Numerical Solution of Ordinary and Partial Differential Equations
Prof. G.P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 30
Fourier Series Stability Analysis of Finite Difference Scheme

Hello, welcome back. In last lecture we had discuss about stability analysis, of course with respect to matrix stability analysis, also worked out couple of examples. So, let us learn to day Fourier series stability analysis. So, as world suggest Fourier series stability analysis. The main idea in this is the initial error expanded in terms of in Fourier series and then look for condition restricting the amplitude in the amplitude. So, let us discuss Fourier series stability analysis.

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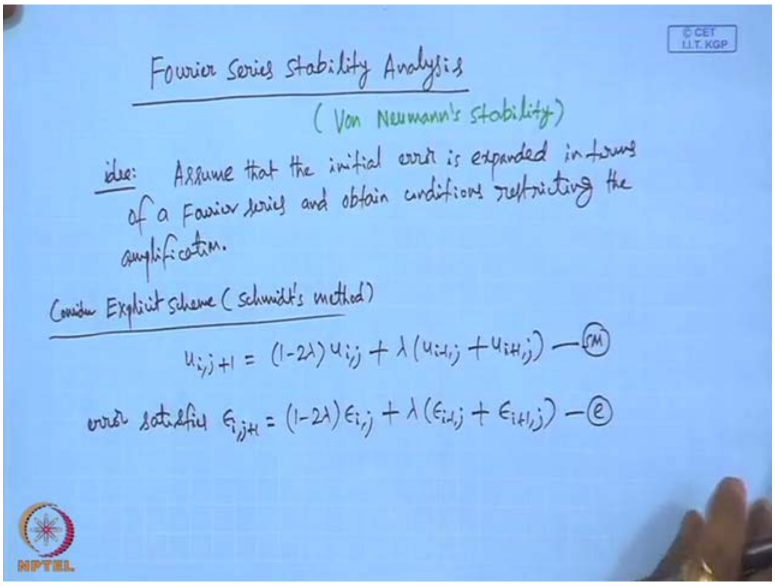

So, the title says Fourier series stability analysis, also this has another name van Neumann's stability analysis. So, what is idea as I mention, assume that initial error is expended in terms of Fourier series and obtain conditions restricting the amplification, this is the main idea. So, let us even instead of discussing more general case, we can consider couple of example and then learn the process. So, consider explicit scheme which is Schmidt's method, if we consider this u i j plus 1 so Schmidt method. Now, we would like to Introduce the error, if you Introduce error satisfies epsilon i j plus 1 epsilon i j. So, this is error, now as I suggested here, we have to assume the initial error in terms of Fourier series.

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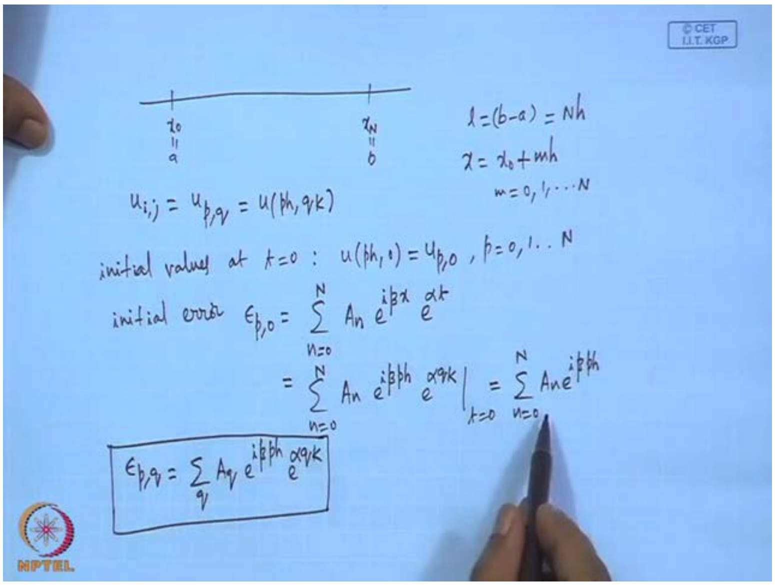

So, in order to do that let us say couple of thinks this is our x naught a, then x N is d assume that one is b minus a and x naught plus m h where m is 0, 1 to N. So, accordingly this will be N h. Now, for convenient purpose, because we have to expand Fourier series, and there we are going to introduce notation i which is due to the complex variable notation. So, not to confuse with normal indices, we introduce this notation. So, this is our usual practice so for we have to following, but this may be introduce like this, where this at any point which is p h q k.

