Numerical Solutions of Ordinary and Partial Differential Equation
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Lecture - 35
Finite Difference Approximations to Hyperbolic PDEs- I

Hello, in the last class we learnt about elliptic PDE’s and few techniques, so with this we covered parabolic elliptic PDE’s. Now, let us move on to hyperbolic, so before we go for final difference approximations, it is worth to spend few minutes on the corresponding analytical treatment in brief. So, the typical example of hyperbolic PDE which is generally taught in first course is wave equation, so let us review wave equation and then proceed to the corresponding final difference approximations.

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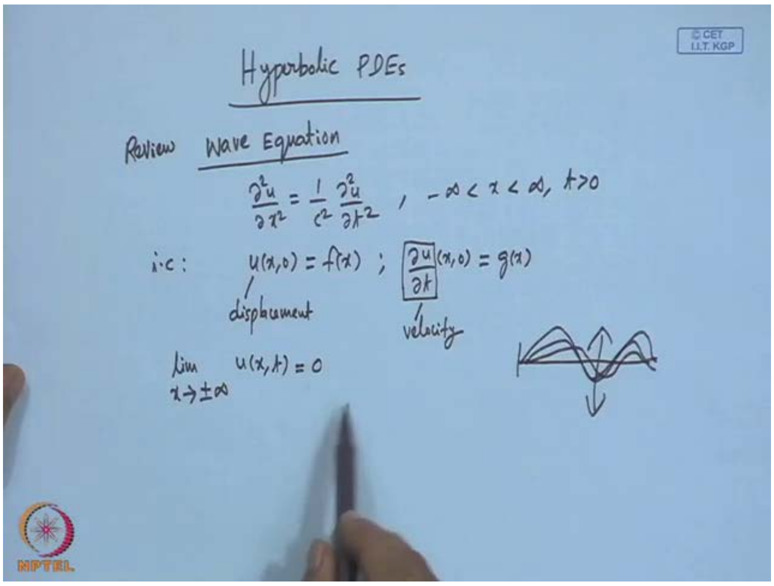

So, we review wave equation which is defined as this infinite string problem and we have various versions fixed at one end. For example, if it is fixed at one end, then one end will be infinity; other end will be the corresponding point where it is fixed at both ends. So, then it will be an interval a to b, now we have to define the complete problem by specifying the corresponding initial boundary conditions. So, we have these are initial conditions, so what are these specifications, so here the dependent variable u denote displacement, so dou u by dou t. So, this denote velocity, so the initial conditions are given initial displacement f of x initial velocity g of x, so we are looking for the solution and in the present case since it is infinity I would like to put it.

So, this displacement is typical though for fixed boundaries one may have a function of t, but in general for field if you see displacements are generally vanishing displacements. If you take for example, if you fix and then pull, so you will have vibrations then maybe they will die down slowly. So, you have some vibrations back and forth and then at far field they die down, so this is a complete definition of the problem.

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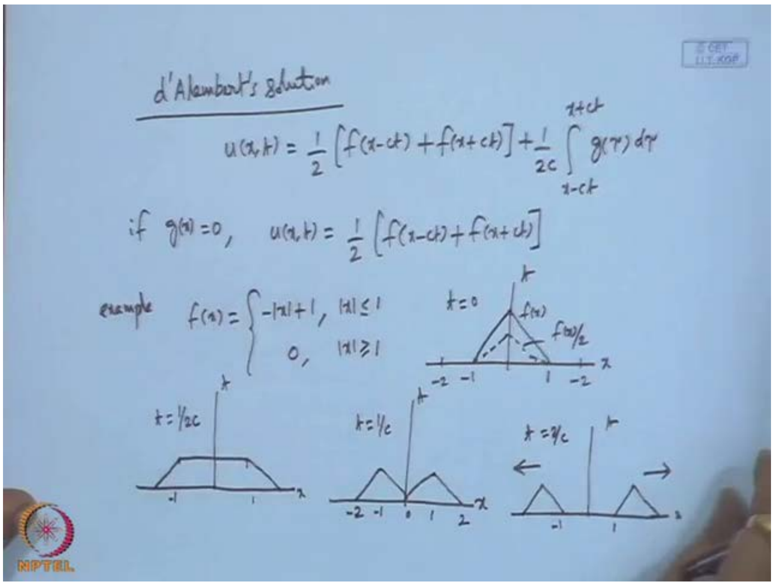

From analytical point of view, there is a popular solution known as d'Alembert's solution, so this is given by plus 1 by 2 c, so with this given initial velocity as f of x, sorry initial displacement is f of x and initial velocity is g of x. Then the solution is given by this if g of x is 0, that is initial velocity is 0, then u of x t is, so this will be the solution. So, this represents two waves each equal to f of x by 2 c, one to the left and one to the right.

So, for example, let us consider some initial profile say f of x is given by such a profile is given, then what would happen, so this is x axis and t axis. So, we have minus 1, 1, so this is f of x, so these are f of x by 2 say this, so this is at p equals to 0 which is this profile. So, then what would happen correspondingly, so this is at t equal to 1 by 2 c and further, so this is at t equals to 1 by c, so then again after some time this is at t equals to this initial profile.

