Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 32

Finite Difference Approximations to Elliptic PDEs – II

Hello, good morning. In the last lecture, we have discussed finite difference approximations to elliptic PDE in particular Laplace equation, of course I have shown it for a Poisson equation as well. Now, the main crux of the problem is the system ultimately reduces to a sparse system of equations, now we must focus on few concept on handling sparse systems. So, yesterday we discussed, in the last lecture we discussed efficient way of storing sparse system, now let us focus on the corresponding iterative methods. So, let us see how a general iterative method works out for solving A x equal to B, the kind of linear system of equations.

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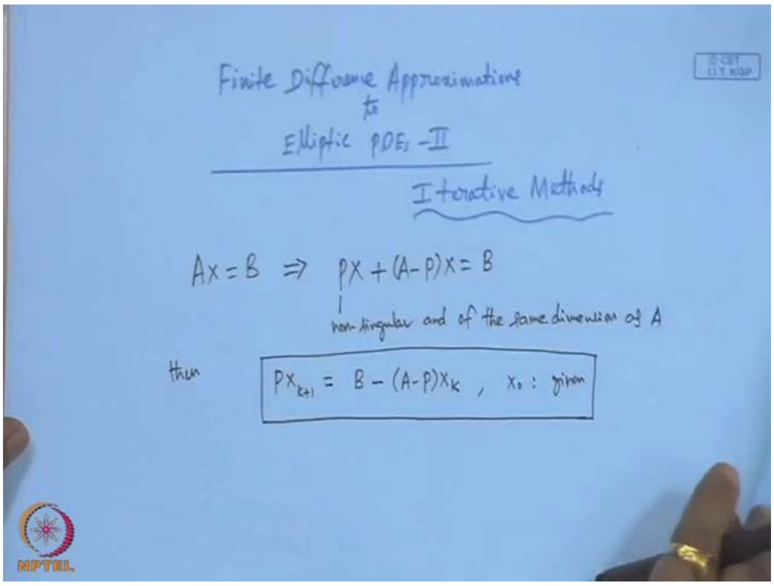

So, the system under consideration is this, and we are aiming at solving this iteratively, so we need to split something like this, where P is non singular and of the same dimension as A. So, if you can do this, then what would happen? One can define an iterative scheme and x not given, so this kind of an iterative scheme can be defined. So however the question is whether any kind of P would do or P should have some specific structure.

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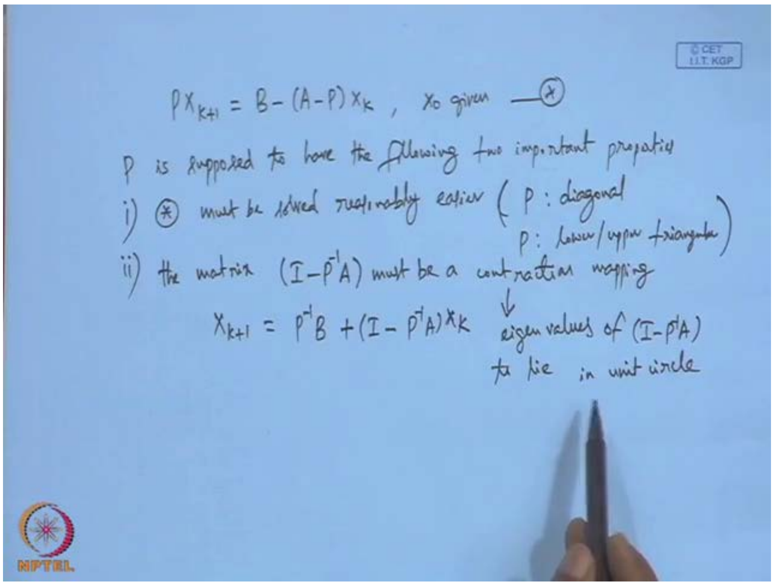

So, P demands the following properties, so if we define this iterative scheme and x zero given P is supposed to have the following two important properties. So, one say this is star must be solved reasonably easier, so in some sense this must be solved with little bit of less effort star must be solved reasonably easier and 2. So, this this kind of structure which supports this that means relatively easier, so possible structures are P is diagonal matrix or P is lower upper triangular.

So, if P has this structure, then one expect that this system is relatively solvable easier way then the matrix must be a contraction mapping such you can see. So, here when we get such a structure we ask for since P is nonsingular, then we can write it I minus there for this must be a contraction mapping so that the method converges. So, in other word, this means in other words Eigen values of this to lie in unit circle. So, I am not discussing the complete theory because you can see any book on algebra, so this is this must be contract mapping that means Eigen values of this to lie in the unit circle, so let us see couple of methods where it has this kind of structure.

