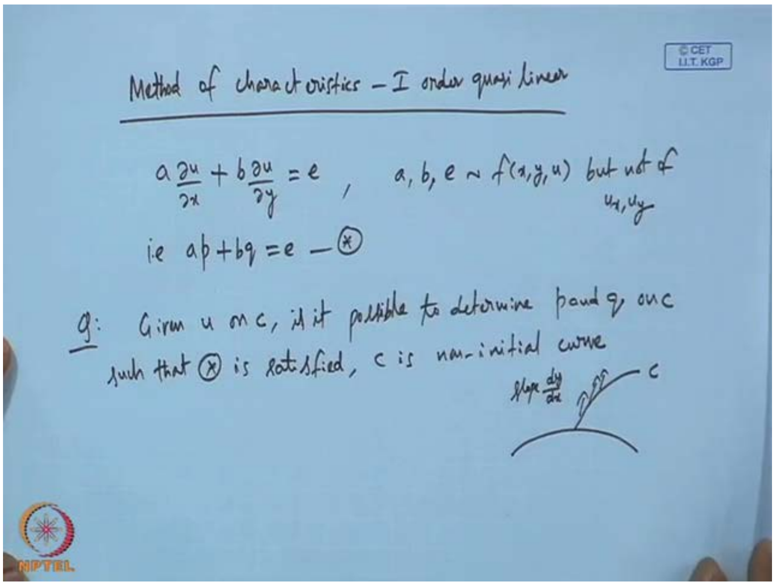

So, method of characteristics first order Quasi-linear, so we consider a d u by d x plus b equals to e. So, as given a b e are functions of x y u, but a b and e so they are functions of x y u, but not of u x u y, but not of u x u y. So, this equation is nothing but a p plus b q equals to e, right? Now what is our aim, the question given u on c is it possible to determine p and q on c such that, star is satisfied where c is non initial curve. So, that means this is an initial curve, than this is c. So, if u is given on this is it possible to determine p and q on c such that star is so the answer is true like second order, so where in this case this slope of these tangents is denoted by d y by d x.

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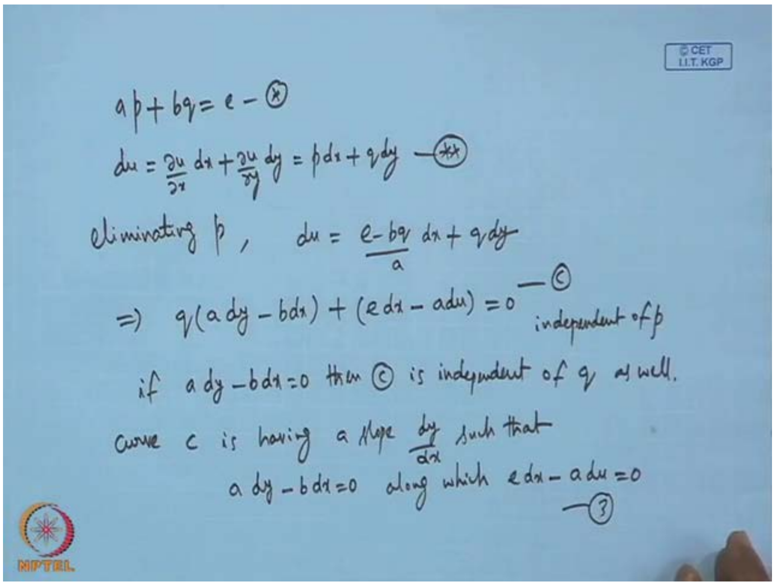

So, we have the following scenario a p plus b q equals to e than d u is q d y. So, then eliminating p, we have this implies, so this is independent of p this is independent of p. Now, this can be made independent of q, if 0 then this is independent of q as well. Therefore, curve c is having a slope d y by d x such that a d y minus b d x is 0 this is satisfied so this is.

