Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 27
Other Numerical Methods for Parabolic PDEs

Hello, welcome back. So having discussed explicit and implicit methods and also consistency of these methods; there are still methods for parabolic PDEs. So, let us discuss one of the important methods. Of course, the title I kept it little, I mean not precise, because I have covered explicit and then implicit, and then the definitions of convergent stability and consistency. And we discussed definitely the stability aspects and then consistency of some methods, but still since there are several uncovered topics, which still belong to parabolic PDE. So, I kept the title as just other numerical methods for parabolic PDEs.

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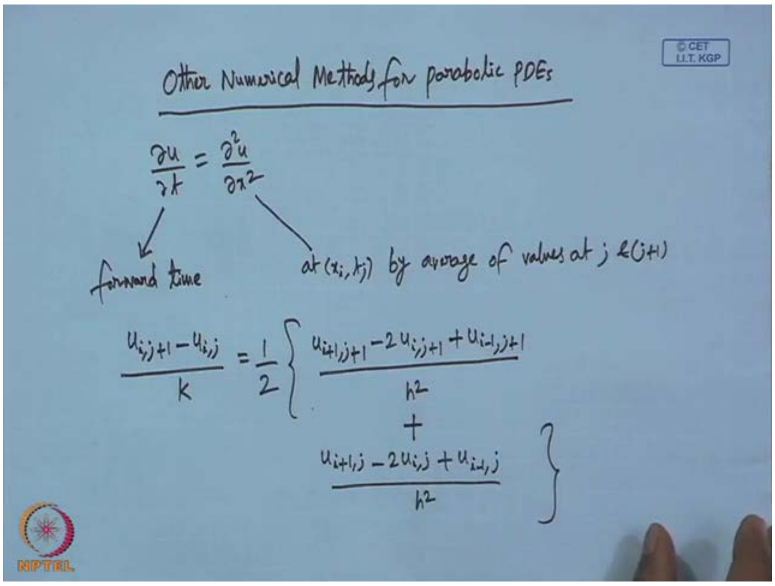

So, let us look at it, so still we are considering the heat conduction equation. So, when I said this it is understood that within a specific domain and then the corresponding initial and boundary conditions are given. Now, let us approximate this by forward time, but this one this at x i, t j by average of values at j and j plus 1. So, this is a slight deviation what we are doing, this we are going for forward, but whereas this at this point is approximate by average of values at j th level and j plus 1 th level.

So, accordingly we have this, now I am going for the average of, so this is standard second order approximation central at j plus 1 plus this. This is again standard j second order approximation for central at j. So, the meaning of this is this. So, you can look at it this is a usual forward, but here this is the value of j plus 1 central for second order second derivative and this is value at j.

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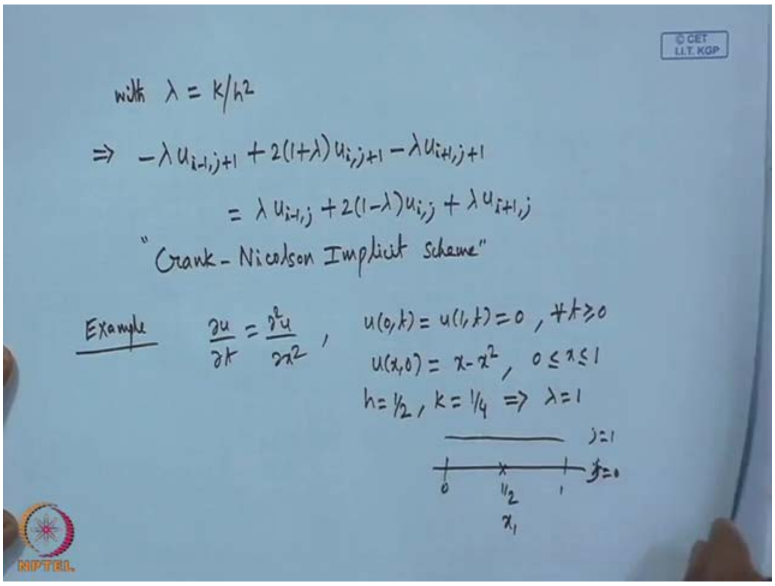

So, if you simplify this we have with lambda equal to k by H Square the discrisation simplifies to. So, you can figure out all the terms at level j plus 1, we are writing to the left hand side then all the terms at level j to the right hand side. So, this is called very popular Crank-Nicolson implicit scheme. So, this is implicit because at j plus 1 we have more than one point, more than one term involved.