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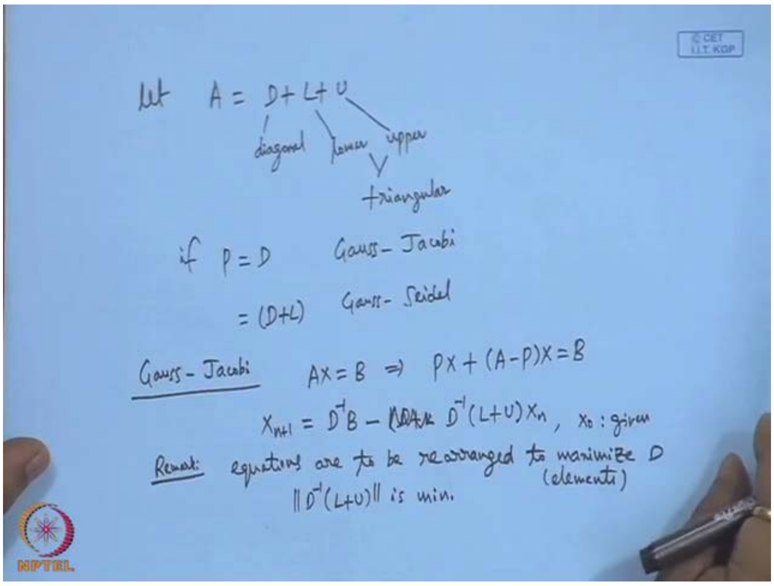

So, let A be so this is diagonal and this is lower upper triangular matrices, now if P equals to D, so then the method is called Gauss Jacobi P equals to D plus L, then the method is called so. We see in detail anyway, so if when P is D the system under consideration is A X equals to B and this we have split as and P, we want to be D, therefore the structure we would have is A minus P will be remaining. I am sorry this is this d, so A minus D will be L plus U, so D inverse D plus u of course X not given however see we need convergence, so this should not group. So, in order to have that one remark one remark is equations are to be rearranged to maximize D, so maximize D means when is say elements of d. So, if you do that, then this is minimum, so this will ensure faster convergence so this is the structure of Jacobi then Gauss Seidel.

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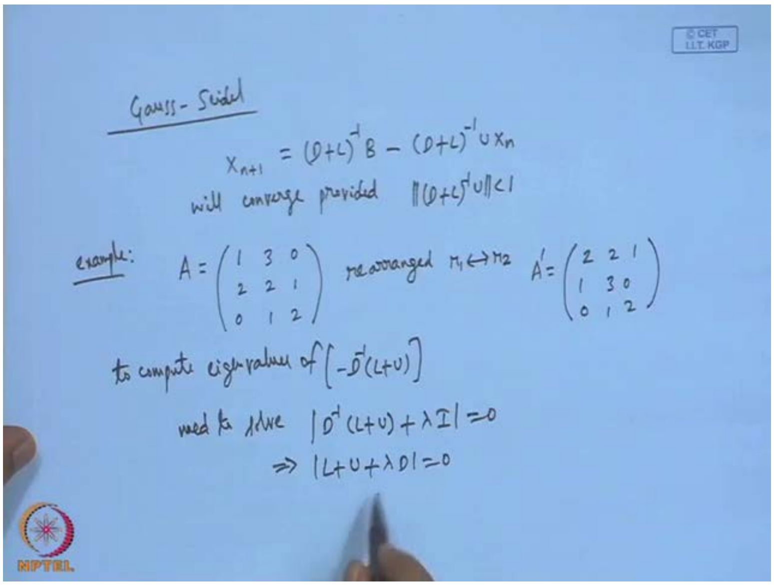

So, this is the case where P is d plus l, so this will converge provided the norm lesser than 1, so let us take some example, so let us take this A, then rearranged by exchanging row one and two, then say call a prime, this would be. So, you can see when we split this into D plus l plus u D contains the larger elements whereas here D contain 1, 2, 2 where as here it is 2, 3, 2, so this is what we need maximizing D. Now, to compute Eigen values of because this is for Jacobi, so we need this matrix to compute the Eigen values of this. We need to solve need to solve since D is nonsingular this amounts to, now for the present case.