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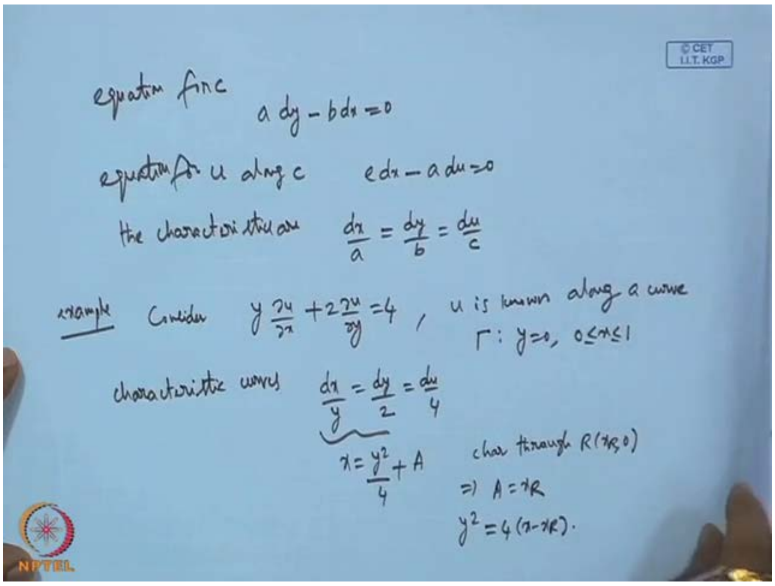

So, that means equation for c equation for c a d y minus b d x equals to 0 and equation for u along c so therefore, the characteristics are the curves. So, before we proceed analytically numerically, let us see a simple case. So, we consider y equals to 4 and say u is known along a curve gamma, which is y equals to 0. Now, the characteristics curves d x by y is dy by 2 is d u by 4. Now, let us consider these two. So, by considering these two we get x is y square by 4 plus A, then if the characteristics passes through R, characteristic through R. So, this if we substitute x R and y is 0 and we get a to be x R so this implies we have A is x R. Therefore we have y square equals to 4 into x minus x R. Now, along this characteristic.

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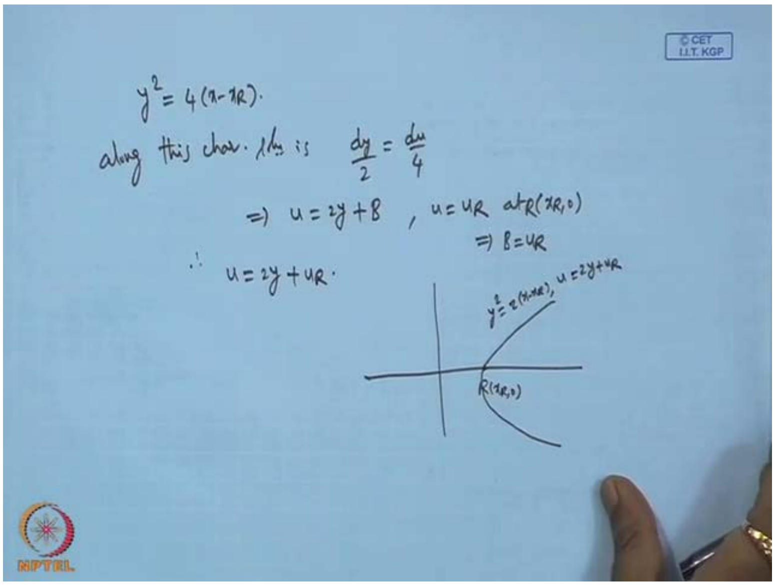

So, we have the characteristic y square is 4 into x minus x R. Now, along this characteristic the solution is 4, this implies now again u is u R at x R 0 which is R. So, this will give us therefore, u is 2 y plus u R, right? So, in this case the curves looks like this, suppose this is the point so these are the curves y square is 2 into x minus x R along which u is 2 y plus u R. So, if u is known than one can compute so this is the analytical scenario.