So, this is implicit nature we have discussed before. So, this is very popular Crank Nicolson implicit scheme. So, let us try with an example. So, the example we are considering, the boundary conditions for all t then we need initial conditions, this is of course for x boundary 0 to 1. Then further we need the parameters say h is half, k is 1 by 4, this will give lambda equal to 1. So, first glance this sounds too much because run the boundary to 0 to 1, we are going for h half so that means essentially you have only one grid point. So, suppose this is 1, so this is just half. So, this is only one grid point, however along the time steps, so this is j is 0 and j one and you can. So, let us see what happens for this case. So, u of 0 t equals to 0 so this implies u of 0 j equals to 0 because this is x 0 is 0, x 1 is half x 2 is 1.

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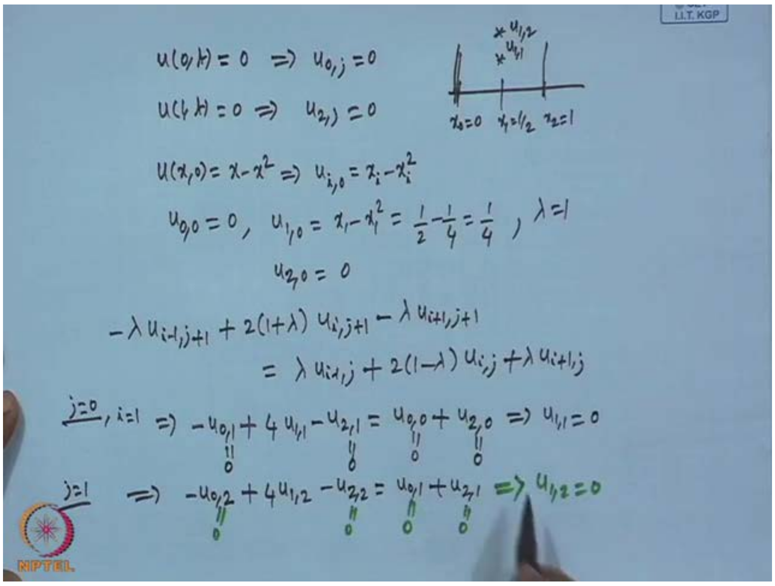

So, the boundaries u 0 t is u 0 j is 0 then u 1 t 0 this is the boundary, so u 2 j is 0 then the initial condition u of x 0 is x minus x square. So, this implies u i 0 is equal to x i minus x i square. So, accordingly the initial values at these grid points we can compute u 0 0 is 0 u 1 0 is x 1 minus x 1 square so this is then u 0 0 is 0. Now, the discretize schemes Crank Nicolson scheme is minus lambda is equal to this we run at, so since lambda is 1 so this goes off.

So, let us consider j equals to 0 then corresponding to i equals to 1 we have so please j is 0, i is 1 and lambda is 1 we have. So, lambda is 1 we have it for the choice we have taken. Now, this implies minus u 0 1 plus 4 u 1 1 minus u 2 1 u 0 0 plus lambda is 1. This term goes of u 2 0 however please observe carefully 0 j is 0 therefore, this term 0 this is 0 and 2 0 is 0 and then 2 1 because of the virtue of this so this implies the only unknown 0. Suppose, we run for j equal to 1, this implies suppose if we do this even now this is 0, this is 0, this is 0, and this is 0 because of virtue of this. So, we are getting what could be the reason, if you proceed all this is u 1 2. So, if you proceed along the time stepping we may end up with getting 0 values. So, the reason is the choice of step size so that is too much.

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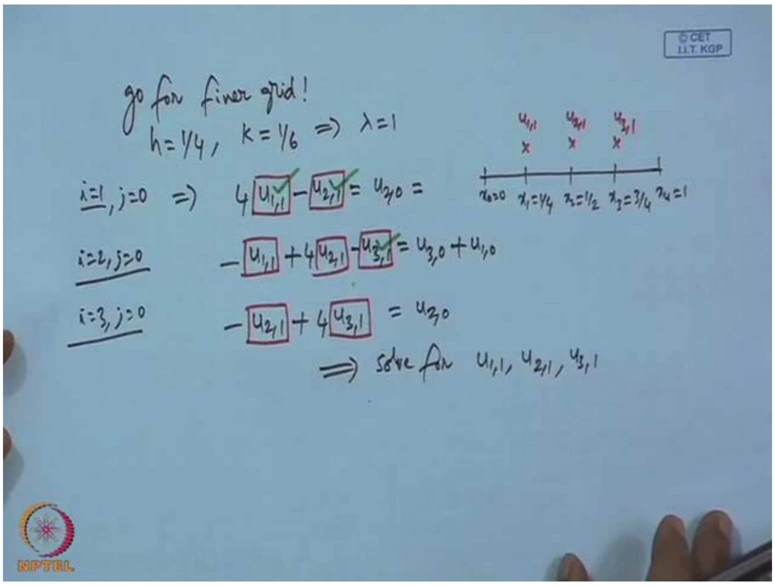