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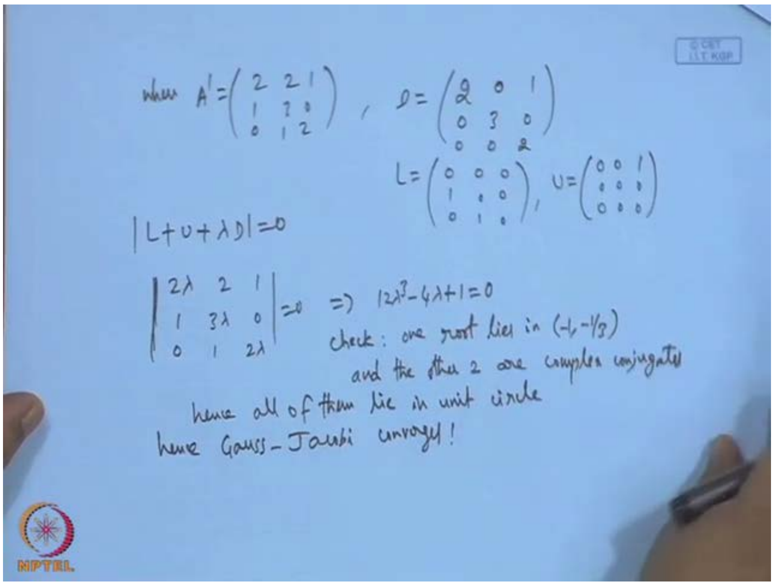

So, when A prime equals, we have D is and L U, now in this case if we solve this, now please check one root lies in this, you can check and the other two are complex conjugates hence all of them lie in unit circle. Hence, Gauss Jacobi converges, this is important aspect, so when you define any iterative method, the convergence is an important issue. So, in case of Gauss Jacobi, we have seen corresponding Eigen values of this particular matrix, so this particular combination there must lie within the unit circle, so there is another convergence criteria.

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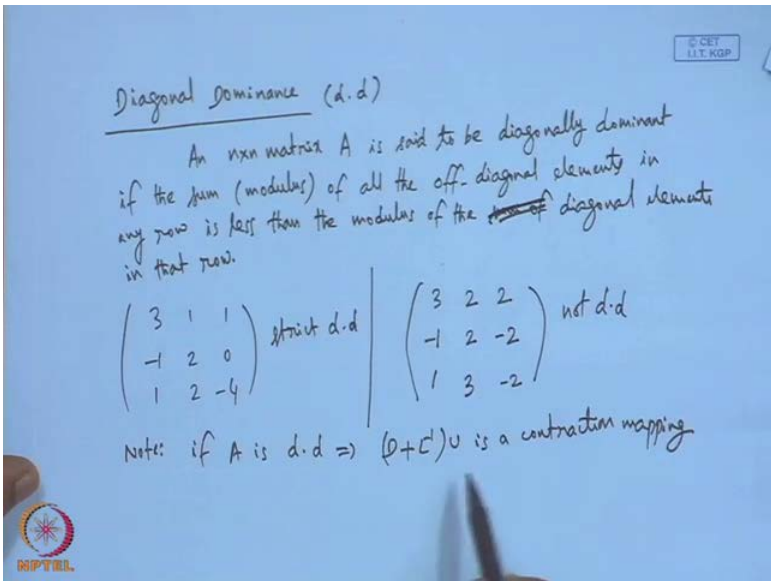

So, that is diagonal dominance, so this is n cross n matrix, A is said to be diagonally dominant if the sum when you say sum modules of all the off diagonal. So, all the off diagonals in any row obviously is less than the sum of the diagonal elements modules of the sum of diagonal elements in that row. So, one property for example, consider if you read this said to be diagonal dominant if sum of all the off diagonal elements in any row. So, if you take for example, sum of these two is less than the modules of the, sorry there is no sum here because modules of the diagonal elements in that row.

So, these two together is less than this diagonal, suppose in this row modules of this is 1 suppose there is 1 here so that will be 2, which is not less than. So, here the diagonal is 4 and then in this row 3, so it is less than 4, suppose there is 5, so we can we can get another matrix. For example, this is 2 is to 3 less than 3, 1 is to 2 less than 2, 2 is to 3 less than 4, so we can call it as in short strict D. Suppose, we can construct this, so construct the diagonal element is this and the sum 4, so which is not less than the diagonal element.

Similarly, the modules of the sum is 3 which is not less than, this is not so strict diagonal dominance means we have convergence so the property if A is diagonally dominant, this implies u is a contraction mapping. So, I mentioned I am not giving much theory because this is required for the iterative methods Gauss Jacobi and Gauss Seidel in particular this is required for Gauss Seidel. So, those who are interested in theory, there is supposed to refer any book on linear algebra, now once it is diagonally dominant this is contraction mapping, hence our gauss sidle method converges.