So, this is notation now we are going to introduce, further initial values t equals to 0 u of p h 0, u of p 0. So, Instead of i we have introduce p instead of j we have introduce q. Now, initial error this we expand it as follows, initial error a n i beta i beta at any points this is x, then e power alpha t. So, this is a Fourier series expansion, however in this case i beta p h, but at any x i or t j. So, x p and p q, this will be e for alpha q k. So, this our initial error. So, this when we have expanded, this should be restricted a t equals to 0. So, this is general expansion therefore, ph. So, which means the general notation is this is what for summation q, a q e power i beta p h e alpha q k. So, this is more general, where as this is fir t equal to 0 initial expansion.

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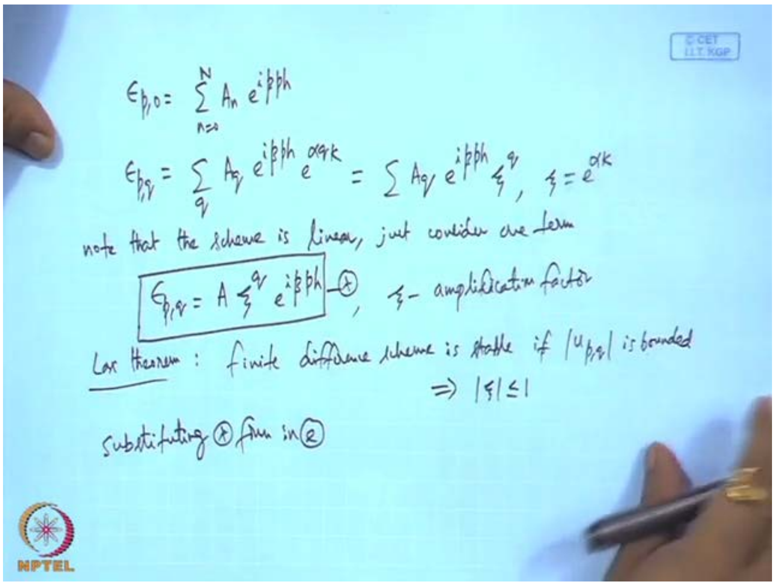

Now, this notation, so we introduce of the simplified notation zeta power q, where zeta is e power alpha k. So, what is need of introducing these variables. So, the error at time t equal to 0 is plug-in to the system and we are observing at subservient time starts the amplification of the error. Hence they variable which are going to observe should be related to time. So, this is variable, now note that the scheme is linear, so we just condition one term as follows and we call zeta amplification factor. Now, Lass theorem which says, finite difference scheme is stable if mode u p q is bounded. So, accordingly this implies when this is bounded only when corresponding conditional amplification factor. So, let us substitute this in our substituting star form in e.

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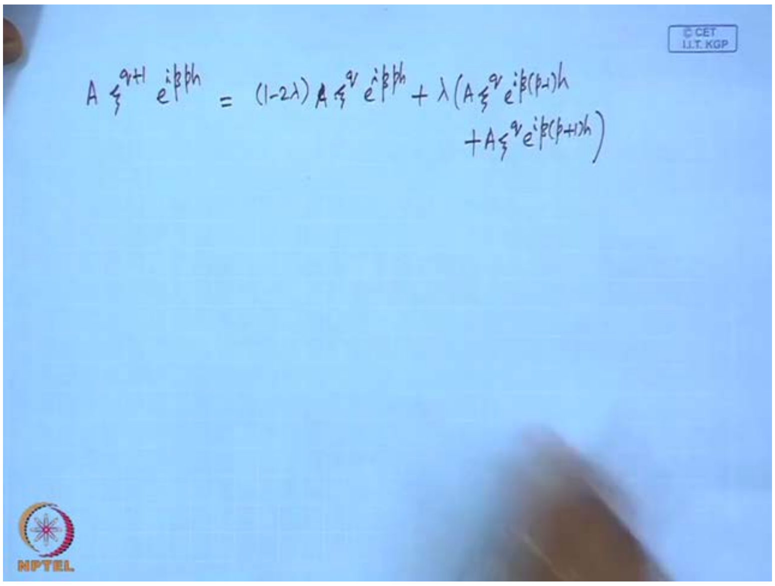

So, what was our e equation satisfy by the error difference equation satisfied by the error. So, if do this what you get we get as follows, A I will show you why this is, A this power q plus lambda A h plus. So, please look at this.

