Numerical Solutions of Ordinary and Partial Differential Equations

Prof. G. P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 3
Analysis of Single Step Methods

Hello, good morning, having introduced single step methods, I kept this title as analysis of single step methods, because if you look at the initial value problem y dash equals to f of x y with given initial condition y of x 0 equal x 0. So, what is y dash, you can think as a slope now slope is equal to some function, so we have to now find what is the corresponding y?

So, this definitely needs some approximation, as you see in the Taylor series method which we have discussed. We were using various terms; second term, third term up to fourth term, fifth term and then corresponding error etcetera, so that means definitely there is some approximation involved. So, can we formulize this approximation? So, that is the question, so this needs some analysis hence the title, so let us recall Taylor series method.

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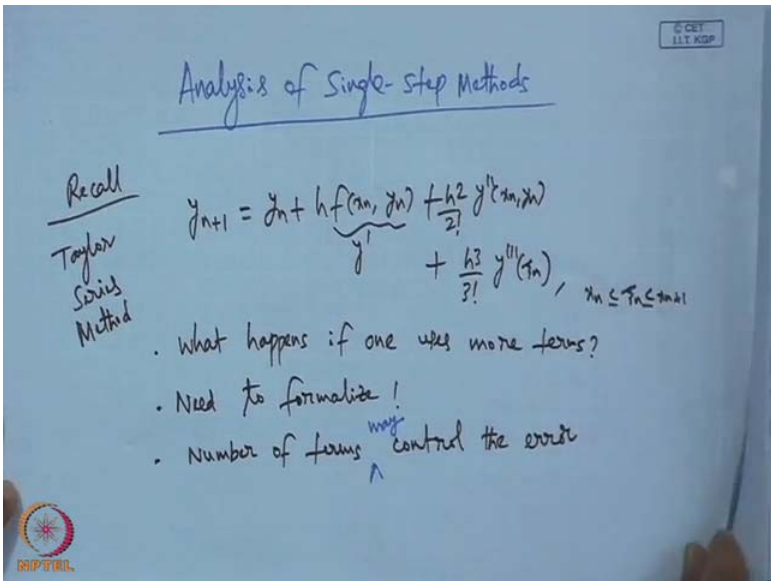

So, this is Taylor series method, for example, stops at third, and so then we write zeta n where this interrupts, so the question is what happens? If one uses more terms. So, the immediate answer is definitely the solution may vary yeah of course you are true the solution may vary, but what sense can we say concretely if we use more number of terms. So, with general intuition the solution will be better and things like that.

So, we need formulize this and one institution says number of terms controls the error, so this may be formulized. So, if you look at this, this is first derivative, we have written it as this is nothing but y dash second derivative and so on. So, that means after some stage you can take for example, you can take out h, but we need at least first approximation, so suppose if we take out h then these terms can be controlled as some function and the entire behavior may depend on this function.

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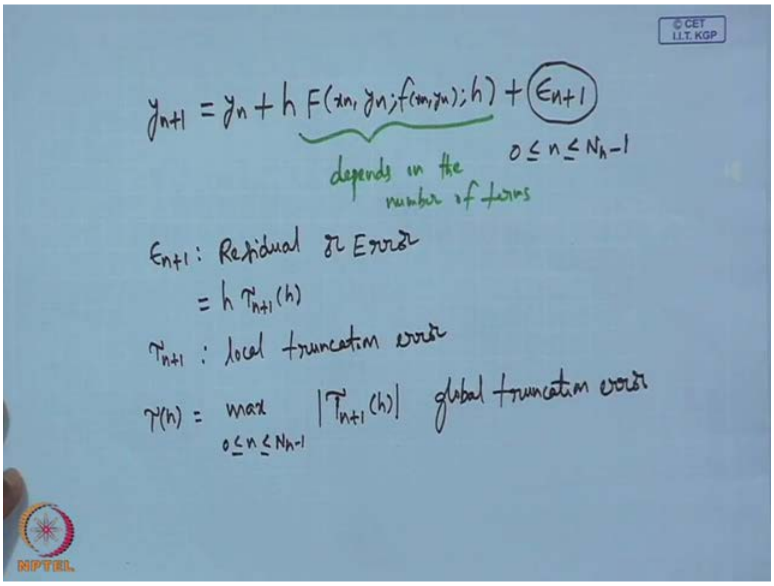

So, suppose if we introduce this notation for a particular h up to some terms, then what kind of structure F may have. So, this depends on the number of terms certainly, so this depends on number of terms, if this depends on number of terms for sure we are approximating up to those many terms and started neglecting from these. So, what is this then? So, this is residual or error, so what do you mean by this if you compare with the exact solution up to these than the r neglecting from here.

So, that is called residual or error and we introduce this notation because the minimum up to first. So, as I mentioned earlier, you can pull out h and express the rest as tau and plus 1, so what is then tau and plus 1? So, this is local truncation error why because you are approximating up to some terms and you are truncating therefore, this is local truncation error then we introduce tau h which is max because in each interval if you consider 0 to 1. I mean n 1 to 2, so we have corresponding local truncation error involved, now we take the maximum and call it tau. So, this is global truncation error, now typically what kind of structure F would have.