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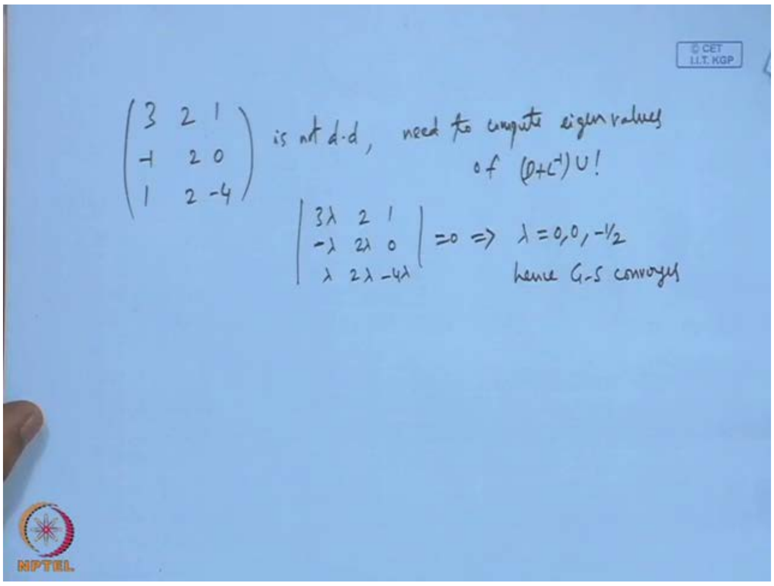

Suppose, you consider 3, it is less than or equal to in this case it is equality and this is diagonal element. So, 3 here, so this is not diagonally dominant because equality holds here, therefore we cannot use the other criterion. So, hence compute Eigen values of that, so all of them are within the unit circle, hence Gauss Seidel method converges. So, for the iterative methods both Gauss Jacobi and Gauss Seidel method, these convergence criteria are important, now let us see a more general Laplace equation.

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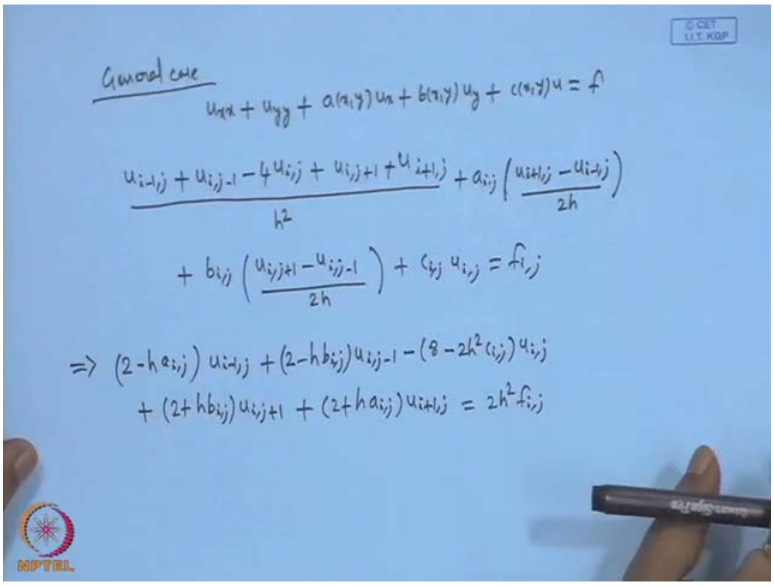

This is given with Dirichlet condition, then if we approximate for this, so i minus 1 j, so for this and this from here we get i minus 1 j, then we need i plus 1 j. Then from here i j minus 1 i j plus 1, then minus 2 i j minus 2 i j, so this we get plus from here a i j, then from here, then from here. Now, clubbing the coefficients we get what we have done, we have just club the coefficients u i minus 1 j coefficient because 2 h 2 goes there, then minus because h 1 h goes there. So, 2 minus h i j for this i j minus 1, we have i j minus 1, so similarly, 1, 2 from here and then from here minus h b i j.