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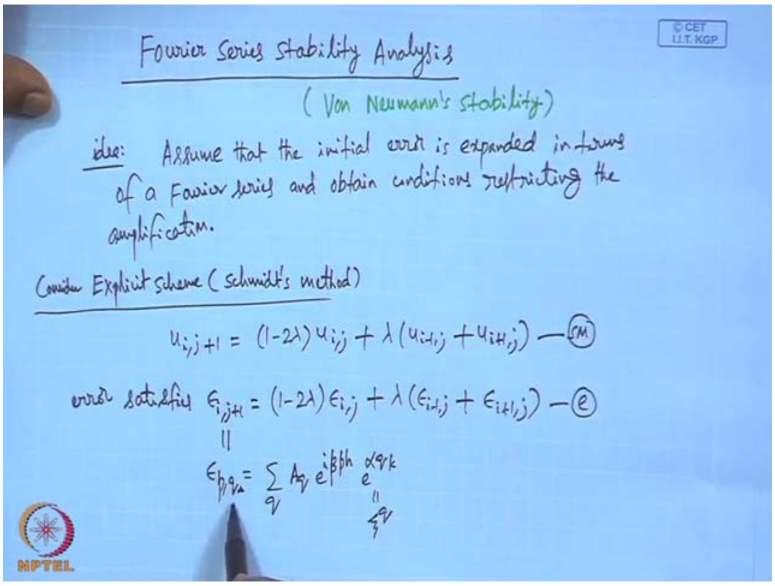

We have this is nothing but epsilon p q and this is expanded as A q and e power alpha q k. This is nothing but p index is for h, q is for k, now we have here q plus 1. So, in some sense we need this. So, I have transferred i to p and this of course we have to adjust. So, this is generic notation, but we need it for q plus 1 here, q here and q here. So, accordingly we can see q plus 1 on the right hand side and q and left hand side. Similarly, i minus 1 corresponds to t minus 1, i plus 1 and i corresponds to p. So, because we have this index earlier and now this is complex notation so I change it.

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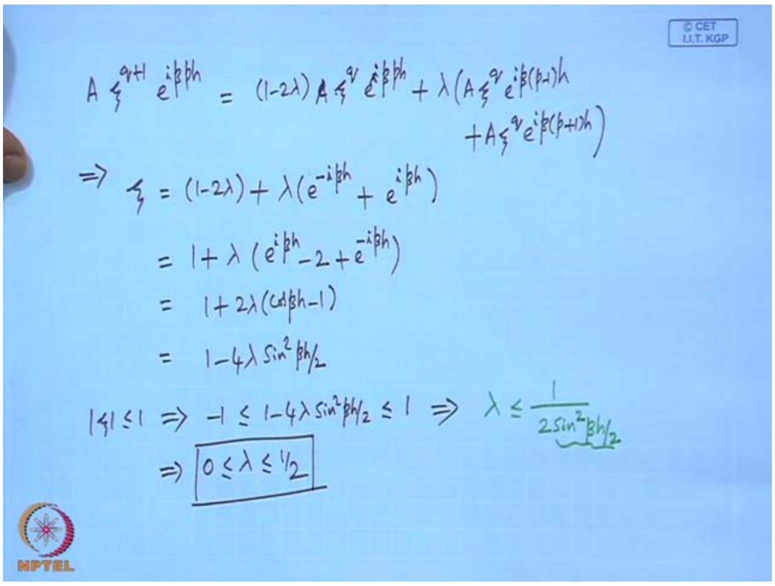

Now, some terms cancel so after cancelling these terms for example one term. So, then we get this entire get cancel 1 zeta left and here of course we have some extra term there is minus, minus i delta h and here we get plus. This can be simplified 1 plus lambda minus two, so I have taken this term inside. So, then this is 1 plus 2 lambda, just an adjustment. Now, for stability we need, this implies, we get lambda is less than or equal to 1over sin square beta by 2 in the since less than or equal to 1.This point is we get this implies o less than lambda less than 1 by 2.

