Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 33
Finite Difference Approximations to Elliptic PDEs – III

Hello, welcome back. In the last lectures, we have discussed about finite difference approximations to elliptic equations. In particular, we discussed Laplace equation, Poisson's equation, and then arrived at system of equations. We also discussed methods how to solve system of equations, because these systems need special attention because these are sparse.

Now, let us proceed further with various special structures. For example, derivative boundary conditions or polar coordinates or when specific grid structure, how do we address elliptic PDEs and how do we discretize? To start with, let us make an attempt with derivative boundary conditions. Of course, I would like to explain with a reference to a specific example.

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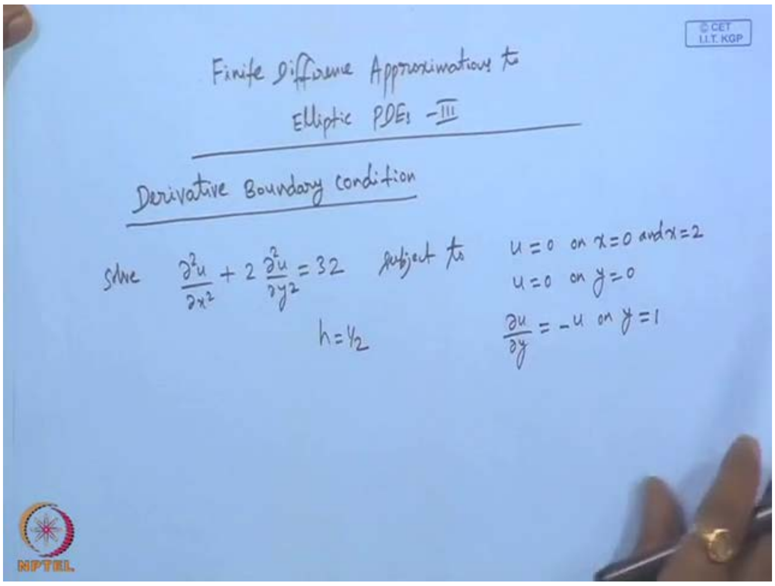

So, the case one is like derivative, derivative boundary condition. So, this may not be so much different from what we have done for parabolic case, but nevertheless, let us see. So, let us consider the example. So, we have solved this subject to say u is 0 on x equals to 0 and x equals to 2. Then u is 0 on y equals to 0 then dou u by dou y equals to minus u on y equals to 1 and say h is half, so with this, the grid becomes. So, you remember x is 0 to 2 and y is 0 to 1. So, it is rectangular and h is half.

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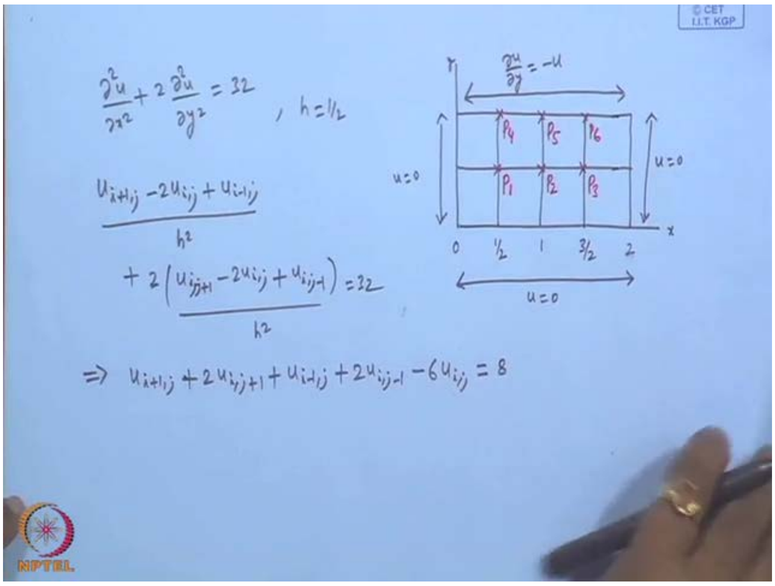

So, the grid becomes, so 0, half, 1, 3 by 2 and 2. So, this is the grid. Accordingly, so we have u is 0. Then we have here u is 0 and u is 0 on this boundary. So, hence in a normal situation, when we have let us say this data on the entire boundary, so then we would know these points as well. However, since we have derivative boundary condition, the unknowns, so we have to number them p 1, p 2, p 3. This also becomes unknown because here we have this is a mixed condition because this is digital.

So, it is in some sense, this is a special case of alpha times u plus beta times dou by dou y prescribed. Therefore, these are the unknowns. So, we have to find the solution at these points. So, let us approximate the given equation. Now, when we discretize by h square plus 2 times u i, j plus 1 by h square equals to 32. So, in the present case, h is half. So, h square is so we have in the denominator. So, in the numerator, we get 4. So, we get the following. Then we have minus 2, and then we have minus 4. So, we have let us u i plus 1, j, so then next one is two times u i, j plus 1. So, I would like to write u i, j plus 1, then u i minus 1, j, then two times, then we have minus 2 u j and minus 4, so minus 6 u i, j.

We have h square that is 4, so 4 takes it, so h square, so we get the right hand side, so this will be 4, so 8. I hope I am correct. So, let me check because we have h square on our h square that is 4. So, throughout one 4 gets cancelled. So, in the right hand side, we get 8. Now, we have to discretize the boundary conditions.