So, better to have go for final grid, so this could be the lesson. So, accordingly let us choose h is 1 by 4 and k is 1 by 6 of course, still for the simplicity grid parameter is 1. Now, in this case if we run the equation this gives equal to even, I am putting the simplified version and u to 0 we have computed to be 0. In this case we have to compute again so the earlier case is 0 and u 2 0 we do not have.

Now, if you go for i equals to 2 of course, j remains the same then we have minus u 1 1 plus 4 u 2 1 minus u 3 1 equal to u 3 0 plus u 1 0 then i equals to this. So, h is 1 by 4 so let us see what happens in this case x 0 is 0, x 1 is 1 by 4, x 2 is half, x 3 is 3 by 4, x 4 is 1. So, how many unknowns we expect, 3 unknowns and what are they u 1 1 u 2 1 u 3 1 so these are the unknowns. So, in this the unknowns are since Crank Nicolson is explicit, we expect system of equations. So similarly, now for i equals to 3 we get so here again these are there are repetitions.

So, if we pick only one time so 1 2 3, this is repeated, 2 1 is repeated, 1 1 is repeated, 3 1 repeated. So, we have three equations and three unknowns solve for so this is an explicit and this is implicit. Again one step method because to compute the values at higher level you need only the past data at one level the immediate on before. So, since it is an implicit it is consent to solve the system this is very simplest. So, we could get the solution, but in general if you have a larger system how do we go about it.

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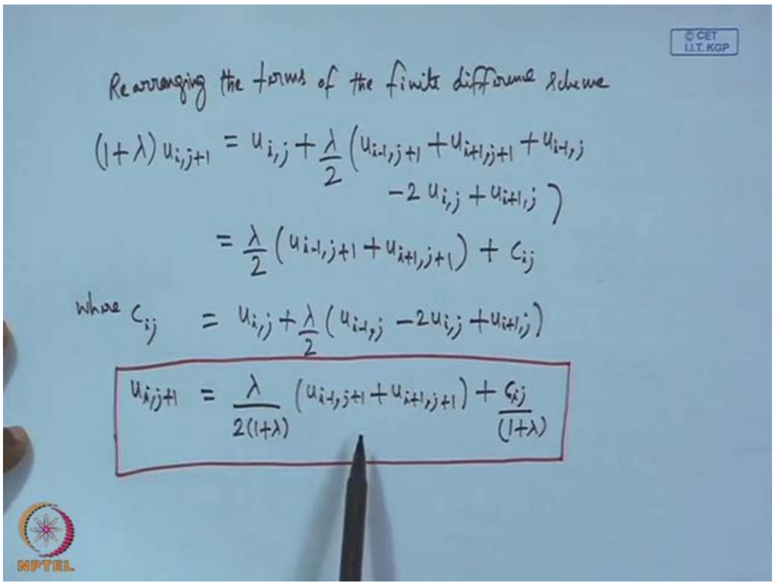

Let us see we rearranging the terms of the finite difference scheme we get say slightly boring, but this give some idea. So i minus 1, j plus 1 i plus 1, j plus 1, i minus 1, j i j plus 1 j, so this is the grouping. Further this can be written as u, so will bring up only the terms at j plus 1 plus c i j where, c i j this term plus lambda by 2 times whatever missing in this, but written here. So, they will be i minus 1 j i j so this is just for convenience. So, then accordingly we have u i j plus 1 is lambda by 2 1 plus lambda because we have 1 plus lambda there. So, this can be put it as a system, so the system has been rearranged in this form. If you observe closely we are looking for i at j plus 1 level, the value at x i j plus 1 level and right side you have j plus 1 comes 2 more level and this entire c i j contains only terms at the j t h level.