As time progresses, one wave pass to the left, another wave passes to the right, so this is popular d'Alembert's solution which explains the physical significance of the solution. So, similarly there could be other case corresponding to f of x equals to 0, so this is a special case.

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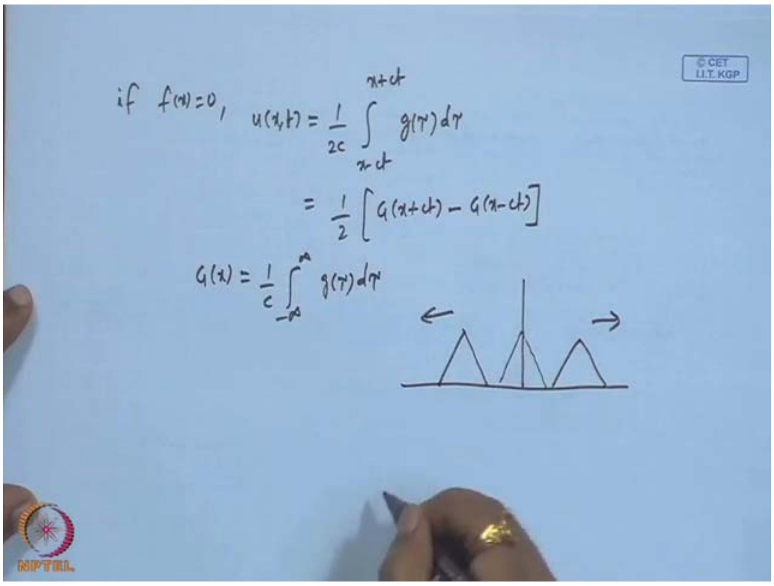

Suppose, if f of x equals to 0, then we have u of x t, so this will be where g of x is again for a given initial g of x. We can see how it progresses, so this at least it is useful to understand the analytical solution which is d'Alembert's solution and the corresponding physical significance. Now, when you see the waves are passing one to the left and one to the right, so initially when we have given initial, so this point has been displaced to the maximum extent.

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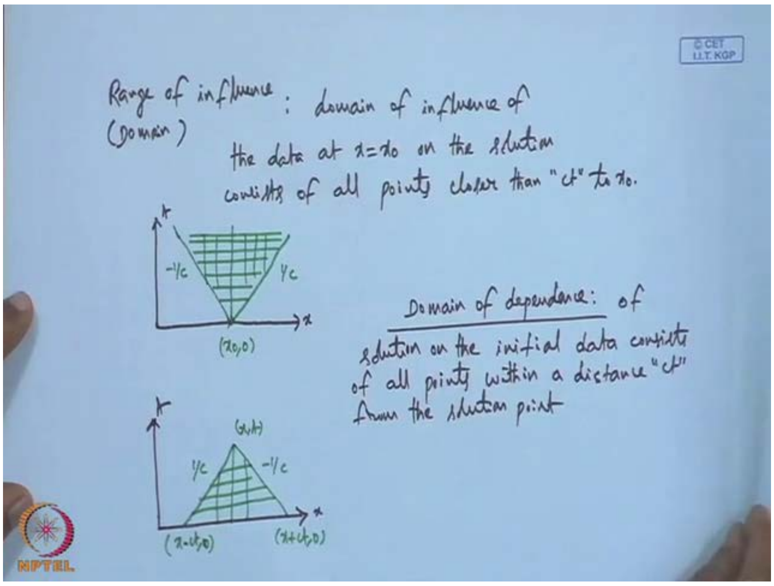

So, now it is useful to understand something called range of influence sometimes domain of influence, so which means domain of influence of the data at x equal to x naught initial point on the solution. So, this domain of influence of the data at x equal to x naught on the solution, so this consists of all points closer than c t to x naught. So, what do mean by that, suppose if we take initial point, then domain of influence of the data at x equal to x naught on the solution.

This means how far this would influence the solution this is with slope; these are the corresponding slopes, so this is the domain of influence of x naught on the solution. Similarly, we have domain of dependence of solution on the initial data, so this consists of all points within a distance c t from the solution point. So, this is range domain of influence of x naught on the solution and here it is the opposite so domain of dependence of solution on the initial data. So, this would be these are x minus c t x plus c t and this point x t solution point.

So, this domain of dependence of solution on the initial data consists of all points within a distance c t solution data, so this solution depends on the initial data within this range. So, these are few important remarks about the wave equation, now let us proceed to the finite differences scheme. So, to start with we go for our usual approximation and then see what will happen, under what condition we go for explicit and under what condition we may arrive at implicit scheme, so let us start with simple normal usual discretation.