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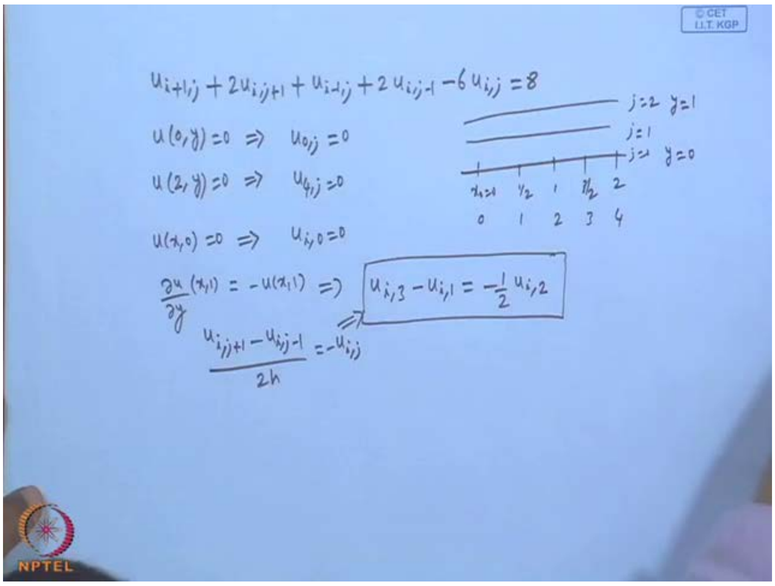

So, let us write down the equation again. Now, the boundary conditions u of 0 y is 0. So, this implies u 0, y is u 0, j is 0. Also, u of 2, y is 0. So, this implies u 4, j equals to 0 because the grid is x 0 equal to 0 half, 1, 3 by 2 and 2. So, this is 0, 1, 2, 3, and 4. So, this will be u i 0 equals to 0 and dou by dou y x, 1 is minus u x, 1. So, this implies, so more general if we do, then simplify u i, 3 minus u i, 1 equals to minus half u i, 2. So, this we can for example, if we do it more general case, u i, j plus 1 minus u i, j minus 1 by 2 h is minus u i, j and this is at x equals to 1.

So, at y equals to 1, sorry at y equals to 1, so y equals to 1 in this direction. So, this is j equals to 1, j equals to 2, so i 3 because here, this corresponds to y equals to 1. This y equals to 0. Therefore, from here, we get this. Now, let us write down the equations at each of the points. Which points? At each of the points, we write down.

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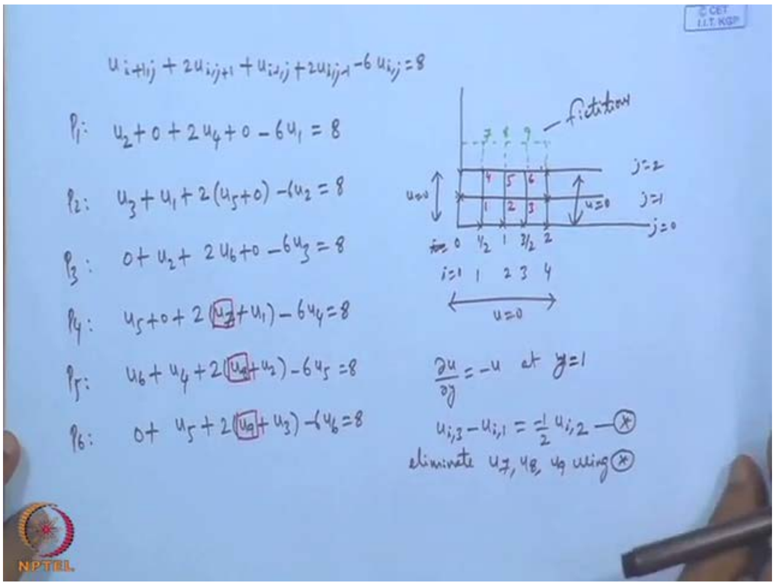

So, we need the grid again. So, this time, I am making a smaller one that is 1, 0, half, 1, 3 by 2 and 2. So, this is p 1, 2, 3, 4, 5, 6. So, j is 0, 1, j is 2 and i is 0, 1, 2, 3 and 4. Therefore, the equation we have to write this equation at p 1. So, p 1, this corresponds to i is 1 and j is 1. So, as I mentioned, we have to number them. So, we know these values. So, I am just marking. So, when we apply this at this point, we need this point, this point, this point and this point.

So, accordingly I am writing u 2 plus here it is u is 0 u is 0 u is 0, so u 2 plus u, this is 0, then 2 times u 4 plus this value is also 0 minus 6 u 1 equals 8. Then at p 2, u 3 plus u 1 plus 2 times u 5, of course this is 0 minus 6, u 2 is 8. Then p 3, so this is 0 u 2 plus 2 u 6 plus 0 minus 6 u 3 is 8. Then p 4, so p 4, if we use, remember what we get. So, we need to introduce fictitious grid. Since, we have numbered like this, let us number this 7, 8, 9, so this is fictitious. Now, at p 4, u 5 plus 0 plus 2 u 7 plus u 1 minus 6 u 4, then p 5 u 6 plus u 4 plus 2 times u 8 plus u 2 minus 6 u 5 and at point 6, so this is 0 plus u 5 plus two times u 9 plus u 3 minus 6 u 6 equal to 8.