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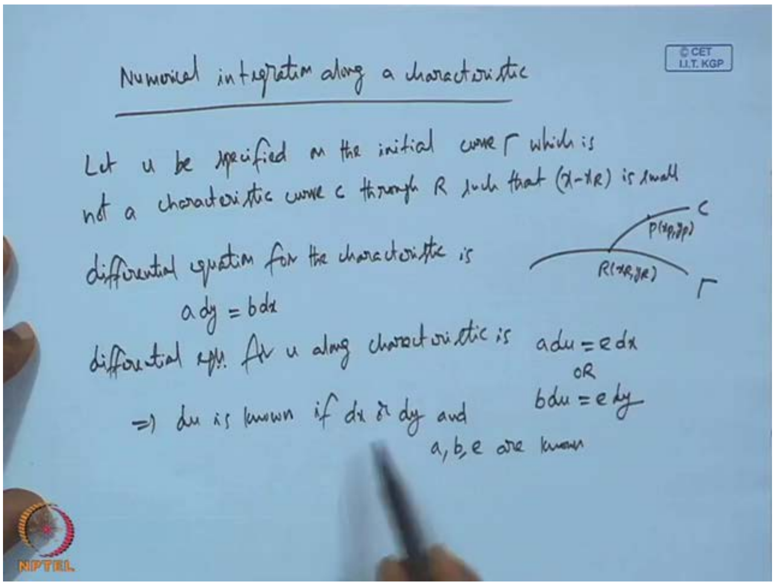

Now, we will like to consider numerical integration along a, so having obtained analytical. So, we want to make use of concept of characteristics then compute the solution numerical. So, this in some sense similar to second order, but let us have a look at it. So, let u be specified on the initial curve gamma, which is not a characteristic curve c through r such that x minus x r is small.

So, that this is the gamma this is c and R and consider P, then the differential equation for the characteristic is a d y is b d x that is what we have seen, then the difference equation for u along characteristic is a d u is e d x or b d u is. So, this implies d u is known if d x or d y and a b e are known analytically also we have seen this. Now, how do we go for the numeric?

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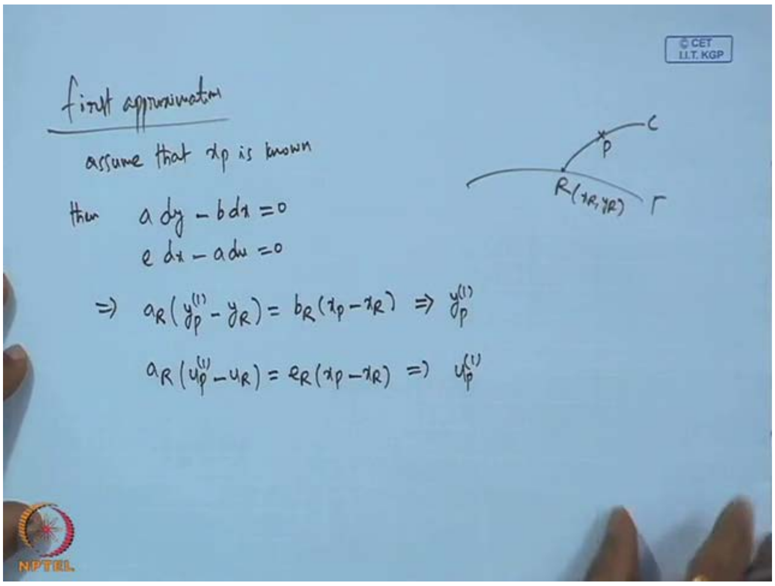

So, first approximation first approximation this is P assume that x P is known then because we have to approximate a d y minus b d x equal to 0, then e d x. So, this pair was alternately another pair could be, so this we are discussing a R y P first approximation equals b R, x P minus x R. So, this would give if you know x P the rest are all known y P 1 can be obtained than from this equation u P 1 can be obtained.