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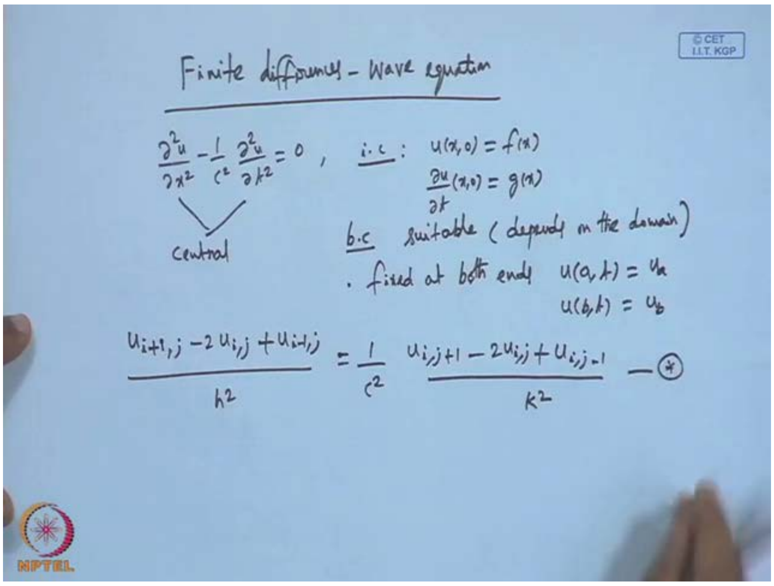

So, finite differences wave equation consider with initial conditions on the boundary conditions. So, when I say suitable, this depends on the problem, so this depends on the on the domain, so for example, if it is fixed at both ends, then what we expect U of a t is some U a, U of b t is some U b. They can be in functions of t, now we discretize this so go for central, so then we arrive at.

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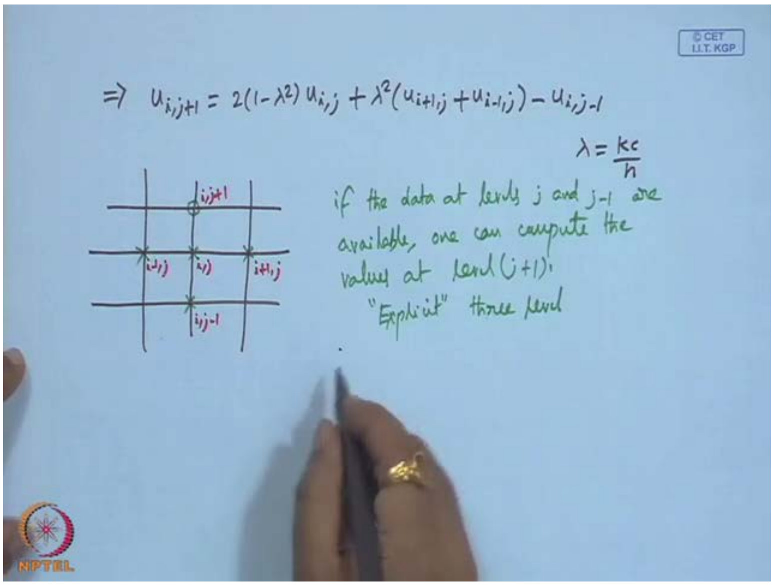

Now, this we can simplify, so if we simplify where lambda equals to k c by h, so if what we have done we just adjusted. So, this is we need at higher time level, so we have written this to the left hand side and the remaining terms got adjusted, so when we take this h square. So, h by c k h by c k is on our lambda, so accordingly that got adjusted c square by k square comes here, so c k by h whole square which is lambda square then got adjusted. So, the remark here is the coefficients are as follows, the grid is as follows, so this is i j i k i plus 1 j i minus 1 j then i j minus 1 and we are looking for the value here. Interestingly, if we see the right hand side it is asking data at two time levels, so if the data at levels j and j minus 1 are available one can compute the values at level j plus 1. Hence, this is explicit in the sense, however three levels are involved, so let us have a quick look at the corresponding error involved, so let us expand this equation.

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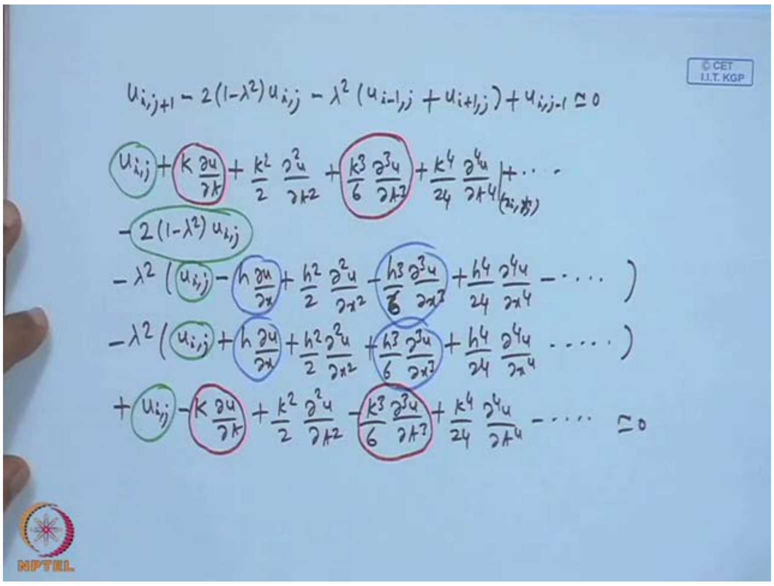