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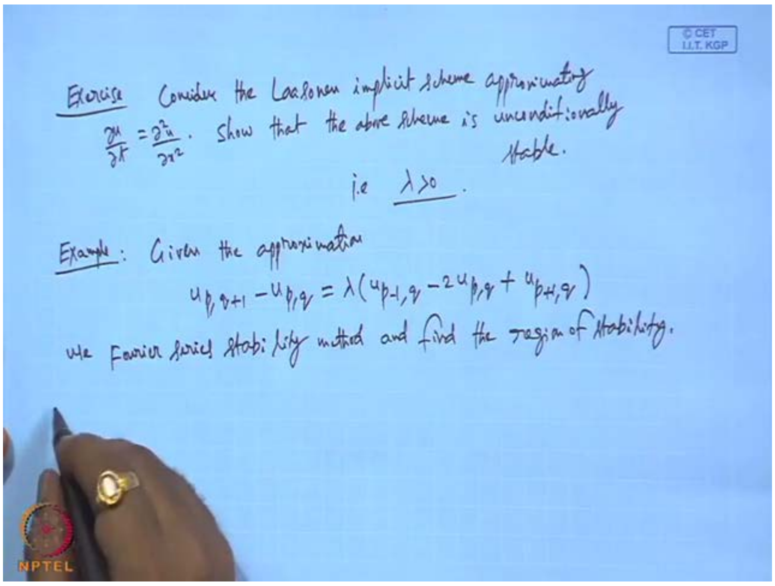

So, this is region of stability. Now, exercise consider the Lausanne implicit scheme approximately show that the above scheme is unconditionally stable, unconditionally stable. So, that means for all lambda, we get stability. The technique is also very interesting compare to matrix stability analysis because the initial error has expanded in terms of obesity and then plug in to the different equation satisfy by the error. Then restricting the amplification we find out the region stability. Let us work on several other scheme so that we get it better. So, example given the approximation, use Fourier series stability method and find the region of stability.

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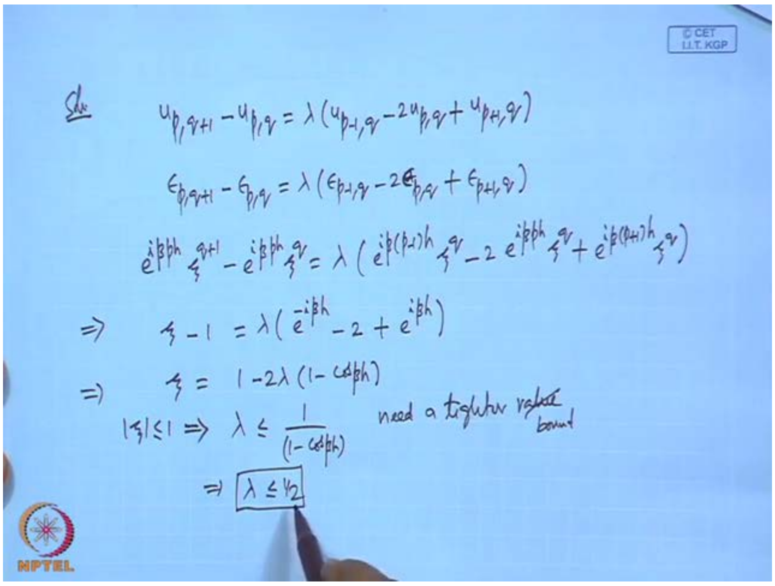

Suppose, this is given then we would like to solve this solution. So, the equation is final difference equation, scheme is linear. Now, accordingly the equation satisfy by the error we have to follow index notation carefully so this is epsilon, considering expansion error this will be. So, I am dropping the constant p h and this will be, so this error equation then after some cancelation we get this and farther simplifying.

So, this will be 1 minus 2 lambda, then imposing restriction on the amplitude on amplification this will be lambda less than or equal to 1 by 1 minus. Now, we need tighter value that means tighter bound. So, correspondingly we may get two in the denominator therefore, lambda is less than or equal to half. So, this the corresponding stability region. So, let us see some of problems.

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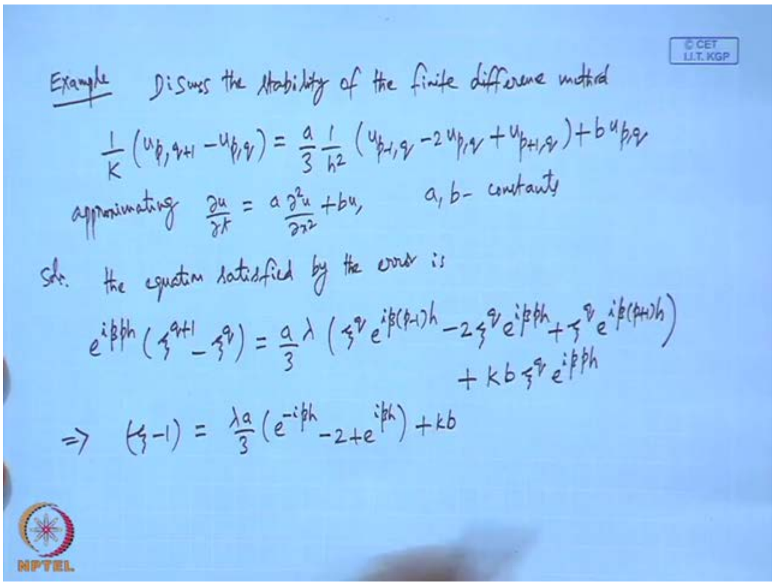

This is general, so general equated is any general question the still the processor reminder same. So, hence I prefer to discuss lot of example. So, discuss the stability of the finite difference method approximating here a and b constants. So, again we do the same process, if do the similar process we get the equation satisfied by the error is. So, it pull out because these two are having index p with respect to x.