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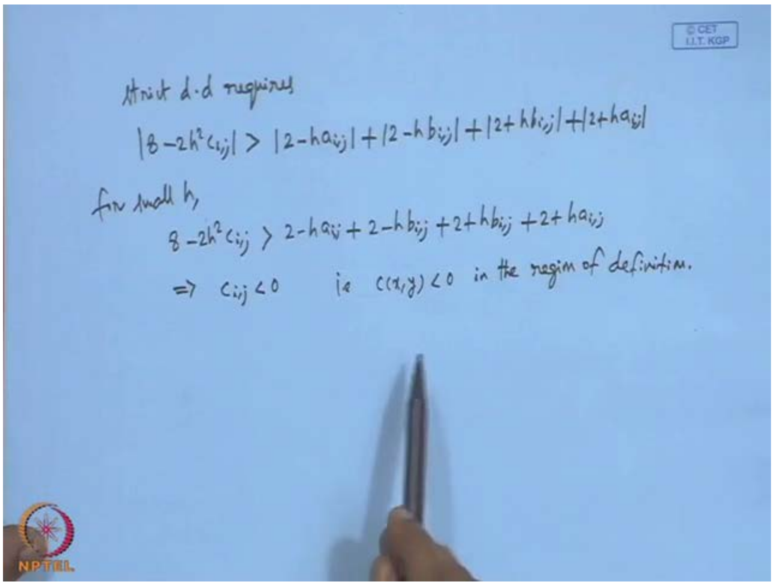

So, similarly for remaining terms now based on this if we consider the corresponding iterative scheme, if we consider the corresponding iterative scheme one can show that strict diagonal dominance requires because this is the diagonal term coefficient of u i j. So, for strict this has to be greater than sum of the modules of this numbers in that row, now general case will be slightly difficult, but for small h assuming this is not shooting up.

So, the condition is 8 minus 2 h square C i j greater than, so this implies C i j is less than 0 so which means C of x y is less than zero in the region. So, that means if you consider such a general elliptic equation and if you want to define a corresponding iterative scheme. So, then it converges provided C satisfies this condition, so this is definitely a useful inference, now having defined the iterative schemes, let us work out couple of examples where the solution is based on one of those iterative schemes.

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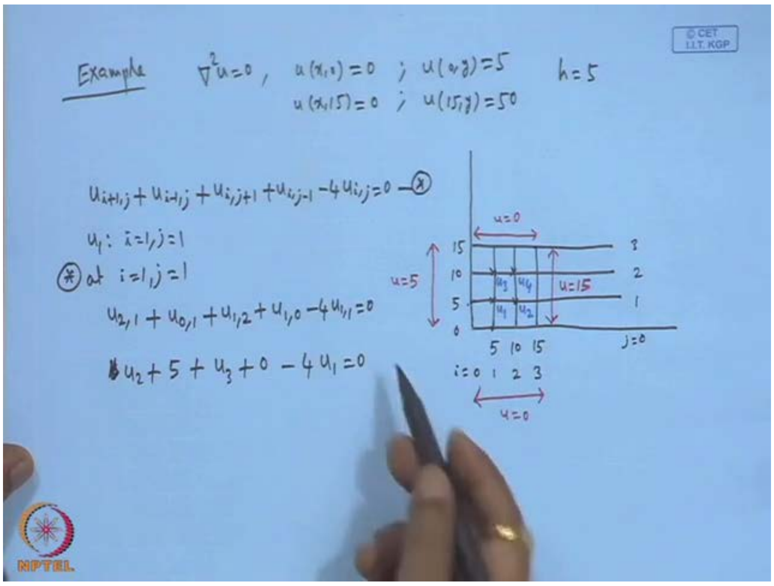

So, let us proceed for the example, so we want to solve Laplacean and say h is 5, so now the grid say 0, so correspondingly now i 0, 1, 2, 3 and j 0, 1, 2, 3. So, now the boundary u of x 0 0, so then u of x 15 is 0, so y equal to 0 and y equal to 15 then x equals to 0 and here now what are the unknowns? Unknowns u 1, u 2, u 3, u 4, so these are the unknowns, now consider the discretized equations, so this is the discretized equation. Now, if we run this equation at this point, it expects this this this and this, so instead of fighting with two indices, we have discussed in the last class one method of renaming method of enumeration. So, according to that this one will be u 1, u 2, u 3, u 4, and now if we see for example, corresponds to u 1, corresponds to i equal to 1, j equal to 1. So, I will show in one case with two indices, so then we switch over, so for example, i equal to 1, j equal to 1, so we have stat at i equal to 1, j equal to 1 will be u 2 1, u 0 1, so according to our notation u 1 1 is u 1.

So, then 1, 0, 1, 0 will be this, so it asked for this point this point and this point which are nothing but this 4, so when I switch over to the other notation simply it will ask for you see the value here. So, the value here is on this boundary, it is 5, so 5 consistent with this u 2 2, first let me write down u 2 1 is nothing but u 0, 1, 0, 1 is this 1, 1 that is u 3 and 1, 0 that is this number, so which is 0 minus 4 u 1. So, what I am trying to do is if you run here, we need u 2 plus the value here which is 5 plus u 3 plus the value here which is 0 minus 4, 1, so we do this way.