So, consider this approximately equals to 0, so if we expand this term I am expanding minus 2 of course, all of these evaluated at x i y t j, so this term is done then this 1 minus lambda square i j minus dot, then we have minus lambda square in this is U and plus h. So, we have done earlier, so by now you should be conversant with this technique plus U i j minus 1 minus K. Now, let us count, so we have this term this term gets cancels and this term this term gets cancels then look at there is one U i and we have another U i. So, we get 2 U I, then we have a minus lambda square U i and another minus lambda square U i so these so this term cancel then which are the term get cancelled So minus lambda square remain so this so these terms get cancelled.

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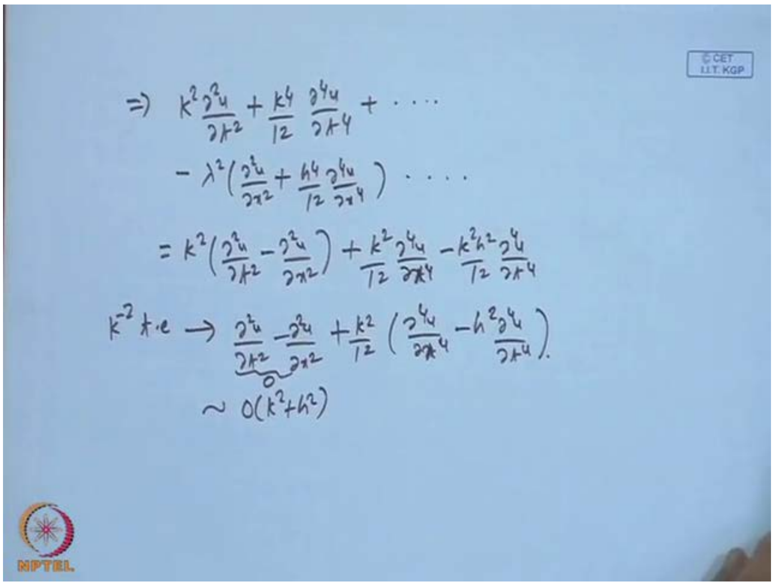

So, if you count it, ultimately we get this minus lambda square etcetera, so this can be written as k square plus k by 12, so we can simplify and ultimately k power minus 2 goes to plus, there is a k square, this is x, 4 x 4 minus h square. So, this is of order because this goes to 0, so the order of the error is this much, so however, let us observe closely the equation which equation this one. So, as I mentioned that I marked, we need two time values at two past time levels j minus 1 and j, so let us see how U h we can proceed with this.

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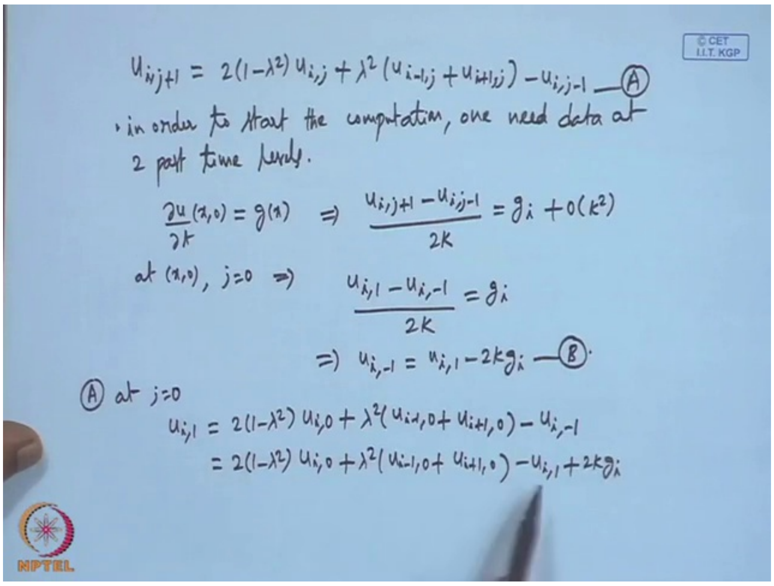

So, I mark is U i j plus 1 is so given this, in order to start the computation, one need data at two past time levels, so this is the remark however consider dou U by dou t at x equals to 0 is this is one of our initial condition. This initial velocity is prescribed, so we make use of this g i, of course this is of order we call this as second order. now this is at at x 0 e have j is 0.