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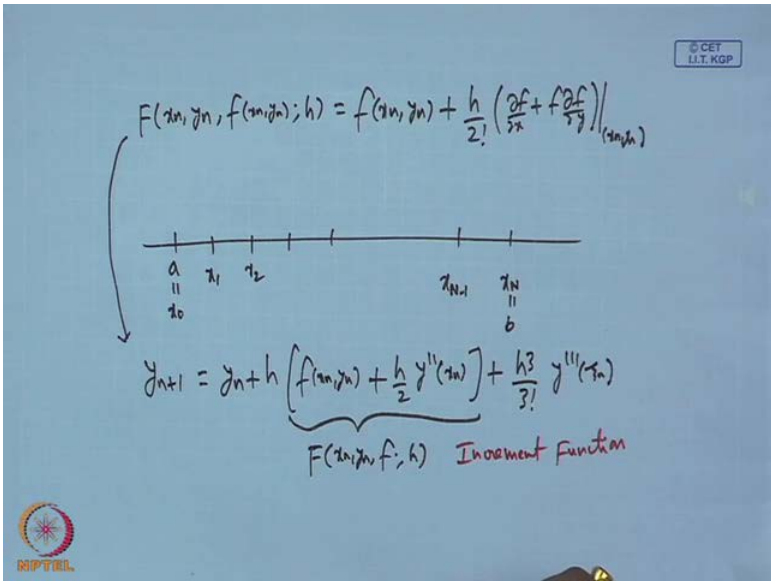

So, the structure F, so why this structure because we have pulled out h, therefore f starts from f of x and plus so on. So, when we start a competition, so when we start a competition like this, so this would lead to say if we put down up to three terms third term is like this. These are increment function, so these are called increment function because you keep on adding more terms.

So, F behaves according to that, so we have to formulize the error and also we have to formulize if ten terms are used what is the behavior and what do you call the method if used twenty terms. So, this ten terms, twenty terms, forty terms, so this number also have significance and same time whenever you have a h truncated up to something then whatever you have thrown away, so that also has a significance we are trying to formulize.

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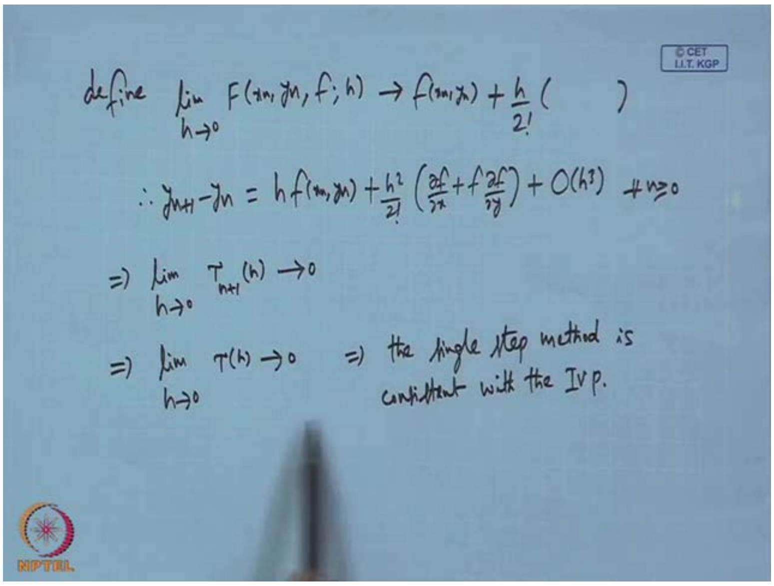

So, to this extent how F behaves as h becomes small, so this is our first term then h over 2, then we can pull out, then we have some remaining terms. So, this would be plus, plus because the remaining will be of order h cube right. So, this employs limit h goes to 0 tau and plus 1 h, so this goes to 0, so this goes to 0. So, this employs limit h goes to 0 tau h goes to 0, so if this happen then we say the single step method is consistent with the IVP. So, why this is happening the way we have defined residual h times tau and plus 1, therefore tau plus 1 goes to 0 because at least one h remain in these. Therefore, h goes to 0, this goes to 0, so if this happens then we say single step method is consistent with the IVP.