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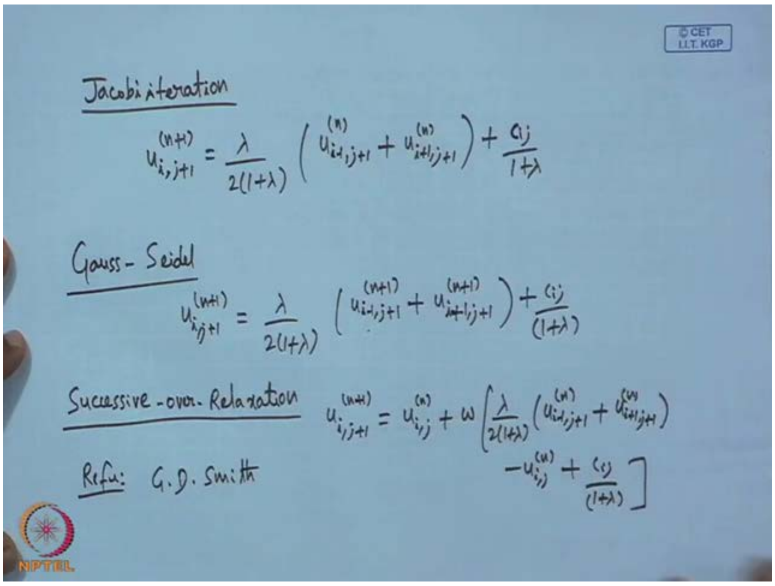

Now, this we can solve using any alternative method. So, for example, we define Jacobi iteration. So, we can keep it j plus 1 or we can drop so this is the index for iteration. So, that means given the values at j t h latest is used to compute at the next level, next iteration with respect to n. So, that means even at j plus 1 time level whatever available you have at the previous iteration use them with this standard Jacobi iteration then Gauss Seidel.

So, in this case whatever is available we are going to use up to that level of iteration. See u i minus 1 u i plus 1 if you have at n plus 1 t h iteration, we use them to compute i. So, the Gauss Seidel and the other one successive over relaxation, this is another standard method to solve system of equations. So, this is successive over relaxation so this is a relaxation parameter. So, you may refer numerical solutions to PDE GD Smith.

So, you get a better idea, so this is for any system of equations. So far the explicit and implicit methods we have discussed including the Crank Nicolson. They involve the values at two time level that means if you give the past time level, you can compute at the next time level. Now, let us see examples of the method where you have more than two time levels are involved.

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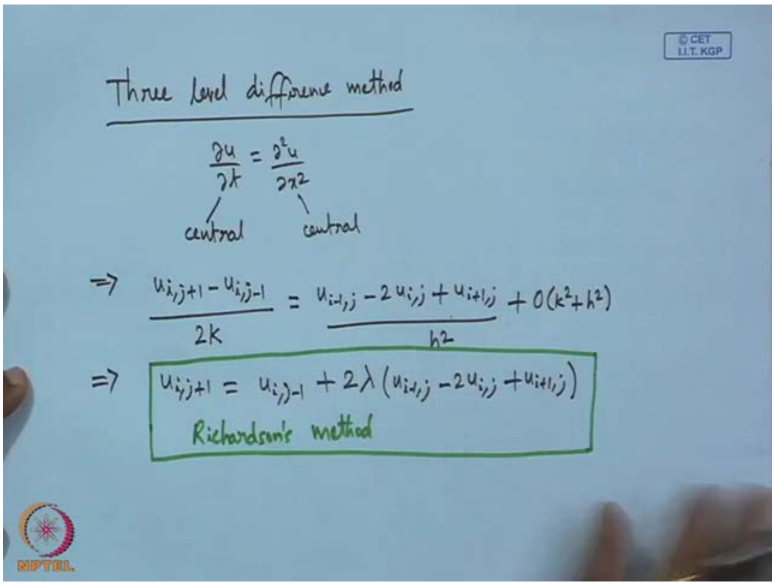

Of course, it depends on the way we discretize, so three level difference method. Still we are with our heat conduction then use central, so first time we are deviating. So, then this is standard central for second order. So, accordingly we have seen for the first order time we are using central. So, this it should be 2 k so this is k square and h squares this 1. So, this we are going to simplify this equal to u i j minus 1 plus k by h square lambda and hence 2 lambda, so this is Richardson's method.

So, when we say 3 level j plus 1 j minus 1 and j, we will see may be in some tutorial try to solve the problem using this. Otherwise the important observation is the discrisation here is bringing one more time level. You can see had it being forward or backward, we would have used j j minus 1 or j plus 1 j, but because of this central we get j plus 1 j minus 1 and right hand side you have j.