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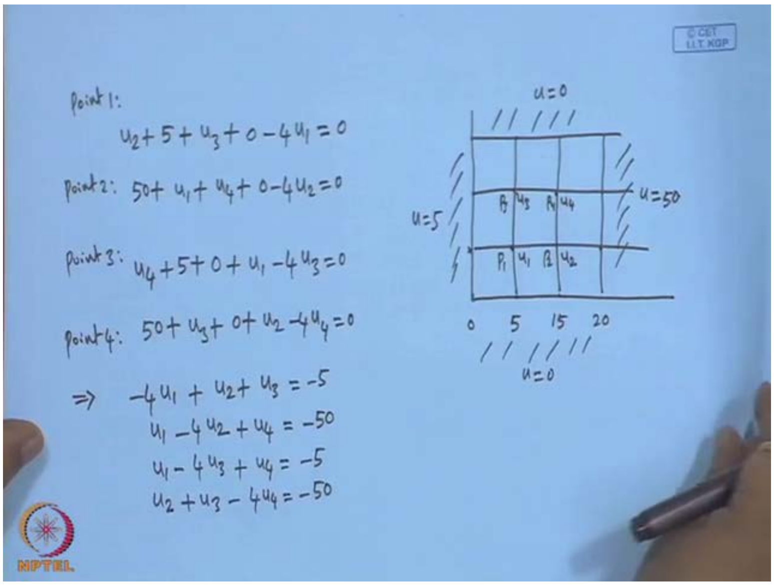

So, slightly narrow the earlier one, so let me write down in this case 0, 5, 15, 20, now this is u 1, u 2 u 3, u 4. So, the boundary here u equals to 0, so here u equals to 50 and here u equals to 0 and here u equals to 5, now the given equation we done at u 1, so point one. So, this is 0.1 P 1, P 2, P 3, P 4, so at point one if you run the equation we get u 2 plus this value 5 plus u 3 plus this value which is 0 minus 4 u 1 equals to 0. Next, we run the equation at point two, so when we run the equation here we get this value plus this value is 50 and this is u 1, then u 4 value here is 0 minus 4 u 2.

Now, at point three, so instead of what I am trying to do instead of following from this, we know and we use at a particular point it needs four neighbors. So, I am just working out based on that now at point three, if you run it ask for this value plus this value. So, this is u 4 plus this value is 5, then the top value is 0, then u 1 minus 4 u 3, similarly point four. So, this value is 50 and this is u 3 the top value is 0 and bottom is u 2 minus 4 u 4 equal to 0, so this reduces to minus 4 u 1 plus u 2 plus u 3 minus 5. Then second equation u 1 minus 4 u 2 plus 4 minus 50, third equation u 1, u 1 minus 4 3 plus u 4, then fourth equation u 2 plus u 3, so we get this, now we need this form here.

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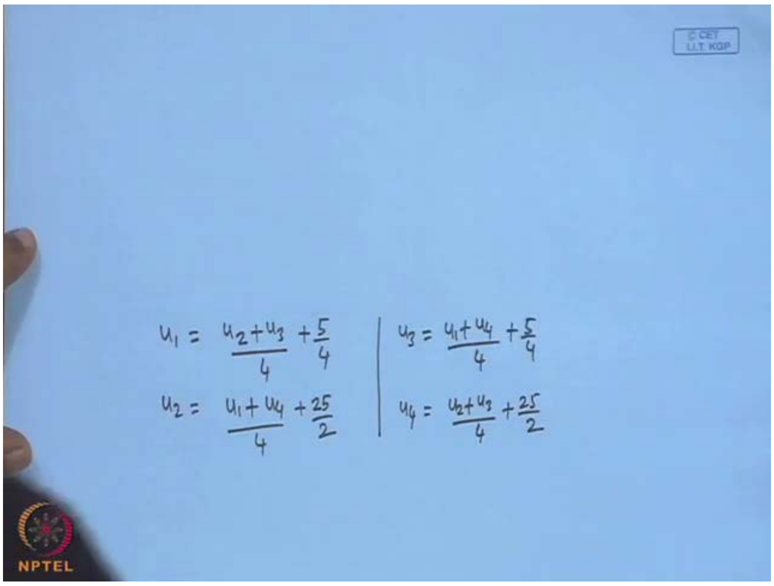

We obtain defined iterative scheme as follows, so u 1 is you may realize when we put u 4, so that is different u 1 is done u 2, u 2 is plus 25 by 2. Similarly, this is iterative scheme, so we are ready to consider either gauss Jacobi or Gauss Siedel, so let us start the computation, now we need any initial guess.