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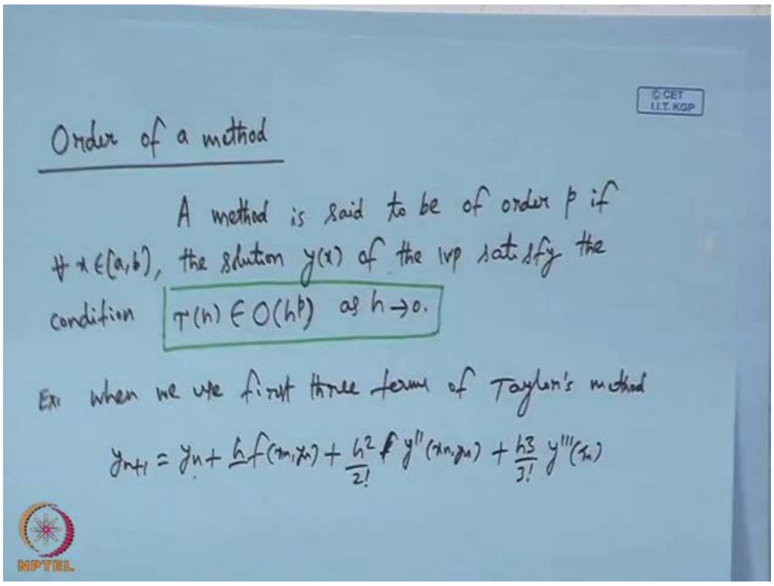

So, up to what terms we consider and then how do we formulize, so we have to discuss that is called order of a method. So, as your intuition says, the number of terms is related to the word order, so a method is said to be of order p if every x a b the solution y x of the IVP satisfy the condition. So, a method is said to be of order p if for every x within in this interval, where the corresponding IVP is defined, the solution y x of IVP satisfy the condition.

So, as h diminishes, your global truncation error is h power p, so then we say the method is a fodder p. Let us say for example, when we use first three terms of Taylors method, so what we would we have h, h square by 2, we have y double plus y 3 at the time. So, if you use first three terms, then we are not using fourth term, see one, two, three, fourth terms we are not using, this means we are throwing it away.

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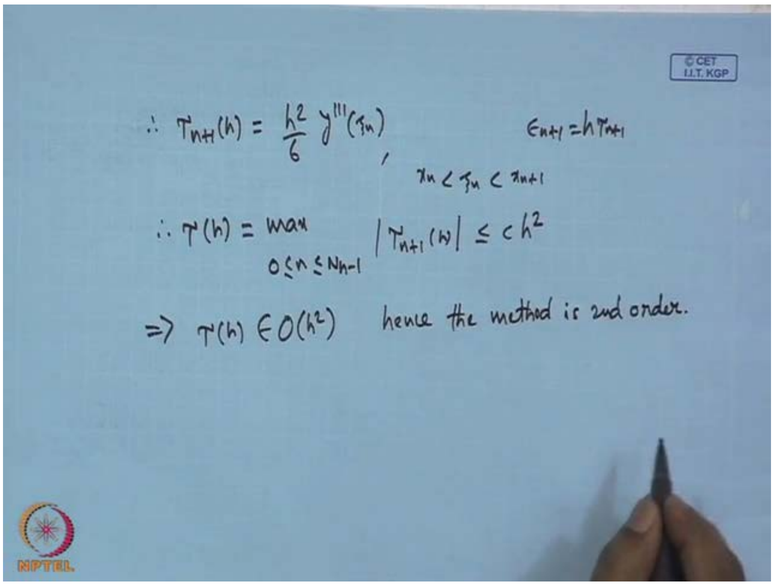

Therefore, this would be h square by 6, so remember we have defined xenon n plus 1 as h times. Now, n plus one, so there is h, so I pulled out one h, therefore the coefficient is h square. So, I made equality, you may correct it of this range, therefore tau h is max of mod, so this behaves like this because we can estimate.

So, we can get a constant that together with the one over 6, we can get constant and behaves like this. So, this employs, hence the method is second order, so this one hand because the number of terms and then the how it behaves. We can construct ten terms, twenty terms, what you call, that is nothing but a direct indication order of the method. So, the next consent is if you consider twenty terms, five terms whether we really get solution that is close enough, so which means convergence.

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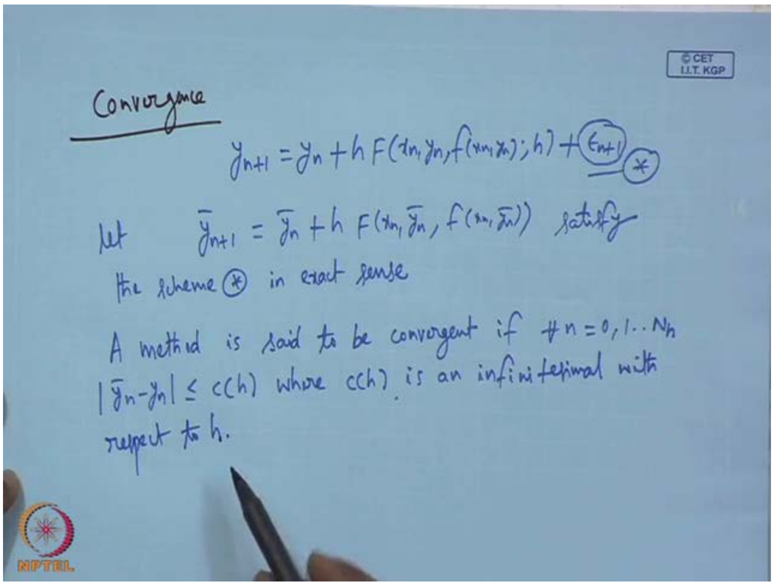

So, let us formulize what is convergence, so we have this approximation, let this satisfy the scheme star in exact sense, then method is said to be convergent if you can easily. Method is said to be convergence, if for every n this is exact of the scheme approximating scheme. So, then y and what we obtain, so then the method is said to be convergent.