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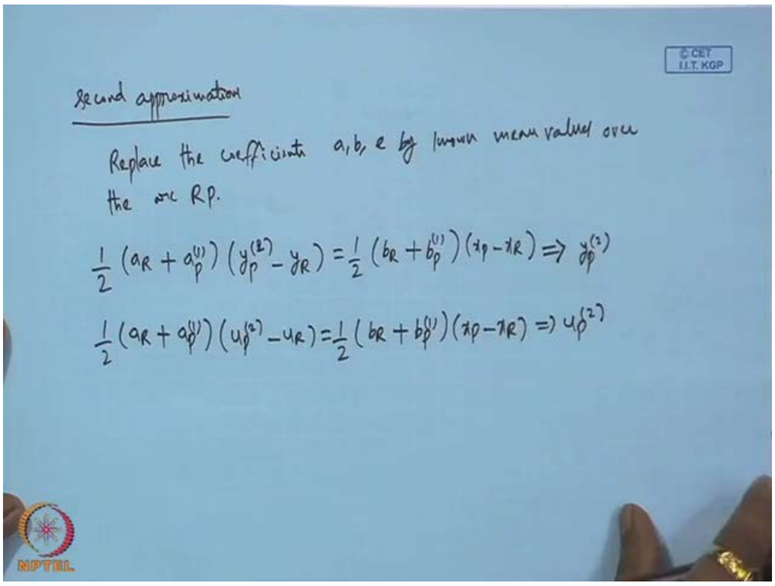

Now, second approximation replace the coefficients a b e by non mean values or the arc R P. So, we consider these two equations and then we replace the mean values, so that would be so this goes like this so you can. So, this is first approximation what we have done we construct this equation than assuming x P is known we can obtain y P 1 than u P 1. Now, we replace this coefficients by average you can see a R and a P 1 so then we get second approximation from here y P 2. Similarly, from here we get u P 2 so that is how we compute.

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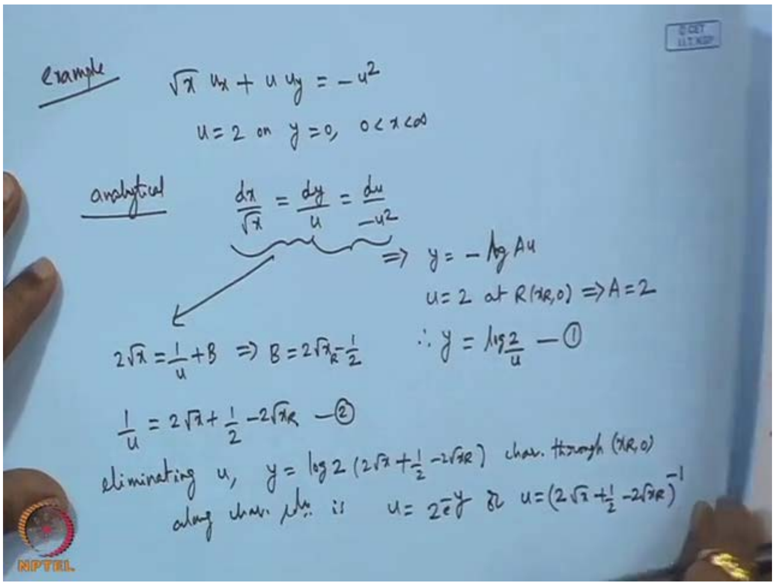

So, let us have a quick look at example one consider this, suppose this is the example then analytical d x by root x d y by u d u by minus u square. If you consider this y is minus log A u then u equals to 2 at R x R 0 this implies A is to therefore, y is log, so this is then we construct these two we get and again we can eliminate. Therefore, we have 1 by u right then eliminating u we get log 2. So, this is the characteristic through, characteristic through x R then along characteristic the solution is u equals to 2 or u equals to 2 root x plus half either from here or here.