So, I have pulled out here this as index q plus 1, this q then we transfer k by h square that lamda here, then corresponding to p minus 1 this is index corresponding to q this is the index, plus we have taken k right hand side therefore, here we have k b. So, this on simplification because zeta for q get cancel here lambda a by 3 we get this.

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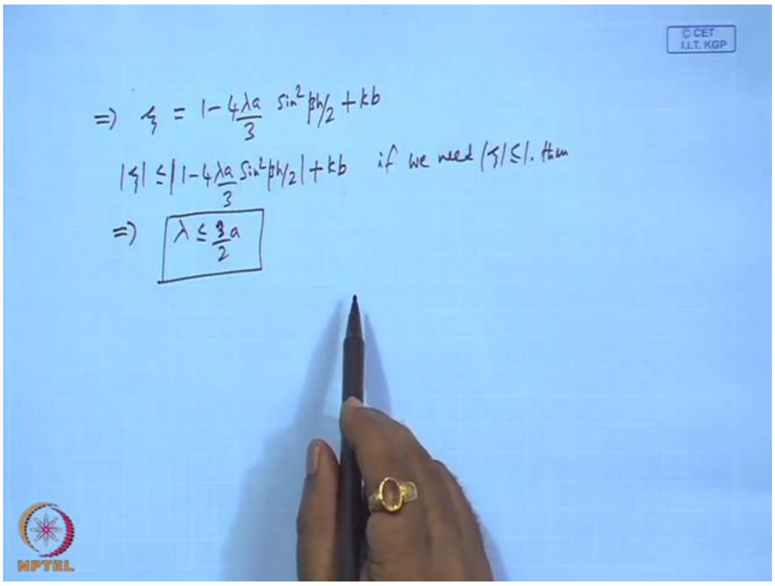

So, can this can be simplified, k b and if you restrict the amplification this will be this. So, k is pose to b is pose to on stem. So, impales lambda less than or equal to. So, if we need then lambda is 3 by 2. So, this is corresponding reason of stability, this gives more or less some idea about both matrix stability and Fourier stability analysis. So, before we go to the next topic, let us spend some time on few example, where matrix stability is consider. So, accordingly my next example where we have matrix stability is considered.

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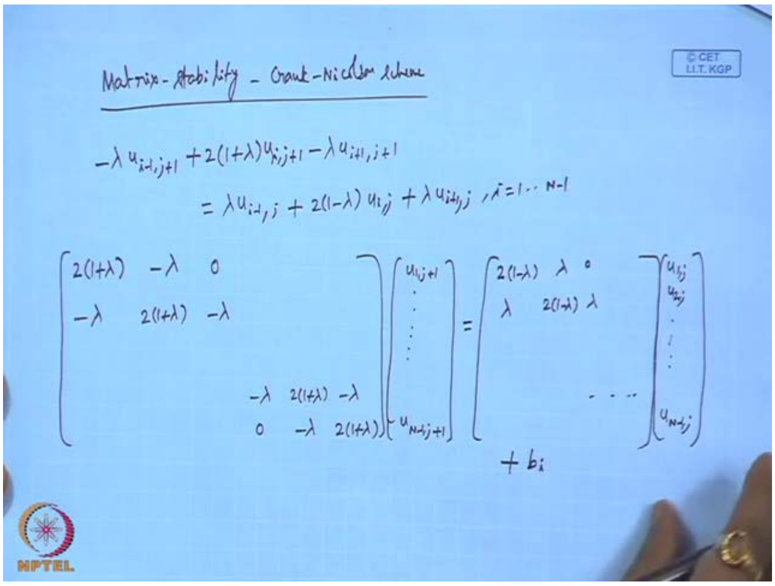

So, matrix stability grant Nicolson scheme we have discuss, but we discussed in different way. So, consider this e, now we have discuss the stability of this method via error equation. In the since we have obtain the corresponding difference equation satisfy by the error. Then we obtain a estimate, but in this case let us go through formal matrix stability analysis. Now, this can be rewritten as when i equals to 1, this become boundary town which is known. So, push to the right.