So, remember we have these fictitious values. Now, these fictitious values, we have to eliminate using the derivative boundary condition, which we have. So, the derivative boundary condition, we have minus u at y equals to 1 for all x. So, from here, we discretized and we have obtained u i, 3 minus u i, 1. So, we have to eliminate from here.

So, because when i is 1, so we have this is u 7. So, let us put a remark. Eliminate u 7, u 8, u 9 using star. So, let us try this.

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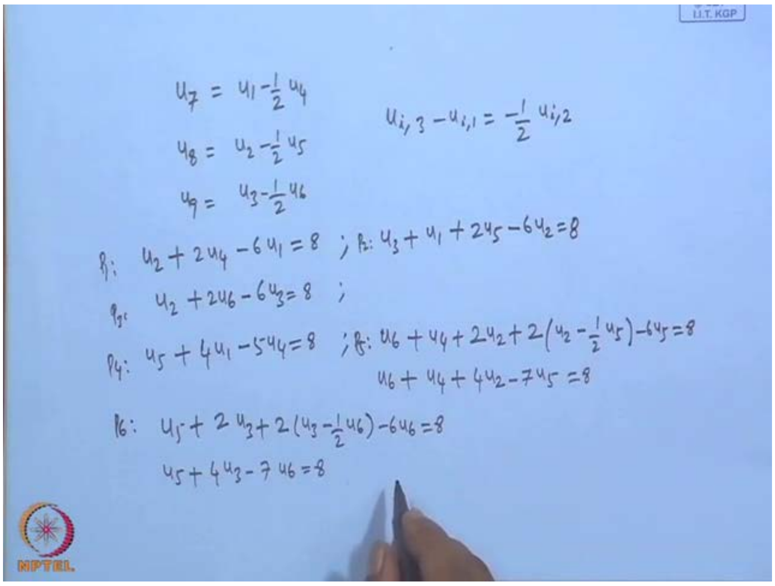

So, if we do that, then we have u 7 equals to u 1 minus half u 4 u 8 is u 2 minus half u 5 u 9. So, from where we have eliminated? We have eliminated from u i, 3 from here and remember from here. So, when for example, if i is 1, u 1, 3 because j is 3, so we are at the fictitious level already with this notation, we are already at fictitious level with 3. Therefore, when i is 1, it is 7 and this is 1, 1. So, that is u 1 and 1, 2. So, that is u 4, 1, 2 is u 4. So, similarly, when i is 2, 2, 3, so that will be 2, 3. So, that is u 8 and this will be u 2. So, similarly, when i is 3, 3, 3, it will be u 9, 3, 3 will u 9 and 3, 1 will be u 3 and 3, 2 will be u 6.

So, that is how we have got these fictitious values. Now, we have to use these fictitious values in this system. So, we eliminate. So, let us do that. If we do that, we get u 2 plus 2 u 4 minus 6 u 1 is 8 then u 3 plus u 1 plus 2 u 5. So, these are nothing but the earlier equations. I am sorry this is 8. Then we have to start eliminating from the equation p 4, so u 7 and u 7 is this, so we eliminate each one. So, then we simplify. So, when we do that, we get u 5 plus 4 u 1 minus 5 u 4 equals to 8. Then next one will be, so let me do one of them.

So, u 6 plus u 4 plus 2 u 8 2 u 2 plus 2 u 8 and u 8, we have and minus 6 u 5 equals to 8, so this gets simplified. So, this get simplified, u 6 plus u 4 plus 4 u 2 minus u 5, so minus

7 u 5 equals to 8. Then the third equation, so this is p 1 p 2 p 3 p 4 p 5 p 6 u 5 plus 2 u 3 plus 2 u 9 minus 6 u 6. This will be simplified 4 u 3, then minus 7 u 6 equals to 8. So, there is some amount of simplification.

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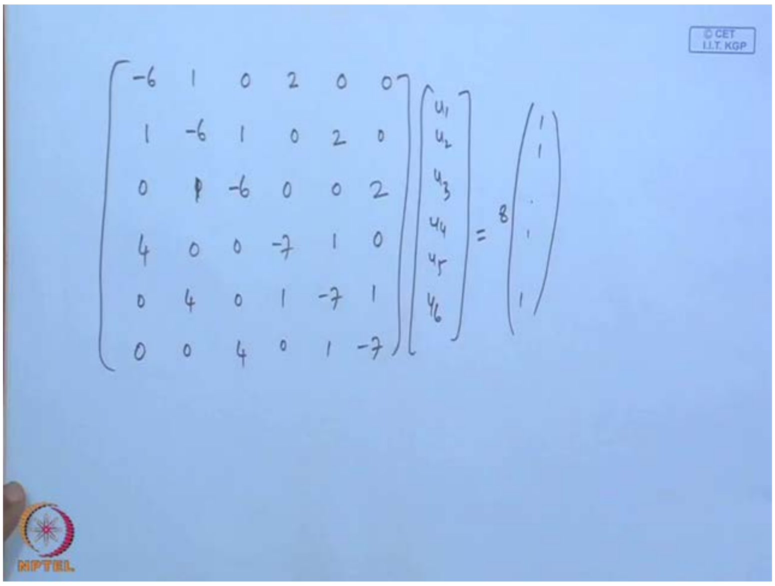

So, then we get system like this minus 6, 1, 0, 2, then 1, 0, 1. So, you may verify this because this is just routine work that I am putting what I have obtained. So, we have this. so this is a sparse system and one can solve and obtain. So, as you can see, this is not so much different from the derivative boundary condition case, which we have discussed in the parabolic PDE. So, all that we have done, we have eliminated the fictitious values by running the discretized equations and also by discretizing the corresponding boundary condition derivative boundary condition. So, now let us see in case of polar coordinates, how the corresponding discretization takes place.