So, when we can obtain, we can see with the corresponding error into consideration. So, then the method is said to be convergent if the difference between the exact of the of the approximating scheme actually obtained after considering the error is less than c of h when we say c of h definitely this depends on h. Now, when do you say this is convergent c h is an infinitesimal, that means this very small with respect to h. So, if this is going to happen, then we say the method is going to convergent, so what is the idea, so you have approximated.

So, what is the guarantee that this would really give you solution with this converges this scheme that means if you keep on complotting and if it blows up, and then it is beyond out of control. So, to reach the solution, so when we say such an approximation is convergent you consider without error than consider with error and the difference should be diminution, it should be very of course depends on h. Now, correspondingly h the number of terms also can be related, so this is convergence.

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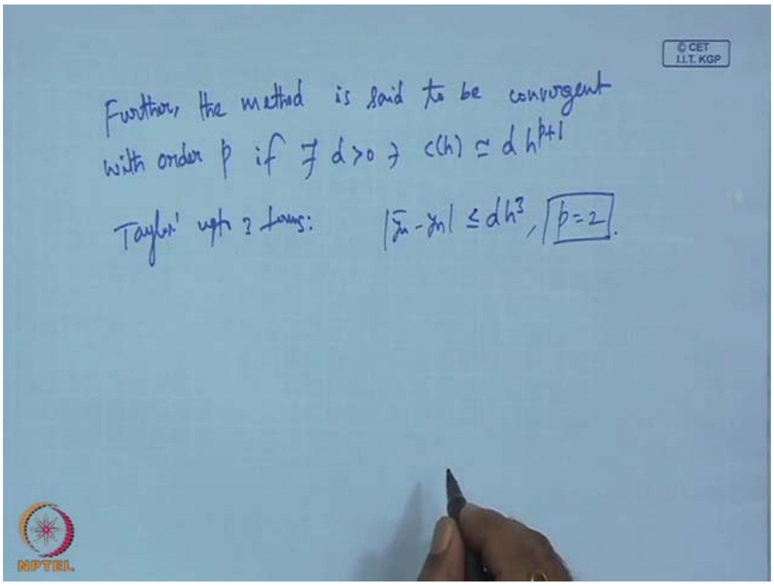

Further, the method is said to be convergent with order p if there exists d great than 0 such that the method is said to be convergent with order p if their exist d great than 0 that c of h behaves like this. So, for example, the Taylors up to three terms d times h cube where p is 2. So, this is convergent with order 2, so what is the motivation, how do we approximate, how many terms we consider? These are the quick questions we may come across, so well we have assumed that F is smooth, then we try to expand and assuming we have as many times as possible to differentiate.

So, we propose Taylors series, but we started compromising because we have restrict within the discrete notation. So, when you have compromised the question are how many terms for this problem if I take this, many terms what happens? So, these are the questions somebody would try to be a little intelligent and then say I would consider h pretty small, so small I mean I can start throwing from h square onwards.

So, if you if you argue like this yeah definitely you get some approximation because h square onwards you are just throwing. Well, definitely what should be your h such that your h square could be thrown it matters, but however if you assume that h square onwards, you can throw definitely you are going to get some approximation, so let us look at it, what is that?

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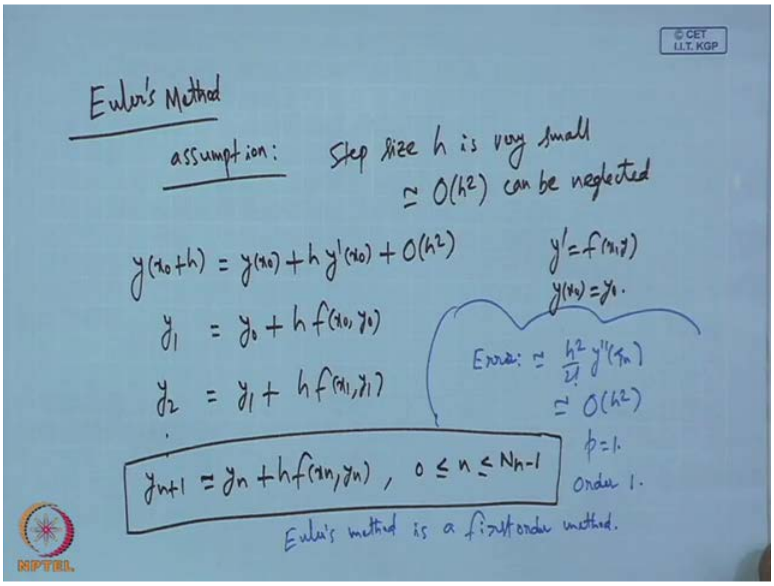

So, this title Euler's method, so what is assumption, the assumption is step size h is very small means order of h square can be neglected. Now, consider because that is our assumption, so this is nothing but y 1 y 0 because our IVP, now if we continue, the next big point. So, we can formulize, so this approximation obviously is your Euler's method. Now, what is the corresponding error, the corresponding error we started throwing from this now as we decided earlier we can estimate this.