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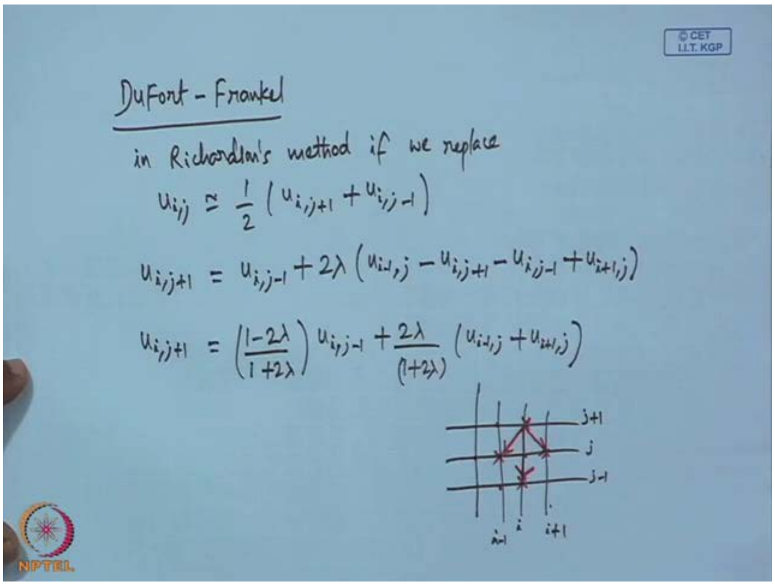

So, this is an observation, and then let us sees one more Dufort Frankel. So, these are the standard names in literature. Now, what we do in Richardson's method, if we replace u i j by the average. So, if we do that we get so then rearranging, what is happening in this case this is say i and this is j j minus 1 j plus 1 i minus 1 i plus 1. Now, j plus i j plus 1 so this point then right hand side i j minus 1. So, i j minus 1, this point is involved then i minus 1 j then i plus 1 j. So, it is implicit this is explicit, but however you can see you need valves at two values at j t h level and one value at j minus 1 level.

So, it is explicit however is demanding values at two time levels so to that extent it is explicit. So, you can see how we mark first because if you want the value here, it expects this, this and this, so we have mark first like that. So, these are some of the variations, which are standard in literature. However if I keep on talking about the error expansions by routine calculation, which we have done for at least explicit and one implicit method then you get bored.

So, in order to discuss about the error and then stability of this methods may be we shall wait and until we learn one more technique to discuss stability other than what we have learnt. So, then we can come back to these methods and discuss under what conditions these methods are stable, how far they these are comparable including the Crank Nicolson method. Because when I mentioned Crank Nicolson method is important, yes it is really important, because many of the problems in field mechanics etc people use Crank Nicolson method.

So, we have postponed the discussion on the stability of this method, until we learn more rigorous analysis of stability. So, before we go to that see whatever problems we have discussed so far, the boundary conditions are just the function values at prescribed. So, only the function values are prescribed on boundary. In case this is replaced by a kind of derivatives boundary conditions then what is going to happen because it is a concern because there is a big difference when derivatives are given on the boundary, what would happen? You do not have explicit function value on the boundary so that value also becomes another unknown for us. So, this is going to be slightly interesting so let us proceed to derivative boundary condition.

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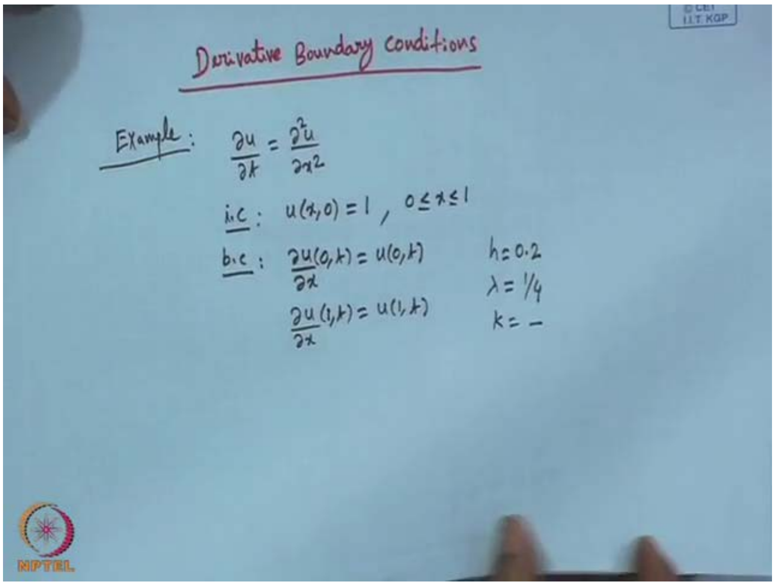

So, derivative boundary conditions let us start with an example because even if we take a general case, this is going to be little kind of hand waving. So, better to discuss this with reference to an example then initial condition u of x 0 is 1, then boundary condition dou u by dou x at. So, there is a big difference you have to observe this, initial condition of course, this is a non-homogenous it is not 0 here, you have non zero quantity. Further instead of the function value we have derivatives prescribed on the boundary both at x equals to 0 and x equals to 1.