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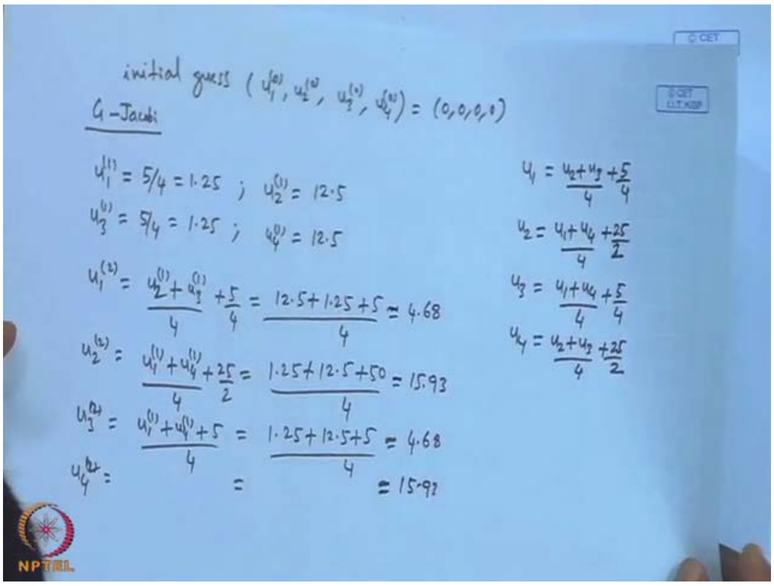

So, let me write down say we start from zero as initial guess so then we have u on equals u 2 plus u 3 by 4 and u 2 is now let us Gauss Jacobi, now in this we define, so are ready to compute. So, u 1 1 equals so u 2 0, u 3 0, therefore 5 by 4, then u 2 1 these are 0, 25 by 2, then u 3 we get this then Gauss Jacobi as you know Gauss Jacobi. Once we get at next level, we use them to compute further, so this will be u 2 1 plus u 3 1 by 4 plus 5 by 4. So, this will be u 2 1, u 3 1 plus 5, 4, so this comes out to be 18.75 by 4, so you can compute roughly, so then u 2 2, so u 1 1, u 4 1 by 4, so this will be u 1 1, u 4 1 plus 50 whole by 4.

So, this is 50 by 4, so this will be 63.75 by 4, so this is then u 3 1, so u 1 u 4 the same, so this would be this would be u 1 u 4 1. Let me check whether I am making mistake because u 1 is u 2 u 1 is u 2 plus u 3, so I have used correct ones, 12.5 and 1.25 and 5. So, this is fine because this turns out to be 12, 17, 18, 18, then u 1 u 4, so u 1, u 4, so this is 13.75 and 50, so 63, so that is this, then u 3 is u 1 plus u 4.

So, u 1 plus u 4, so 13.75 plus 18, so it is same as 4.68, then u 4 1, so this is u 2, u 3, so I am not because you can compute u 2, u 3, so these are 12.5 plus 13.75 plus 50, so 13 plus 50. So, this will be so this is up to two iterations, so these are up to two iteration, then let us try one more iteration, then we compare with Gauss Siedel or we can try Gauss Siedel straight away.

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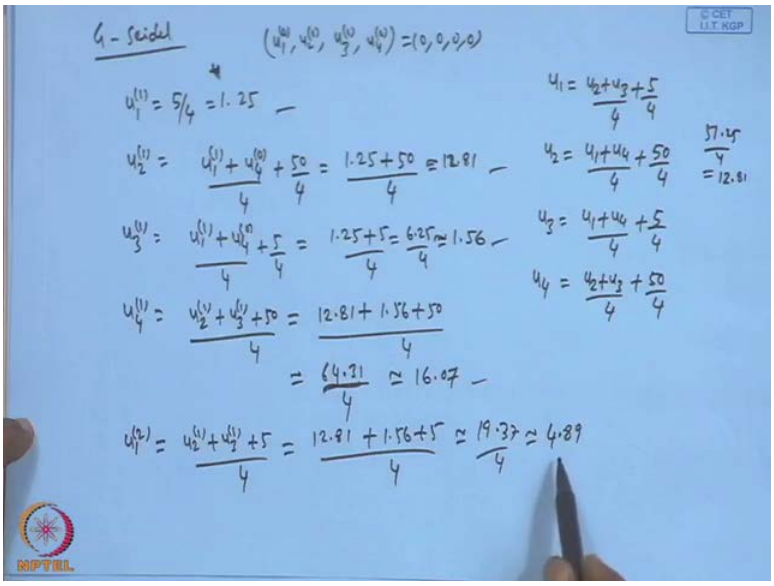