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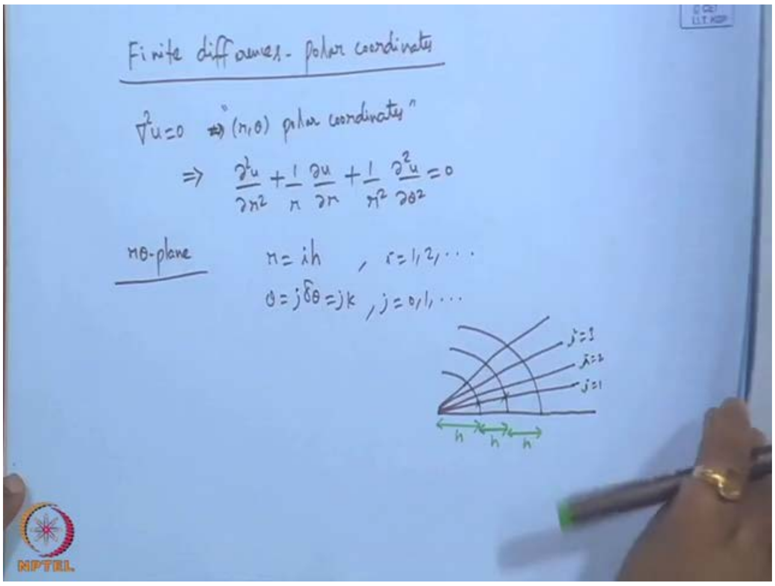

So, finite differences polar coordinates, so consider laplacian in r and theta polar coordinates. So, this if we introduce, so if we introduce, then we have this which means u is the dependent variable, r and theta are the independent variables. Now, definitely if we discretize partition in this coordinate system, it is not going to help. So, we need to discretize the corresponding r theta plane. So, if you consider r theta plane, r as i h and theta as j delta theta, so we can call j k; so where i is this and j is 0.

So, what would be the corresponding discretization? The corresponding discretization could be so r is discretized, 1, 2, 3. So, this corresponds to i 1. So, this corresponds to i 1, i 2, i 3 so on. Now, theta is also discretized. So, that means we have this. So, this will be this. This distance is corresponding to r so and this will be corresponding to theta. So, this is h, this h, this is h and whereas so this is delta theta, this is delta theta, this is delta theta. So, we can number the grid points. So, for example, this point is 1, 1, so whereas this point is 2, 1 because this is on r 2 and j 1. So, I will be 2 here. So, we can number them. So, let us discretize the corresponding equation.

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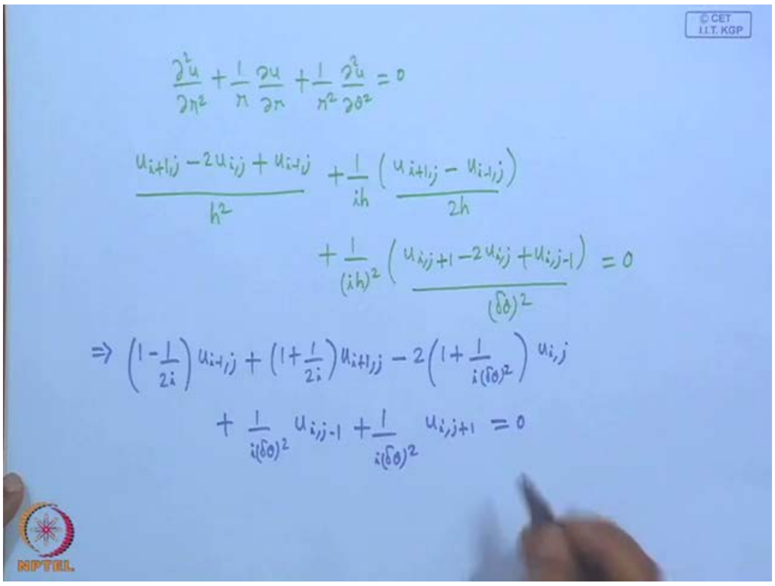

Now, if we discretize the corresponding equation, we get. So, the equation let us write down the equation first. Now, if we discretize plus 1 over i h because r, then whenever we have first derivative, we go for central. Then this is our equation. So, this can be simplified further. If we simplify, let us collect this is i plus 1 j and we have i plus 1 j and we have 2 h here.

So, if we take h square common 1 minus 1 by 2 i u i minus 1 j, so u i minus 1 j first, then 1 plus 1 by 2 i u i plus 1, j, then minus 2 times 1 plus 1 by, so h square is cancelled i delta theta square u i, j plus this j plus 1 and j minus 1 terms. So, this is our discretized equation. So, this can be, so this corresponds to at each grid point.