So, the first non zero term is, this next one is this accordingly. These are the unknown, this is equal to you can see here u 1 j, u 2 j, u n minus 1 j, plus b i. So, this contains boundary values zeroes. Then this can be expressed as in recall our earlier notation. I will explain. So, what is don this is split up into 2 matrix 1 is 2 times indent matrix because where you have one, one, then the other one is lambda this suppose to be this terms.

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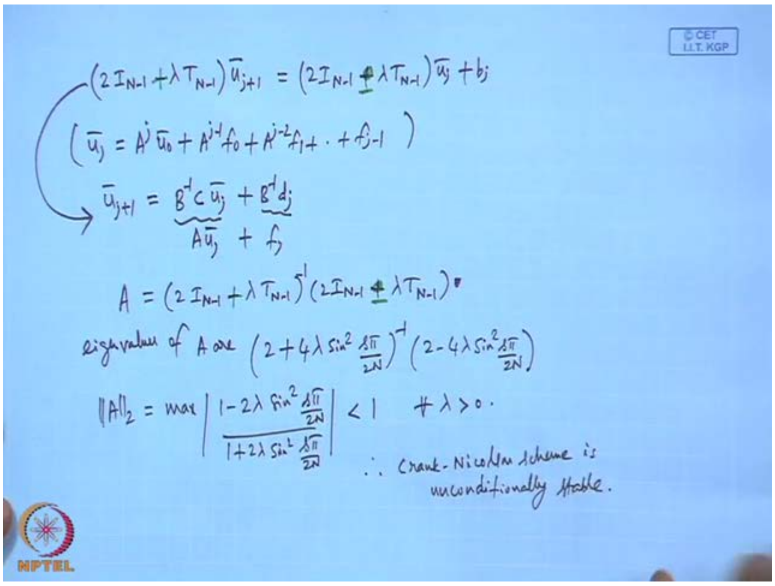

So, we have silted in to two different matrix. So, accordingly two times identity matrix the other 1is n minus 2. So, we get this, then u j is a j, u 0 f 0 f 1 f j minus 1, how did you get this. So, this is our notation that what I suggest to look for the other notation introduced earlier. So, this is a u j plus f j. So, this can be first put in this form, then we alternatively expand. So, while comparing this two we get A is. So, this is A, because b inverse c. So, when we split this 2 into 1 plus lambda and here 2 into 1 minus lambda. So, we split this, please correct it as this plus this is minus.

So, accordingly minus, now we have to compute the Eigen value. So, one can show Eigen value of a, this is an exercise for you. So, these are the Eigen value. Now, respect to this two nom this max of 1 minus 2 lambda sin square p by 1 plus 2 lambda is just simplified version. These 2 get cancel this is less than 1 for very lambda greater than 0.Therefore, Crank Nicolson scheme is unconditionally stable. So, already we have seen, but we have seen via matrix stability analysis.

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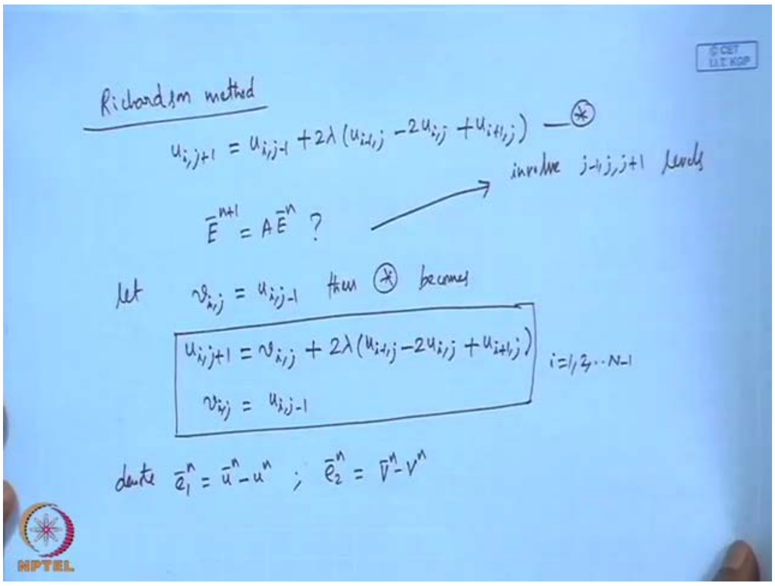

Let us cheek Richardsm method, we have discuss this method. Now, we are talking about stability. So, this is Richardsm method with note we have j plus 1 j minus 1 and j. So, this is going to be slightly different, so then note is involve, j minus 1, j, j plus 1 levels. So, obtaining general alteration slightly difficult, in the sense what I am trying to say. So, this kind of alteration is bit question mark because since it involves three levels. So, we have to do some patch work.