Now, if error behaves like this, then that means Euler method is order 1, so Euler method is first order method. So, you can probably guess because h is quite small started throwing h square, so it is a first order method. So, definitely your solution you are having big compromise in some sense, so we have to really think of better approximations that where we are going to formulize better approximations.

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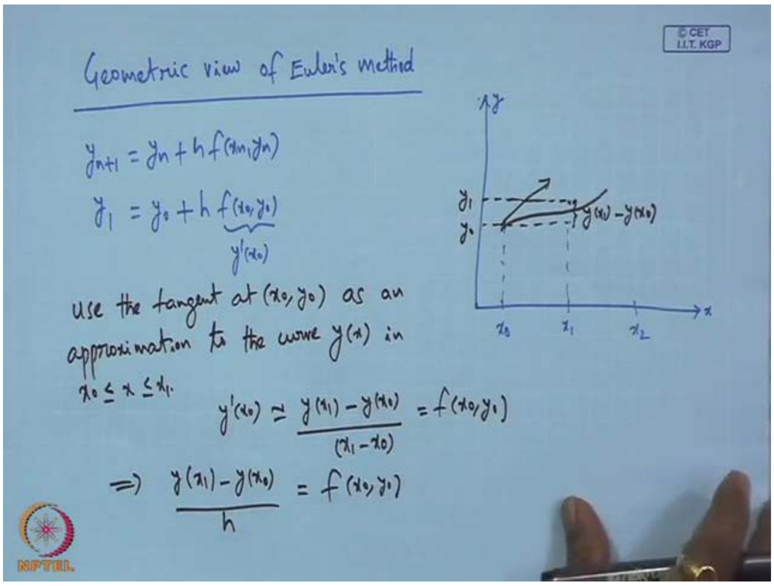

So, before we go because in order to go better approximations we should know what is the interpolation geometric views of this, this would be given a hint for our generalization. So, say we have this x 0, x 1, now this general Euler method, if we write down the initial point and what is this? This is nothing but y dash of x 0, so what is exactly happening, y dash of x 0 has been approximated by this. So, use the tangent at x 0 y 0 as an approximation to the curve y x in this interval.

So, that means we have said function like this then this is y 0, so this is y 1 use the tangent at x 0, y 0 at the approximation to the curve y x in this interval. So, this is nothing but according to this tangent which is by approximation to the function, so this employs, so this we call h so this will give us y of x 1.

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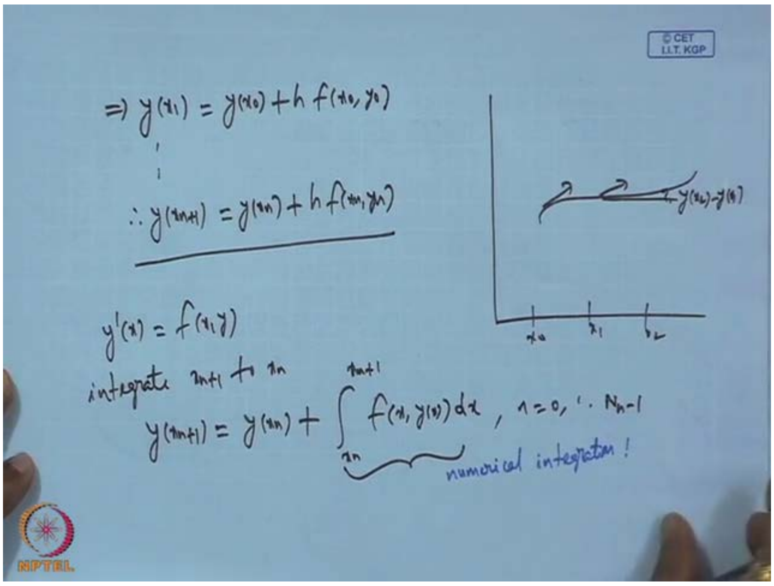

So, this give us that means you started x 0 x 1 x 2, so we started approximating so next step if we approximate, so then we have correspondingly this difference will be y of x 2 minus y of x 1. So, we can march accordingly, so then in a general sense, we get this, so this is a general idea. So, what exactly happens numerically, so this is geometrics so you consider y dash of x equals to f of x y. So, then if you integrate this range we get, so this what now this has to be we are having approximation of this in different patterns leading to various method that is what is exactly happening. So, in order this needs to be numerically integrated, so this needs to be numerically integrated.