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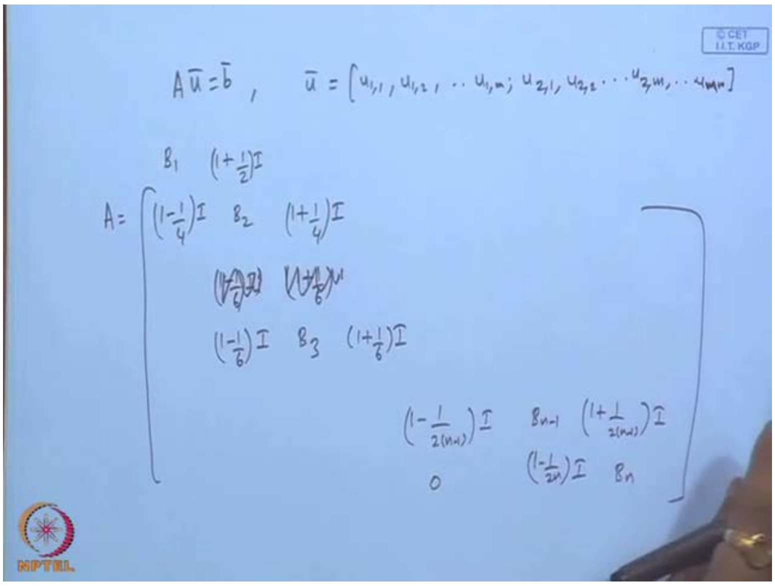

So, this can be expressed in the form of system where u bar is so on so forth. So, this is n m and the matrix A, this is just to know the structure. I am sorry this is to just know the structure. I am sorry. This is third row. I have to do it. So, this will be, so similarly, we get. So, the matrix A would have the structure.

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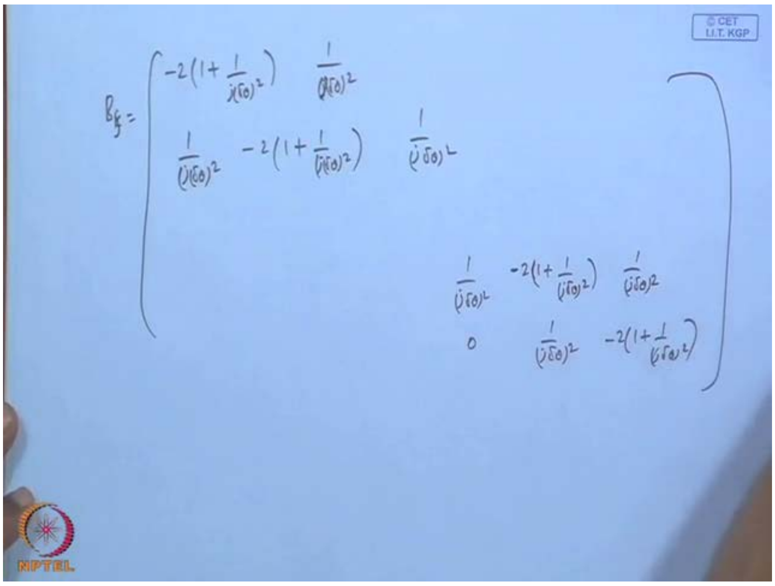

The matrix B, each B 1, B 3, so they also have a suitable structure 1 plus, so you can call Bj. This is for the entire j j delta theta square. So, we have a similar structure. So, this is coming because if you start writing at each of the grid points we get, for example, if i equals to 1 and j is 1, we get 1, 1, 2, 1, 3, 1, so these coefficients are sitting in the matrix B. So, they get multiplied with u 1 1, u 1 2, u 1 m, so it is like that. So, this is just for procedural sake. So, accordingly if we take a specific problem, then we have to use the corresponding discretization. However, when we are programming of course, you have to use these notations, otherwise like partition, you get a simple system and one can solve.

So, this is general scenario with polar coordinates. Of course, I have not solved any problem. So, may be if the time permits, we will see some specific problem involved in polar coordinates. Now, let us see so far what we have discussed. We have discretized nicely so that the domain is discretized in a regular grid and we have discussed the solutions at each grid point.

However, in these problems so far, the boundary has also been discretized and some of the boundary points are also grid points in case of derivative boundary conditions. In case of derivative conditions straight away, the entire boundary is known in that sense, they are not, and some of them are grid points. However, the entire boundary is known, but whereas, if you have derivative boundary conditions involved, so then even though we do not know the data, complete data, but we have obtained by discretizing the boundary, we have obtained the functional values at those grid points.

Now, suppose for a specific geometry, what would happen? May be the grid point is not coinciding with the boundary discretization. So, in that case that means we have discretized the equation. Then when we demand may be in the corresponding discretized formula, we are not getting the boundary points at all. So, then we have to take special care. So, you will realize what I am speaking when I put it properly.