So, what is patch work, let via is u i j minus 1 then u i j plus 1 then above equation v i j, of course then star became this together with. So, it is kind of system, couple system. So, we have to discuss stability right. So, denote usual way e 1 vector corresponding to u then error corresponding to v. So, we have introduce v i j. So, let us denote the corresponding V vector as capital V U vectors capital U, this is the value satisfying exactly by the difference and this around of error.

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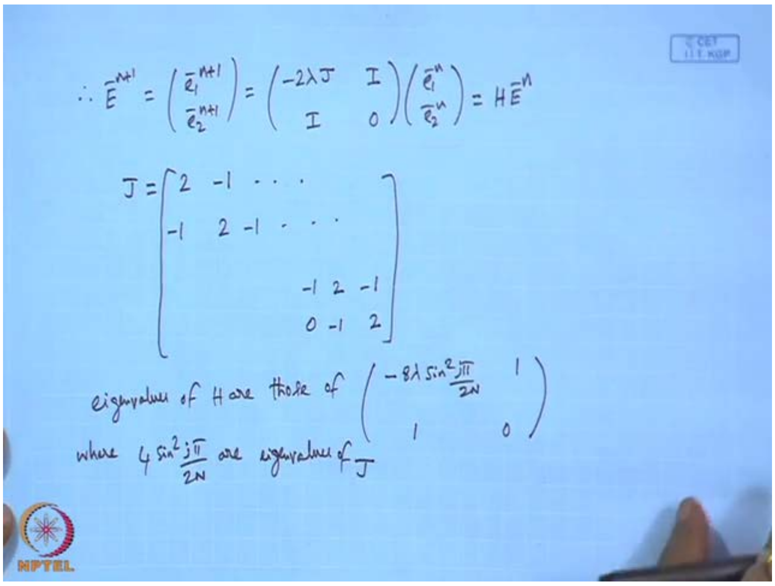

Since, we have v, we get corresponding error we u this is corresponding. So, this is reduce to. This will be minus 2 lambda j i. So, let us see closely this is when we introduce this notation e n plus 1 here. So, we get e n plus 1 and here e n n is multiplying see when i equal to 1 this is boundary comes this is minus 2 lambda. So, this whole think put it to a matrix j, which has specific behavior.

So, if you pull out minus and attach to minus 2 lambda the behavior of these can be given to matrix j. Accordingly j will have so j has this structure. So, this call it now h this is patch work we have done in order to obtain this kind of alteration. So, it not the matrix A directly like earlier case, Then Eigen value of h r those of, so Eigen n value of h r those of this matrix where. So, these are Eigen value of j.

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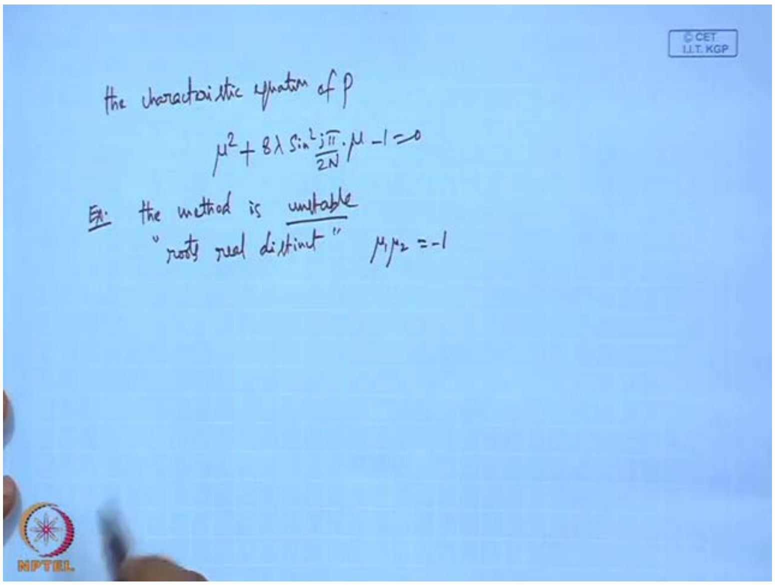

Now, the characteristic equation of this matrix call it as P, so the characteristic equation of P. So, good exercise to do from here that the method is unstable. So, I can give a hint of example if roots are real distinct. So, you can check with detriment you can check with detriment of this and then mu 1 mu 2 minus 1, then one of them is positive. So, I am just giving hint you can true that the method is unstable. So, I hope you are able to now you are in position to discuss stability of any final difference scheme via either matrix stability or via Fourier series stability method.