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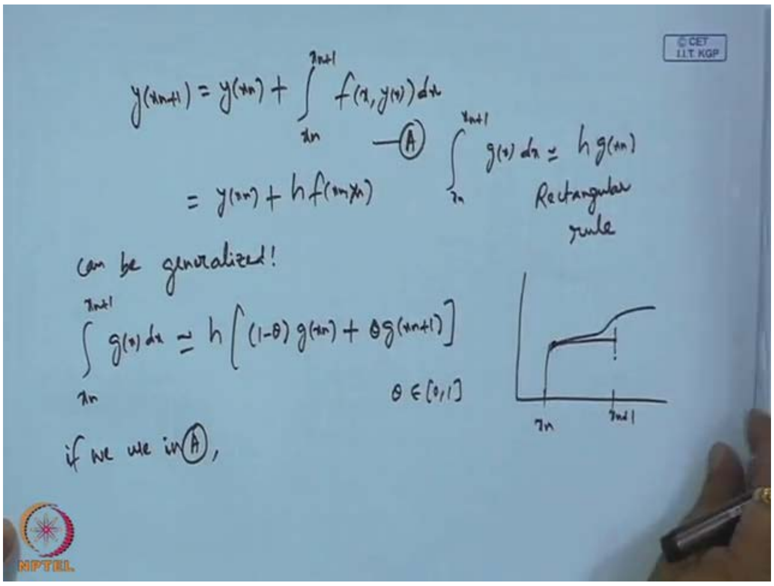

Suppose, the simple situation is say we go for is approximated by h times, so this is called rectangular rule. So, g of x between x and x n plus 1 you are approximating by the difference is h, so h times g of x n you are approximating by rectangular rule. So, if this is the case, then we get from here h times. So, this Euler instead of rectangular can be generalized, so how do we generalize? So, I explain what is this, so we are generalizing not using simply rectangular like this, you are giving some weightage 1 minus theta at g of x n and theta g of x n plus 1. So, restrict theta in this range, now this if we use it in A, so we get corresponding approximation for f then we may get different method.

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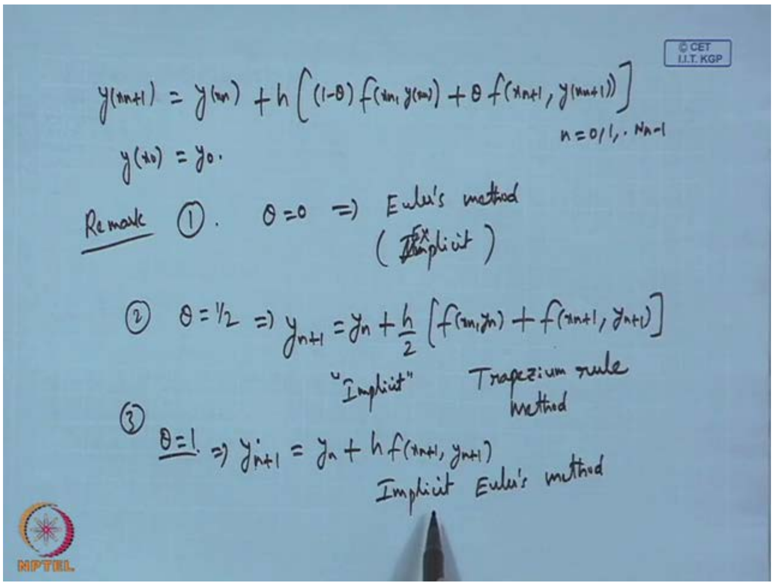

So, we get plus h remember this is at x n plus 1, so with this more general if theta equal to 0 employs definitely Euler's method theta equal to 0. So, this sometimes called implicit Euler method, suppose theta equals to half, so both terms survive we get, so this is trapezium rule method. So, we discuss geometrically another observation we see to compute y n plus 1, right hand side is also demanding y n plus 1. So, this is then I am sorry this Euler method theta equal 0 this explicit. So, this is not a implicit explicit and this is implicit, now say theta equals to 1, so this goes away so we get y n plus h because theta 1 goes away this survives.

So, this looks like Euler method, but this is implicit Euler method, now the idea is if you keep on changing, then we are getting different approximations. So, we have to really generalize what are we doing geometrically we started an interval. Within in the interval, we approximate the slope with the corresponding function value at the initial point the beginning if the interval is x k, x k plus 1 then y dash of x k, we have approximated by the function value at x k.

So, definitely this is giving up to first order, so then we started splitting and giving it 1 minus theta by it to left and theta to right and we tried to estimate depending on theta value we get various methods. So, what is generality in this, it is nothing approximating with various slopes, so you can now think of how to generalize that means we can approximate using two, three, four, five slopes etcetera.