So, let us say for this problem h is 0.2 and lambda is 1 by 4 so accordingly we compute k. So, you assume that we are solving for that type step. Now, if we use as far as discrisation of this is concerned, you can discretize either with explicit scheme or implicit scheme. However there will be a deviation when we discretize the corresponding boundary conditions compare to the earlier case of function value prescribed on the boundary. So, let us see how this deviation will play a role.

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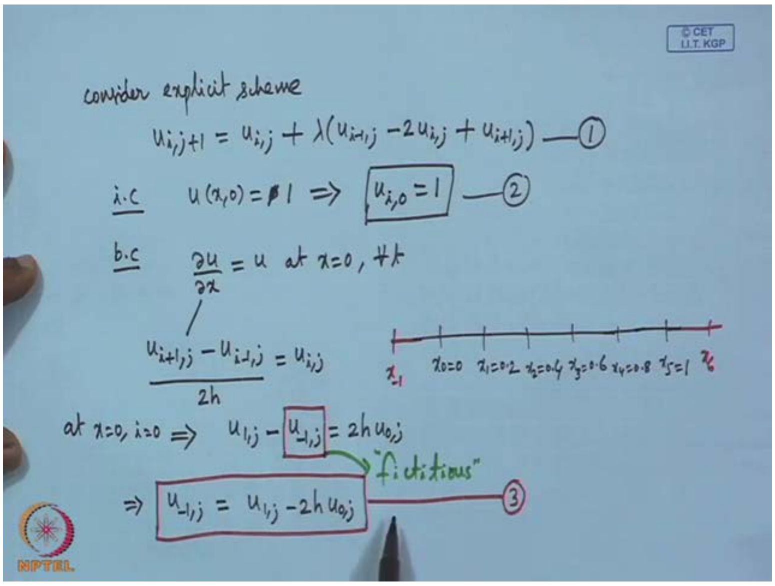

So consider explicit scheme so we have considered this as explicit scheme, now call this 1. Now, we have to discretize the initial conditions, so initial conditions if you discretize sorry this is 1. So, this implies u i 0 equals to 1 and call this 2. Now, consider the boundary conditions d u by d x equals to u at x equals to 0 for all t, so this was the condition we have one of a condition at x equals to 0. Now, if you look at grid x 0 is 0 x 1 is 0.2, x 2 is 0.4 x 3 is 0.6 x 4 is 0.8 x 5 is 1.

Now, what is going to happen here, if you look at this and then discretize, this is the derivatives with respect to x. Now, at the boundaries there is a big concern whether to use forward or backward or central, it is standard practice to use central, so one can justify that. So, if we use we are going to get u i plus 1 j j 2 h equals to u i j. Now, this value this is at x equals to 0 so that means at i equals to 0, so at x equals to 0 i 0 this implies u 1 j minus u minus 1 j is equal to 2 h u 0 j.

So, an important observation when we have this line, this is going to be x minus 1 similarly, this is going to be x 6 and x minus 1 x 6 both are outside the domain. However this quantity appears in our equation, since this is outside our domain we call this fictitious value, we have to handle this very differently. So, how do we handle putdown explicitly is equal to u 1 j minus 2 h u 0 j, call it 3. So, we have fictitious value getting introduced because of the discretization of the derivative boundary condition at the one of the boundaries, so that fictitious value we have written explicitly.

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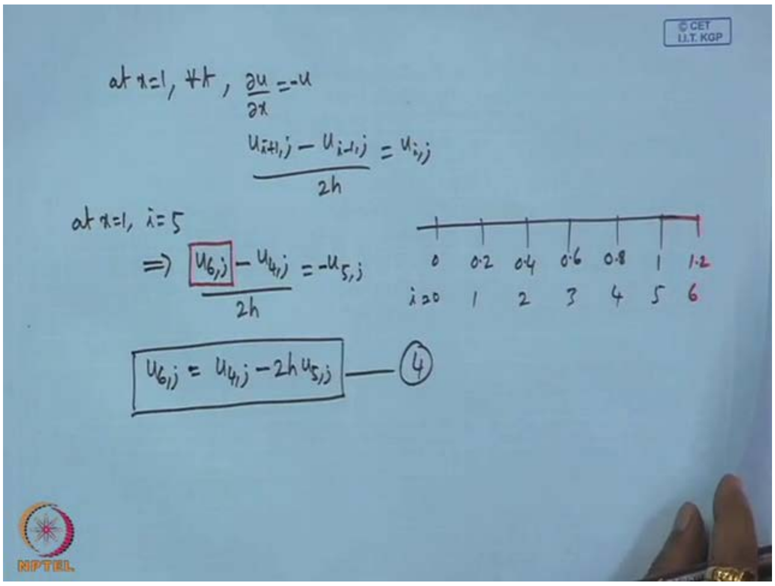