So, this implies U i 1 minus 1 by 2 k is g I, so i can write this as sum a, so this implies these are the values past values so U i minus one equals to u i one minus two K g i, so this is b. So, when we, let us say we want to compute at j equals to 0, so this is at 1, then 0, 0, then we need at minus 1. So, these fictitious values can be eliminated from here, now let us consider a at j equals to 0 because that is where we get fictitious values. So, we have U i 1 u i 0, so then we can substitute we can substitute this value, so this is so we can 2 times and then we can divide.

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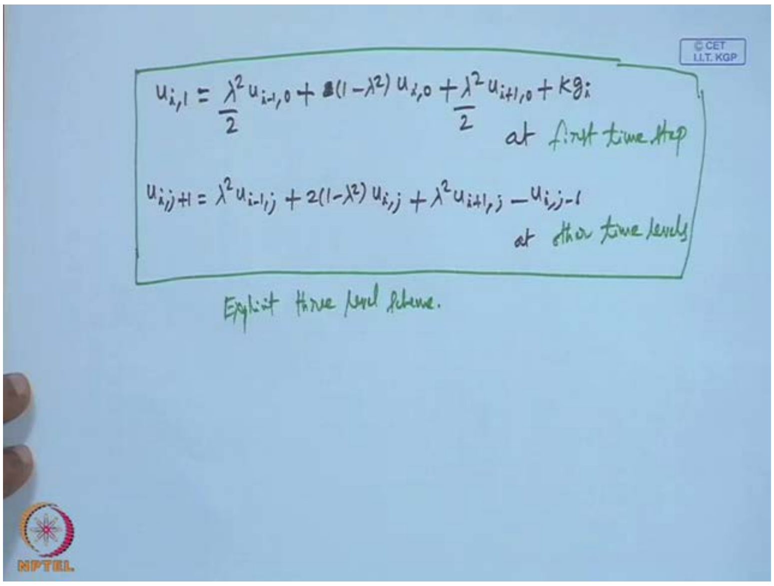

Ultimately, we get U i 1 equals to this 2 g 1 plus K g I, so I did not do anything, so one step I skipped, so this will be 2 times, so everywhere you divide by this, so you get this. Therefore, this is at first time step because you can see at first time step using at zero time step, we can get explicit values. So, this is explicit scheme for the wave equation, so at first time step, we used this for the higher time levels you can see we can compute with this.

So, this is an explicit three level scheme, so let us solve a problem, so as you can see this is very complex. Initially, it looks little difficult task, but when we have displaced the corresponding initial velocity, the fictitious values are removed and hence we got one equation at first time level and then for the remaining we have another equation.

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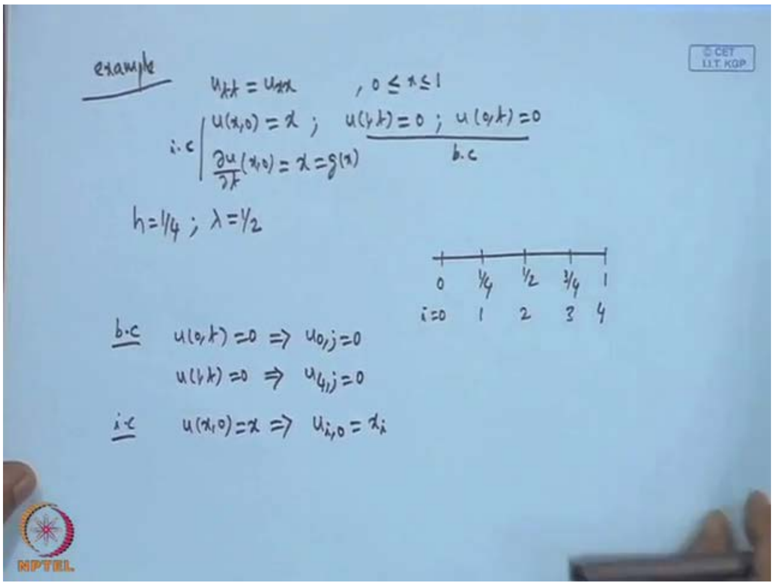

So, let us consider an example, so an example I have changed the notation and then c is 1, so can understand then so this is of x. So, these are initial conditions and these are boundary conditions, now we need to know h is one fourth lambda is half. So, accordingly we solve for the time levels, now consider the boundary conditions U of 0 t is 0, so this implies U 0 j is 0 U of 1 t is 0, one corresponds to 4, 4 j is 0 and initial condition U of x 0 is x. So, this implies U i 0 is x i and the corresponding discretization I would like to show the next sheet.