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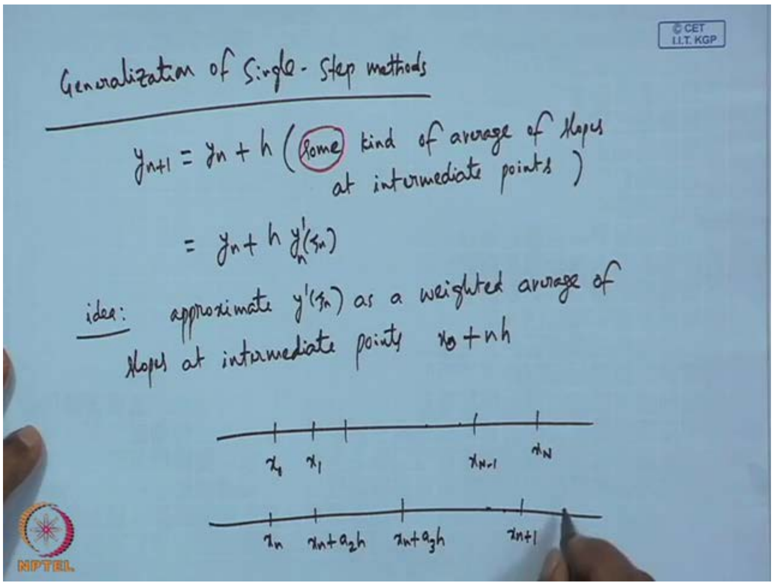

So, let us more carefully, so this generalization to single step methods, so generalization of single step methods h times some kind of average of slopes at intermediate points. So, this is nothing but I put some kind definitely the method would be different depending on what kind average. So, what is general idea approximate y dash of zeta n as a weighted average of slopes at intermediate points x n plus say x 0 plus n h. So, you have say x 0 x 1, so we approximate. So, if you want to generalize this so may be this is x n and x n plus a 2 h and then x n plus a 3 h, then somewhere we get x n plus 1 that means we have changed slightly.

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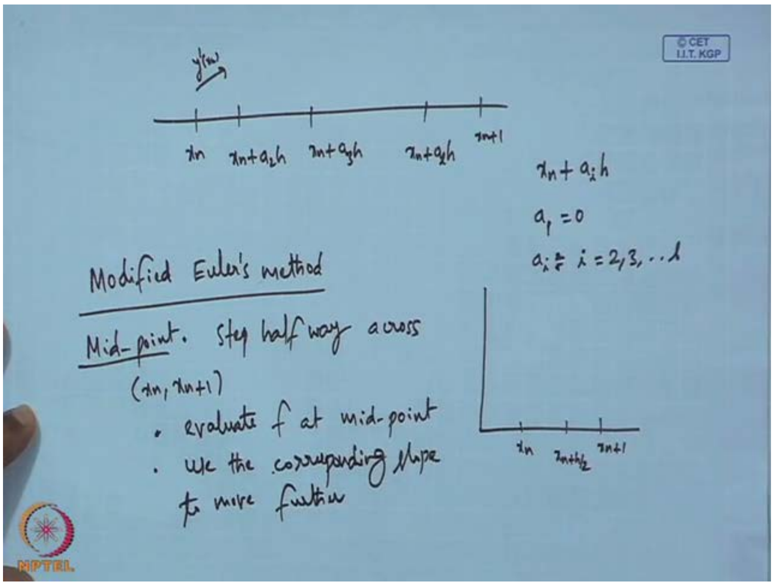

So, because within an interval x n to x n plus 1, we have introduced an a i h where a i is 0 then a i equals to a i depends, so it will be i equal to 2, 3, l. So, within interval x n plus a 2 h x n plus a 3 h x n plus a l h. So, that means within interval, so first time you recall, so this is y dash of x n, we have approximated only using this slope at y dash of x n. Now, within one interval slopes act several intermediate points say in this case l slopes hope you get this is generalization.

So, let us see a simple case in this namely modified Euler's, so this is x n, then you pick up x n plus h by 2. So, where you are getting x n plus h by 2, so this look at this notion, so I picked as half, so it is not equal, so 8 x value depending upon i, now a i, so here so a i is zero, so a 2 is chosen as half you can see, so we are using midpoint. So, we are step half way across this point, goes step half way across this then evaluate F at midpoint, then use the corresponding slope to move further, so which means y n plus 1 is y n plus h by dash of x n plus h by 2.