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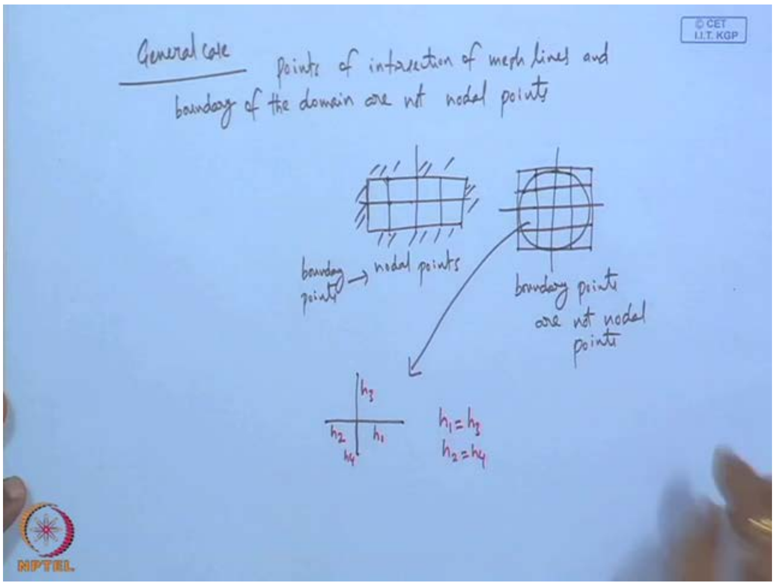

So, let us proceed. So, this is more general case. So, this is more general case, which is called points of intersection of mesh lines and boundary of the domain are not nodal points. So, what do we mean? For example, suppose I take this domain. So, then when I discretize, so this is the domain. When we discretize, you see this is a nodal point. So, when we run the discretized standard formula, it asks these two values and these two values. So, points of intersection of mesh lines and the boundary of domain are not nodal point. In this case, these are nodal points, which are nodal points? Boundary points are also nodal points.

Now, suppose you discretize, so then what would happen, you can see clearly. So, this is this is a boundary point, but this is not a nodal point because if we ask the discretized equation here, it expects these two values and these two values so this is not so this is the case where boundary points are not nodal points. So, we have to have special care. So, what is the care? If you see suppose I magnify this, so let us say, let us take this portion, then if you magnify, what kind of structure it is having? So, this is the structure you can see.

So, this is giving a skewness. So, these two are skewed compared to these two. However, these two are equal and these two are equal. So, that means, if we start naming the size, this grid size is h 1, h 2, h 3 and this is h 4. This is more general, but however in this case, h 1 equals to h 3 and h 2 equals to h 4 in this case. But, one can discuss more general scenario. So, with this, let us talk a more general Laplace equation.

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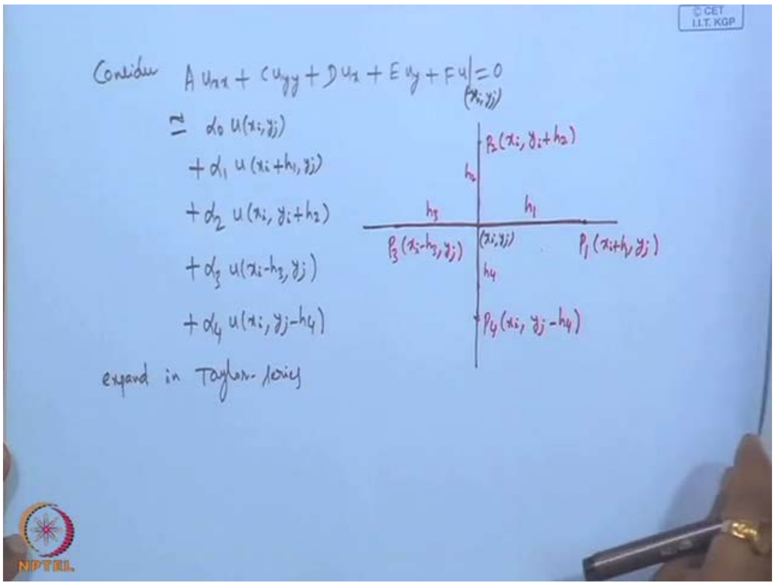

So, consider, suppose we want to, we would like to approximate this. Then say this is approximated by more general case. We are talking about more general case. So, identify point p 1, then point p 2, so this is h 1, then p 3, p 4, so means this is h 1, h 2, h 3, h 4 when I say these lenghts. Now, we want to discretize this equation at these grid points. So, accordingly we have this must be approximated by alpha 0. So, this is, so this stands, then alpha 1, so p 1, alpha 2 p 2, p 3. So, this at xi, yj we would like to approximate by this. Now, how do we determine the coefficients? Expand in Taylor series and collect the coefficients.

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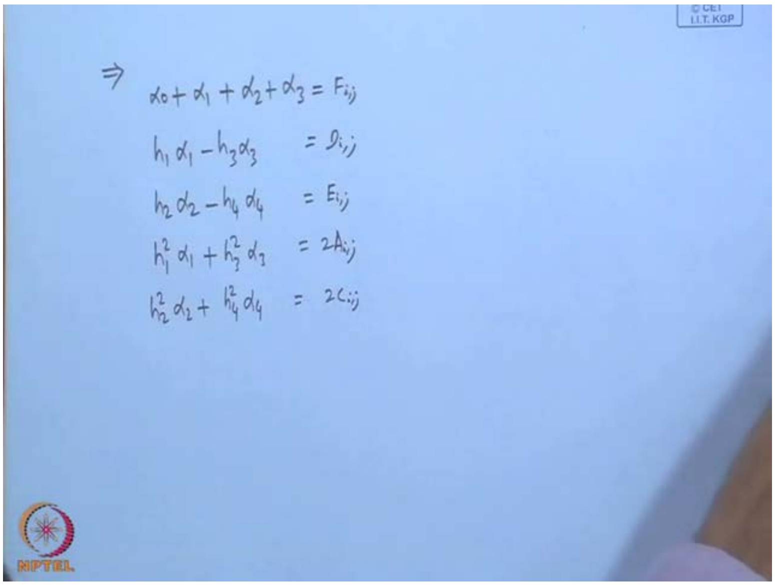

We get, by doing so we get this. This is more general case where h 1, h 2, h 3, h 4 are different. We get this system. How do we get it? As I explained, we just have to discretize this. So, we want to fit like this. Then you expand these in Taylor series and collect the coefficients of equal orders that mean coefficients of u xi, yj.