So, then now we have these values u 1 is u 2 plus u 3 by 4, u 2, so better we write 50 by 4, so with 0, 0, 0, so when these are 0's, we get u 1 is 5 by 4, so which is 1.25. Now, as you know having obtained u 1 at next level we must use that to compute u 2, therefore u 2 1 would be u 1 1, u 4 0 and we have u 1. So, that is how we have computed this then this will be, so this is 0, so this t by 4.

So, this this is 51, so 51.25 by 4, so it is roughly 12.81, then u 3 one you can see we have computed. So, this also latest value we have to take, so this would be roughly this, then u 4 1, so we have computed u 2 1 and u 3 1. So, we must use them so u 2 1 is 12.81 u 3 1, so this is roughly 51, 63, 64.31 by 4, so this is so if you look at it one point, initial one point 2, 5, 1, 0.2, 5, 12.5.

Later, now here second level, we got first level itself 1.25, 1.56, 12, 16, so next iteration we have to use these values and compute. So, you can see for example, I do one value, so u 1 2 see u 2 1 plus u 3 1 plus 5 by 4, so u 2 1 is 12.1, so u 3, 1 12.5, 6 plus 5, if there is a mistake kindly correct it. It dictates the method, so this is 6, 18, so this is roughly 19, so roughly this much so you can see which is Gauss Jacobi is 4.68, this is 4.89, so that means calculation is correct. So, its closely, however you can see we are updating the latest values in u 2 you would realize the benefit of that, so this is supposed to converge faster.

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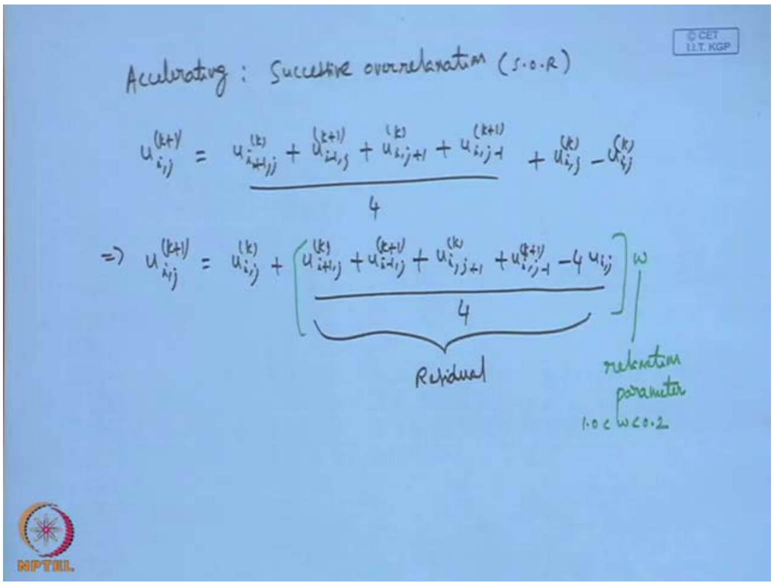

So, then having discussed Gauss Jacobi and Gauss Seidel, that is windup this session with successive over relaxation. So, this is popularly known as SOR, so convergence can be improved, so we do we add and subtract u i j k for this. So, then readjust hope you are able to recall why we are putting k plus 1 because this is we are computing i past values we should use the latest. So, this is nothing but residual, so residual is a kind of adjustment and further more the general method is defined as we add relaxation parameter.

So, this is in general for optimal convergence it lies this lies within this, so this is little faster compared to other iterative methods. So, as you can see the numerator goes to 0 as method converges because this is nothing but the desecration of the Laplace equation. So, these are the iterative methods because it is worth spending on iterative methods. The reason is straight forward as I mentioned for Laplace case because of the second order approximations for the second derivatives the neighbors are diagonal.

So, you have one above and one below one left, one right, therefore what would happen, the tri diagonal structure is lost. Hence, the system is sparse, therefore you need definitely you need better solvers for these sparse systems and hence instead of simple Gaussian elimination, we propose Gauss Jacobi, Gauss Siedel or successive or relaxation. So, in the summing classes, we will see may some examples where derivative boundary conditions are also involved until then.

Thank you.