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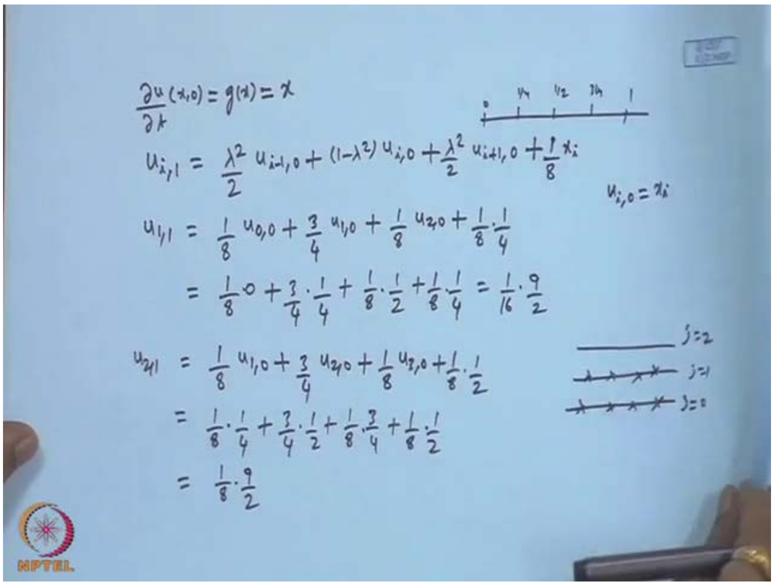

So, dou U by dou t x zero is g of x which is equals to x in our case and from the previous discretization we have obtained you can see. So, we make use of it, so U i 1 equals lambda square by 2 plus K x i and in this case K becomes 1 over 8. So, we have, now let us compute the values at each nodal point, so we have this 0, one fourth, half, three fourth, 1. So, U 1 1 equals lambda square, lambda square is 1 over 4, so this becomes 1 over 8 U 0 0 plus 1 over lambda square 1 minus lambda square.

So, this will be 3 by 4 U 1 0 z 1 by 8, so x 1 is this, so 1 over 8 into 1 over 4, so this can be obtained U 0 0, so this is 0 plus 3 by 4 U 1 0 U 1 0 is x 1, so we have U i 0 is x i, the remaining are 0. So, this is one fourth, this is half, so this is calculated, please check it, so then U 2 1, so U 2 1 means the grid is like this. It is asking, so this is j equals to 0, j equals to 1 j equals to 2. So, in our case, the method asked j equal to 0 and j equals to 1, so in order to compute j equals to 1, we made use of this equation that is what we are doing at each grid point. Using j 0, we are computing at j 1, so that we use the other formula and compute, so U two one U 2 1 is 1 over 8 U 1 0 x 2, so this will be 1 over 8 U 1 0 that is x 1 that is 1 by 4 is half, three by four, so this is obtained as.

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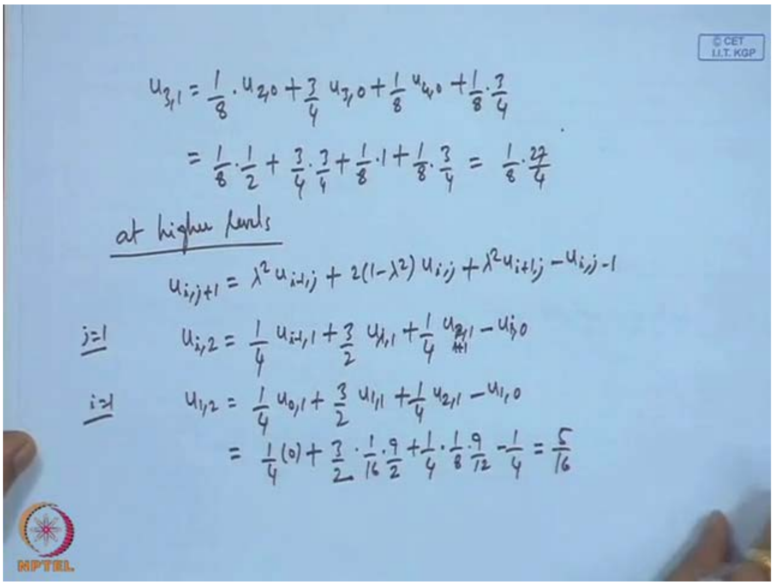

Similarly, we obtain U 3 1 U 3 1 we obtain 1 over eight U two 0 U 3 0 U four 0 x 3, so this is 4 0 this is this U 4 0 is x 4, so that is 1, as the second. So, this is 3 i equal to 3, so this is 4 0, so this is 1 plus, so this may be computed. Now, we have obtained all the values required to compute at other time levels, these are the first time level, now we proceed at higher levels, so at higher levels, so let us consider j equals to 1. So, this is at second level at all grid points now we can use this and compute at each, so for example, you want, this must be i in general case, I am sorry this is i plus 1 i. So, in this is in this case, these values are substituted, so just now we have computed U 1 1, so these values will be used, so you verify so this is the value at this point.