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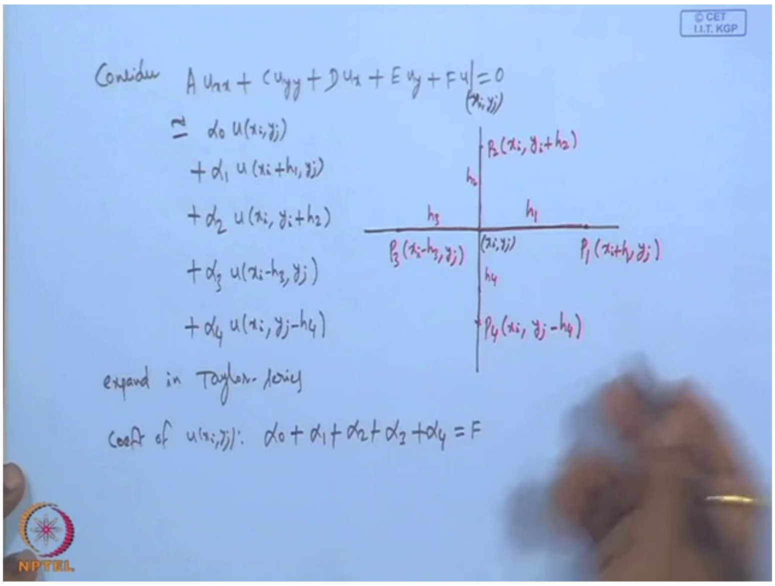

So, I will do coefficient of, so from here alpha 0, alpha 1, 4 and from here. So, this is so minus F equal to 0 otherwise. So, similarly, we do it for other terms.

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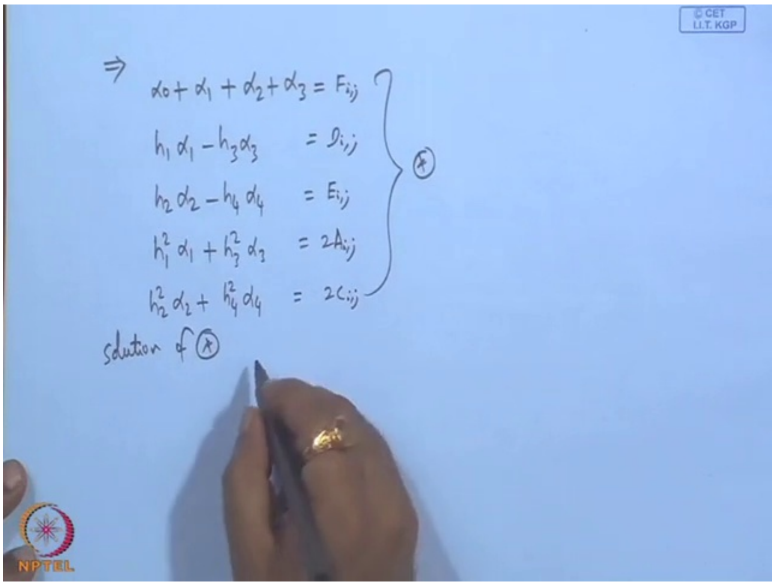

Now, solution of star, so solution of star is given below.

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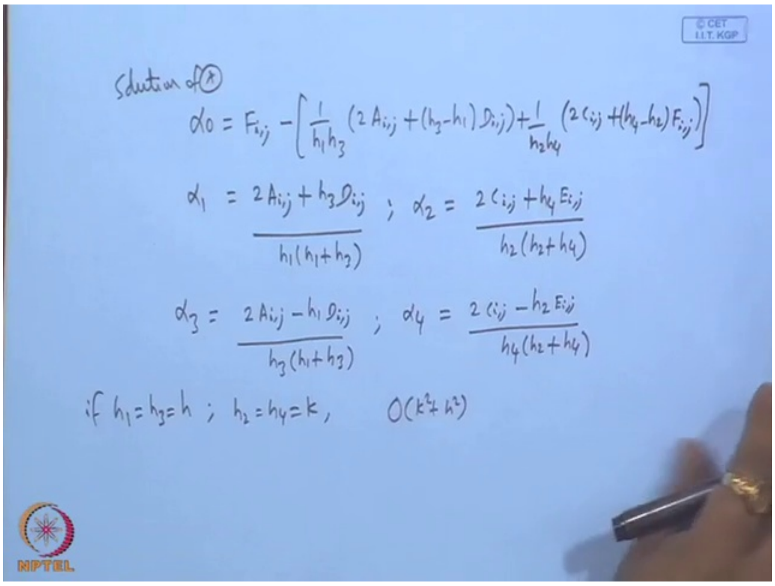

Solution of star alpha 0, so for completeness, I am writing. So, better you derive and then feel for it. This is alpha 0, quite big, alpha 1, alpha 2, and alpha 3. So, this is system. Hence, we have the corresponding finite difference approximation. However, if h 1 is h 3 is h, then h 2 is h 4 is k. so then the method becomes of order k square plus h square. So, this is one remark, otherwise we have general method. Now, let us see how this method can be applied to a typical situation.