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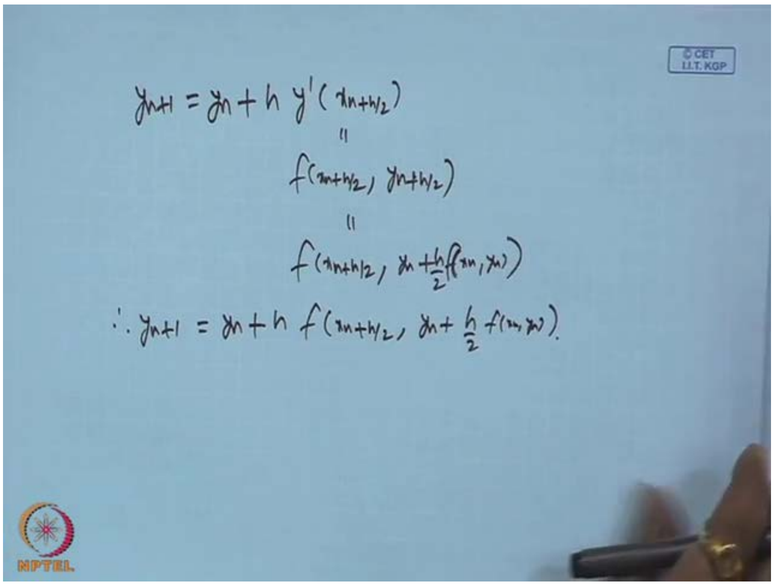

So, this is nothing but f of x n plus n by 2 y n plus h by 2, so this is equal to f of x n plus h by 2, y n plus h by 2 times. Therefore, we get the method, so this is modified if we use midpoint, now there is no necessity that we should use midpoint.

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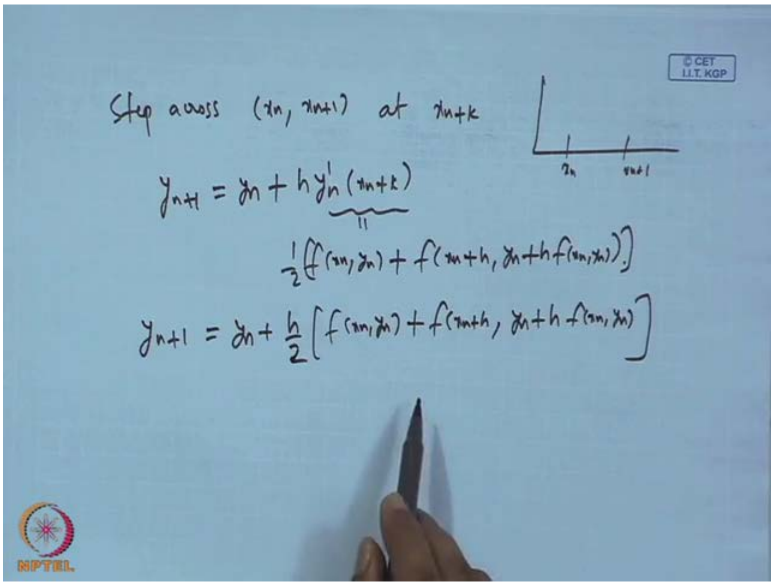

So, let us say step across x n, x n plus 1 at x n plus k, so then we get y dash of n, x n plus k now depends on what kind of approximation we go for. Suppose, somebody goes for approximation x n plus h, so we get suppose we go for this approximation by average at x n, so at x n plus 1. So, then we get approximated like this then we get, so this is your trapezoidal rule.

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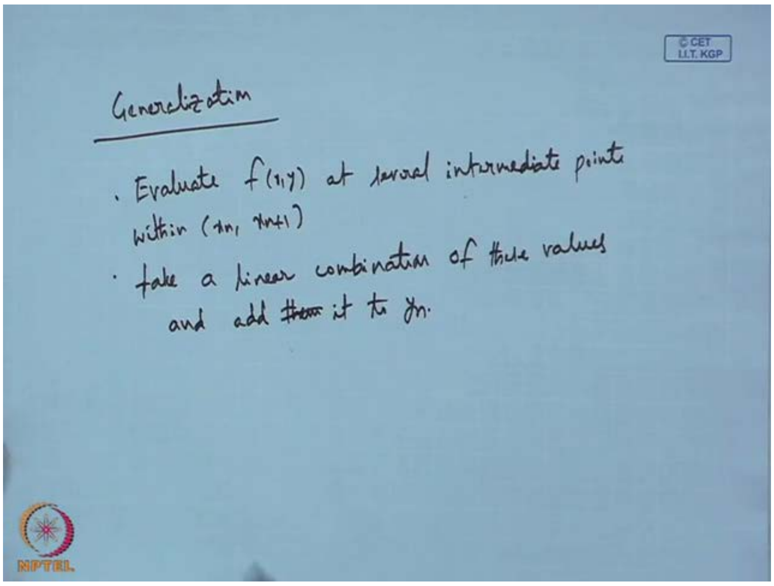

So, we are coming to generalization, so what is generalization evaluate f of x y at several intermediate points within. Then take a linear combination of these values and add it to y n, so you have seen how considered initially the weights on theta. So, similarly instead of considering only the end points, you can step across several intermediate points within the small interval x n, x n plus 1.

I am not talking about Euler's method from x 0 to x 1, x 1 to x 2, x 2 to x 3 and then we got the generalized method I am talking within the interval x n and x n plus 1. You go for several intermediate points and then you start approximating tau, those slopes and take a linear combination, so that it would give better approximation. So, this is this generalization is leading to something called arcade methods, which we discuss in the next lecture until then, bye.