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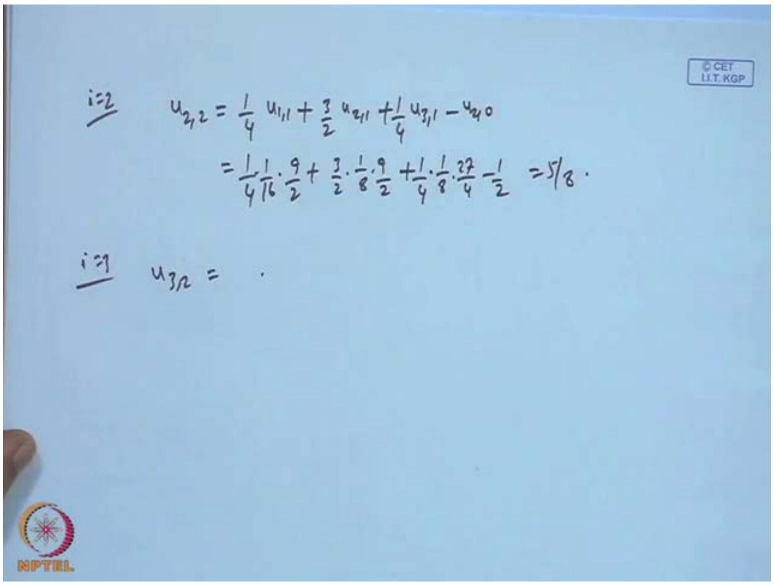

Then, i equals 2, so this will be 3 1, we have calculated earlier, so this we get, similarly we can compute, so this is very straight forward once you get the values at first time level. So, then higher time levels can be obtained by the final differences scheme which is more general, so it is a combination of two expressions, first one is at first time level then the rest is for any general j j. So, with one can match fast and compute at higher levels, so this is very explicit method, now let us review let us see under what situations we get implicit method.

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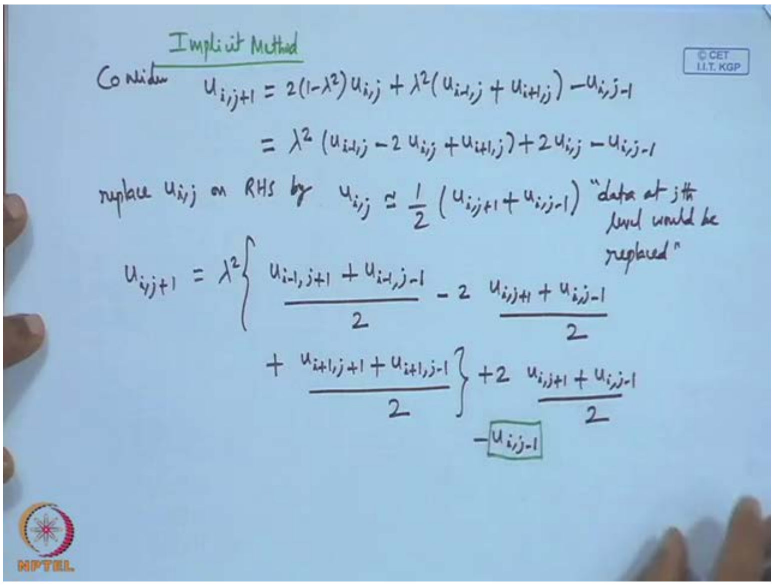

So, consider U i j plus 1, so this is a three level explicit method which we have derived, now this I would like to just put it in a slightly different form, so the same one. Now, what we are going to do is replace U i j on r h s by, so we want to bring in more time levels, so then only this becomes implicit. So, what we are trying to attempt, we are trying to attempt implicit method, so all that we are doing is introduce more time levels. So, let us see what happens, so here when I say r h s U i j, that means data at j th level would be replaced.

Now, this is at j th level, so we manipulate the index for i j this is a rule, so correspondingly for i minus 1 j. So, this remain, then we have j plus 1 j minus 1 by 2, then we have 2 times U i j by 2, then we have this is lambda square, then we have plus 2 by 2 minus. So, this is not replaced because this is containing j minus 1 time level, so what did we do we have replaced all the data point from right hand side which are at jth level by average of j plus 1 and j minus 1. So, you can see we are expanding the scope, so we expect more accuracy, so this is general idea.

So, accordingly we have manipulated the corresponding index notation for the U i j, this one for U i minus 1 j i minus 1 j plus 1 i minus 1 j minus 1 accordingly. Now, we can simplify, so we can simplify and see you have various terms for example, this is U i minus 1 j plus 1 and this is i plus 1 j plus 1. Some terms are expected to be cancelled, let us see U i j plus 1 U i j plus 1, so we do not have any.

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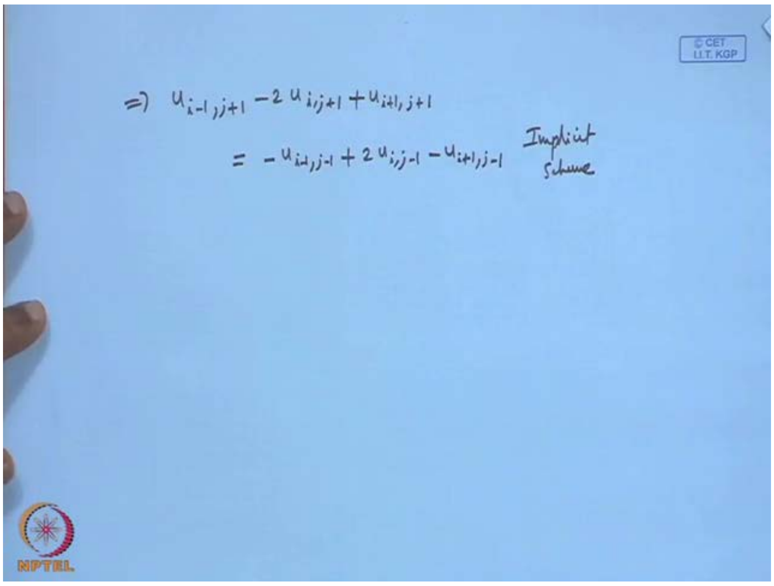