Let us do it for the other end as well, we have at x equals to 1 for all t, Dou u by Dou x equals to u, this is another boundary condition. Now, u i plus 1 j so this is the general discretization then at x equals to 1 i is so let us have the grid because we need it. So, this is 0, 0.2, 0.4, 0.6, 0.8 and 1 and this is i is 0 1 2 3 4 5. So, at x equals to 1, i is 5, now if you put that we are going to get u 6 j minus u 4 j by 2 h equals to u 5 j. Now, the boundary condition which I have defined at 1, this is minus sign here.

So, accordingly if we consider this minus sign we get a minus sign so that was the slip because I have the worked out example for this case. Now, as I discussed this suppose to be 1.2 for i 6, but this outside the domain. So, that is the fictitious value as well therefore, we write down u 6 j is equals to u 4 j minus 2 h u 5 j. So, to brief it we have the following, we have 1 the explicit scheme, which we are suppose to run at the grid points and we have the initial condition, then we have one boundary condition, we have discretize and as a result we got a fictitious value.

And we have expressed this in terms of a, another additional equation then another boundary condition, which we have discretize, we got another fictitious value and we have expressed in terms of another equation. So, let us summarize this equations, so we have u i j plus 1 where this is valid, I will get back to you in a movement.

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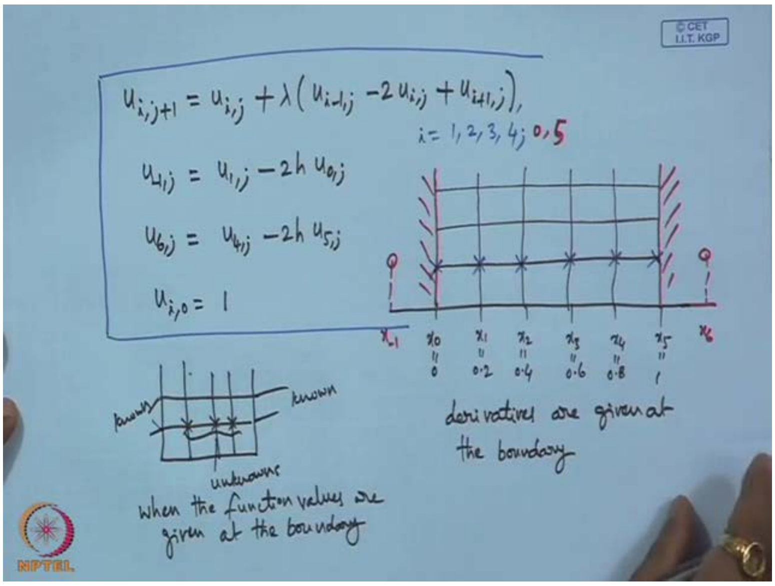

So, then we have u minus 1 j then we have u 6 j of course, we have the initial condition. Now, in order to decide where this is valid 0, 0.2, 0.4, 0.6, 0.8 1 recall the earlier cases where we have the boundary data explicitly in terms of the function value. So, then the values along the boundary are known to us at every time step. However in this case because of the derivative boundary condition, we do not know so let us say the time stepping corresponding to the time stepping.

So, in the earlier case when the function values are given explicitly I will give an example. So, the function values are given explicitly let us say then these are known similarly, these are known. So, the unknowns are in this case unknowns what is this case, this case is when the function values are given at the boundary, but what is this case derivatives are given at the boundary. So, in this case the unknowns all of them are going to be unknowns because in this case we know this, but in this case since the derivative is given we do not know the value here, we have to determine.

So, for the present example 1, 2, 3, 4, 5, 6; so there six unknowns now how many equations we have, if u look at in some sense you have four equations. So, we need two more equations how we are going to get in the earlier case the discretize equation we were running for i equals to 1 2 3 4, but we were not running for i equals to 0 and i equals to n. But in this case the derivative boundary condition introduced fictitious values which are outside. So, this is x minus 1, corresponding fictitious value and in this case x 6 and the corresponding fictitious value. So, if you run this equation for i equals to 0 as well as 5, so let us see what happens if you run i equals to 0, there we are. We are going to get u minus 1 and if you run i equals to 5 we are going to get u 6. So, that means two more additional equations we are going to get, so that may it is all the issue.