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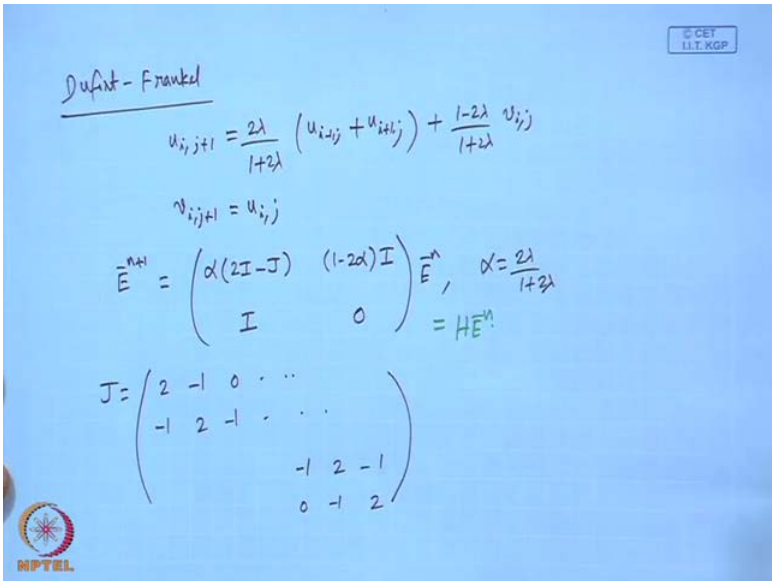

So, let us go for few more example Dufirt Frankal, so Dufirt Frankal is given by. So, this is also similar to the earlier one, then this can be written as where alpha is and j matrix. Now, this is our h this further can be written as H E n. So, we need to compute the Eigen values of h.

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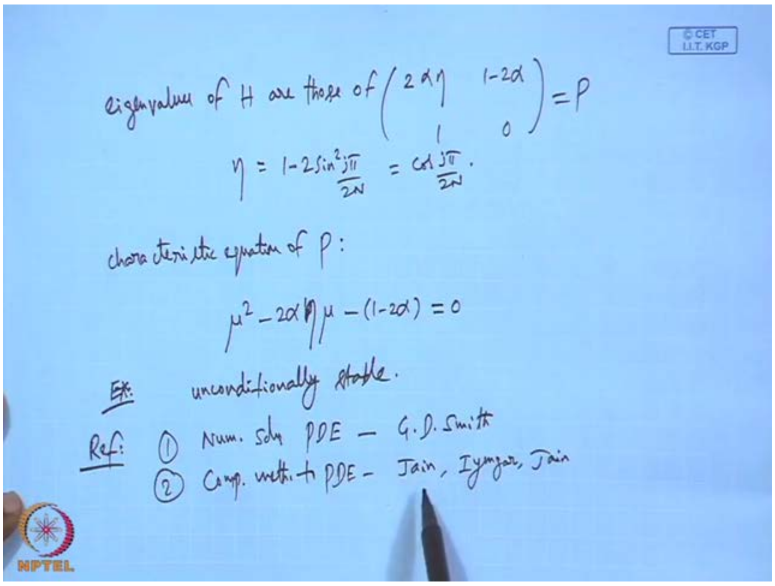

So, Eigen values of h are those of 2 alpha say eta 1 minus 2, alpha 1 0 where eta is 1 minus 2 sine square. So, this is cos the characteristic equation of so this our p matrix of p eta mu equals to 0, then we get from here exercise for you. You can show that this method is unconditionally stable. So, here I would like to mention a point references, I have been mentioning G D Smith. So, numerical solutions two PDE G D Smith, 2 is computational methods to PDE. So, this is Jain Iyengar Jain.

So, the formal references these are just a brief description formal references will be listed in the course content. So, these two books are very much helpful whatever the notations I am using and some other problems off course i have changed the data here and there. So, you can see these books. So, that you get a good idea of these methods. So, please practice as many methods as possible especially computing the stability. So, with these at least parabolic case we have come to a conclusion kind of, before we discuss any kind of parabolic methods for example, other various other special cases, we may switch over to hyperbolic and elliptic coming few lectures until then.

Thank you.