So, this reduces to U i minus 1 j plus 1, so if we adjust these, this is lambda square here. So, we have replaced, then we have to rearrange these terms, so then when we rearrange these terms, we get implicit scheme.

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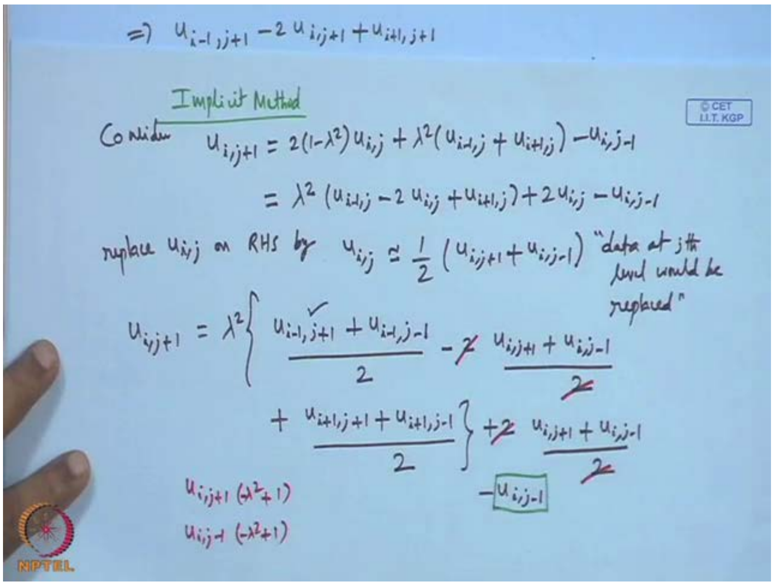

So, we can see how this would come up, so we have variety of terms i minus 1 j plus 1, so this term then we have i j plus 1 two times. So, we have two get cancelled, so we have to do some exercise here, so two get cancelled, then two get cancelled. So, we have U i j plus 1, so let us collect the coefficients U i j plus 1, we have lambda square here, then plus 1, then U i j minus 1, so we have, sorry minus lambda square because there is a minus sign.

So, similarly here minus lambda square and plus 1, so these terms are done, then U i i minus 1 U i j plus 1 U i j minus 1 here. So, this I will come back to you with on the lambda, then similarly we have other terms, so these four terms are taken care. So, U i minus 1 j minus 1 so you can this is the term then U i j minus 1, so these are the terms, then similarly, i plus 1 j plus 1 i plus 1 j minus 1, so i plus 1 j plus 1 i plus 1 j minus 1.

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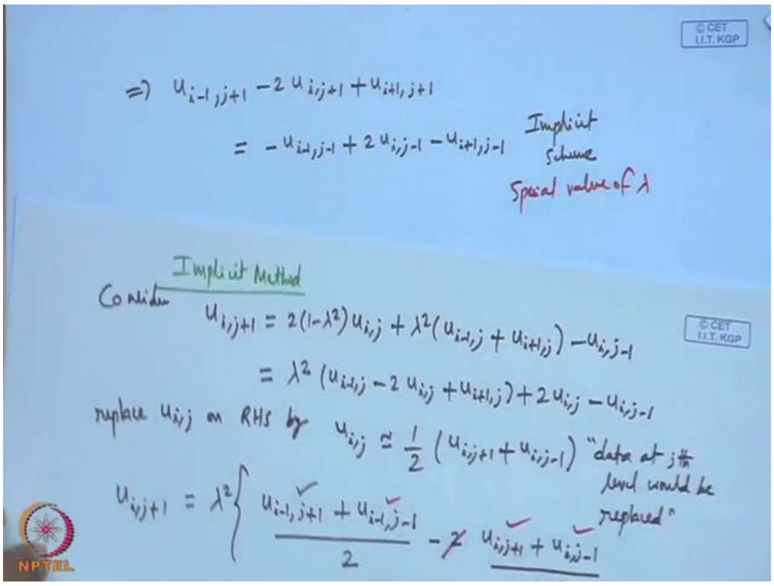

So, this is for a special case, so this is a special value of lambda, but we can get more general formula from here that involves lambda. So, this gives implicit method and then you can see left hand side, you have values corresponding to j plus 1 and right hand side you have values corresponding to j th level. Hence, you get a system of equations, so we will pay more attention to the implicit method in the coming lecture, until then bye.