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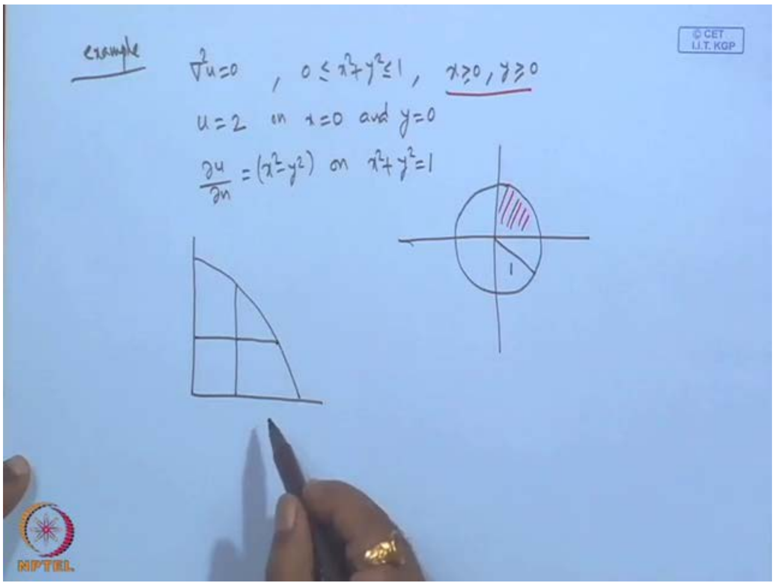

So, example, we would like to solve laplacian circle. However, the first quadrant u is 2 x is 0 and y is 0 dou u by dou n, which means of course, this is a circle with a radius 1. However, we would like to solve the problem only in this because of this restriction and u is 2 x equal to 0 y equal to 0 here. However, on this portion, this is your arm data on this. Now, as I explained earlier, if you discretize, we can discretize the entire one or only this portion say this is our portion, then if we discretize, if we discretize, then you have these points.

So, you can see this arm and this arm of equal size and this arm and this arm equal size. So, we have to obtain the solution. However, these points are not coinciding with grid points because I have to draw a better one. This gives an impression that these are equal, but in a circle, we can obviously see. So, how do we deal with this, this kind of thing? So, let us have a look.

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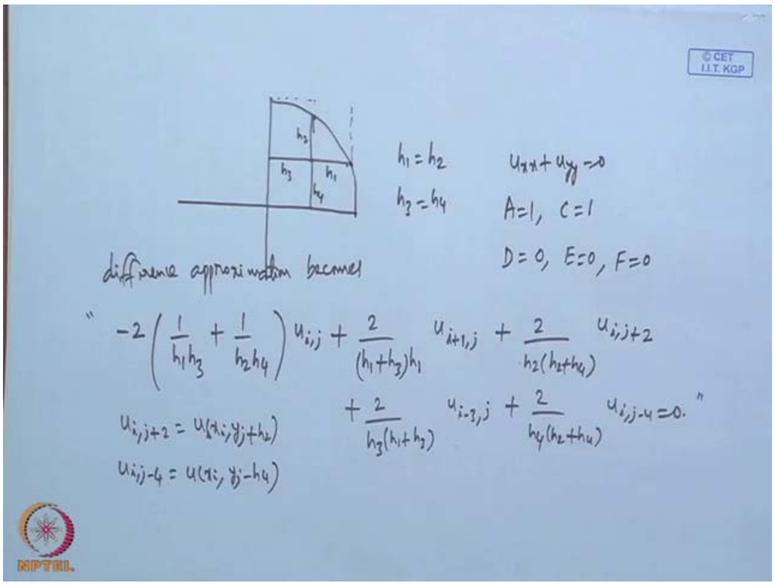

So, what we do? So, you can see this up to here, this would have been complete. So, the arms in this case h 1, h 2, h 3, h 4, so in this case, h 1 equals to h 2, h 3 equals to h 4. So, accordingly the equation difference approximation becomes for the Laplace case. If we compare Laplace case, which is u x x plus u y y equal to 0 with our original equation, A is 1. So, this suggests A is 1, C is 1 and D is 0, E is 0 and F is 0. So, with these things, we get minus 2, so u i plus 1, j plus 2 by j plus 2 plus 2 by, so when I say i plus 1, this is at h 1 j plus 2 means j plus h 2 and i minus h 3 j minus h 4 j plus 2 is u x i, y j plus h 2 u i, j minus 4.

So, the equation get simplified, but however, the challenge is when we approximate this at last these points, so we do not know because this is not a grid point, we have to pay special attention. So, when we solve the problem, we will see, otherwise this is a simple approximation, which on using h 1 equal to h 2 equal to h 4. Further, it gets simplified. So, this will, this appears definitely interesting. However, we have to pay special attention because these are not coinciding with grid points. So, we will attempt this when we are solving a specific problem. So, this is a scenario when your normal points and boundary points are not coinciding. So, when we solve specific problems, we will see how to handle, until then bye.