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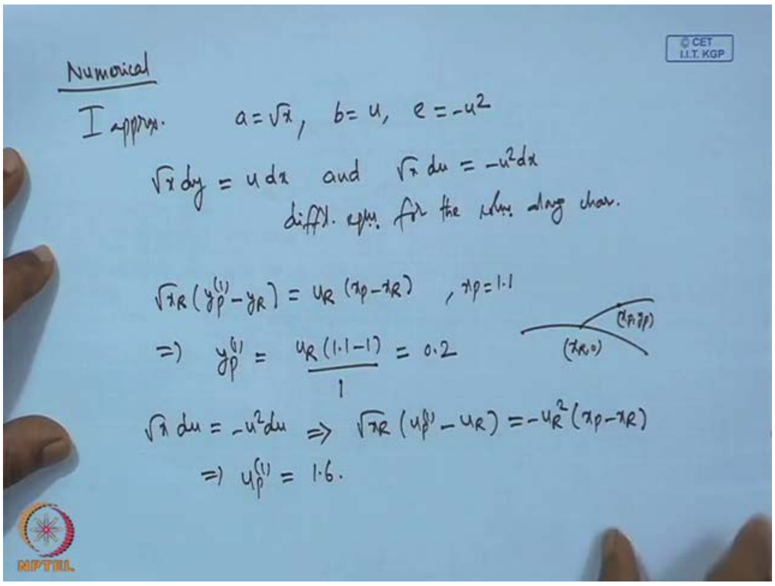

Now, let us see numerical first approximation a is this b is this. Then consider the characters solutions this, so these are the definitely equations for the solutions. So, the differential equations for the solutions along, I am using short forms. Now, if we approximate first approximations u R x P minus x R, so this implies y P 1 is u R 1.1 minus 1 by x R is 1. So, we have considered x p is 1.1. So, this is so this is x R 0 and this is x P y P. So, our initial condition is x R and x P are close enough. So, the decrement is or increment 0.1, so this is now we consider this implies now from here we get u P 1, so y P and then we got this, right?

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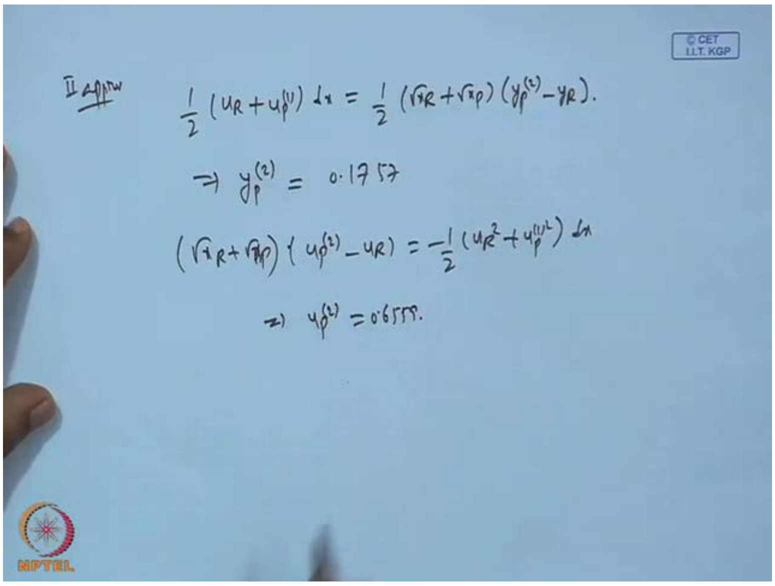

So second approximation, so we go for u R plus u P 1 d x x R x P y R. So, this will give y P 2, so you can substitute the values than so this into u P 2 minus u R. So, this will give u P 2, so this is the numerical procedure along the characteristics. So, the example is little I mean. Since I have a pre calculated I have done it you can verify. So, next lecture will talk about finite difference method and till then bye.