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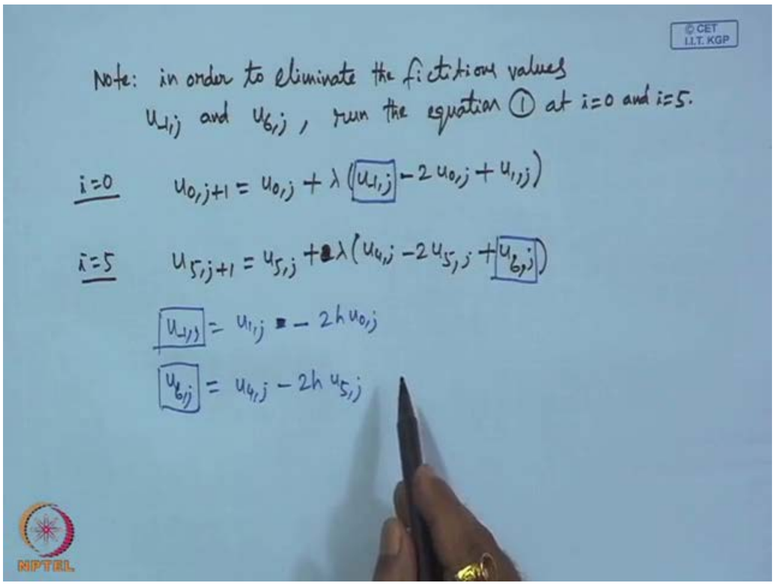

So, let us try to do that note in order to eliminate fictitious values u minus 1 j and u 6 j run the equation one at i equals to 0 and i equals to 5. So, if we do that then similarly this is lambda u 6 j. So we have the fictitious value here we have the fictitious value here then we had minus and u 6 j equals to u 4 j. So, we can eliminate these two fictitious values.

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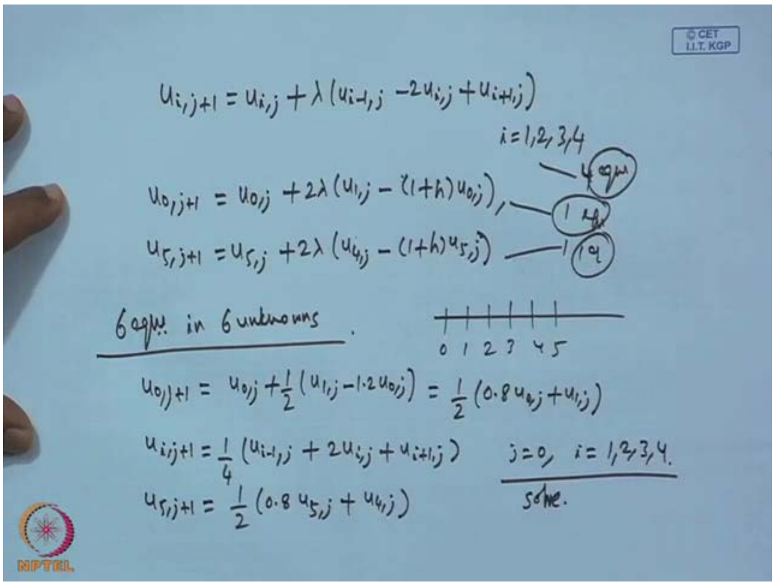

By eliminating we get this is original equation, which we have to discretize and obtaining which is still valid at 1 2 3 4 then we have eliminated the fictitious values, so this is all the unknowns. So, I am making it shorter 0 1 2 3 4 5 at these values we can compute. So, this is for 1, 2, 3, 4 and this is for 0 and this is for 5 and you can see there is no fictitious value. So, four equations here, this is one equation, this is one equation so total six equations in six unknowns.

So, we can compute for the current case, we can compute this is going to be some value in this u i j plus 1 and u 5 j plus 1. So, we can compute by considering j equals to 0 and i 1 2 3 4 solve. So, the main idea in this is the derivatives when you discretize they get additional terms, both to the left and the right and these are called fictitious terms. And in order to eliminate them what we do we run the discretize equation at the end points as well, which was not the case when the function values are prescribed. So, when we run at the end points the fictitious values are introduced in the equation and we eliminate this fictitious value across and we obtain a consistent system. So, may be the tutorial we see the exactly how problems on this can be solved.

Thank you.