Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 26
Consistency, Stability and Convergence

Hello, having discussed both explicit and implicit methods for a parabolic PDE. Also discussed about the local truncation error and then how to estimate it for a particular approximation, we must discuss more of these approximations. So, what are the properties any approximation any method should have. What kind of a main features? We have to pay attention. So, the main features are consistency, stability and convergence of particular approximation.

So, when I say consistency stability and convergence which comes first. So, there could be a debate, but in general when we talk about any numerical method people immediately talk about a convergence, whether your method converges or not. So, may be this is more intuitive, so immediate concern for us. So, let us start with a convergence. So, what do you mean by convergence, for example, we come across several incidence where the convergence has to be talked about. So, when do you say something converges.

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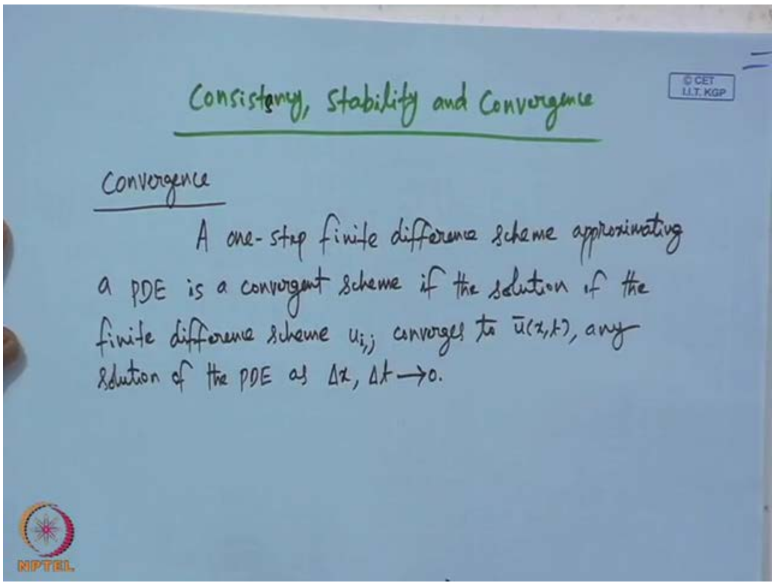

So, for example, in this context we have approximated the corresponding PDE by a particular finite difference method. So, then we are solving, the solution obtained should converge to the original solution. That is the exact solution of the equation. So, let us put down formally and then proceed further to talk about consistency and stability.

So, the title I put consistency stability and convergence. So, as I mentioned I would like to talk about convergence first. So, as I mentioned briefly a one step, we need a formal definition that is why I would like to put down here. One step finite difference scheme approximating a PDE is a convergent scheme if the solution of the finite difference scheme say call it u i j converges to u of x t, which is any solution of the PDE as delta x delta t goes to 0 so, this is a formal definition.

So, one step finite difference scheme approximating a PDE is a convergent scheme if the solution of the finite difference scheme u i j converges to the exact solution which is any solution of the PDE as these parameters goes to 0. So, as I mentioned once we talk about the convergence, there are two more you can see stability and consistency. So, even consistency if you see consistency for example, when you talk about stability of a method, see suppose somebody is little perturbed it is ok, but somebody is more perturbed then definitely we say the person is not stable or the system is not stable.

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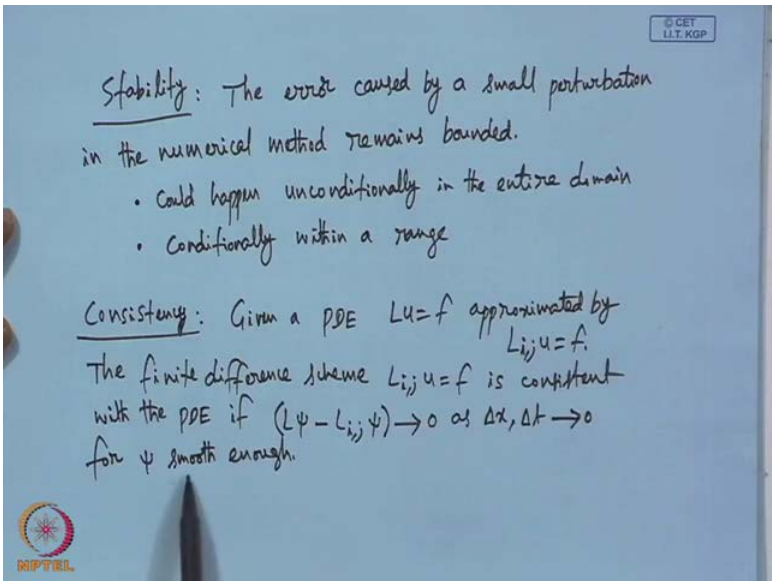

So, that means it is a general usage and even in this context as well it is same. So, in the sense if you give small perturbations and then if the small perturbations remain small then there is no issue, however if the small perturbations grows up, grows up, grows up and then blows up. So, then definitely the particular system is not stable. So, let us see the formal definition of stability.

Stability, the error caused by a small perturbation in the numerical method remains bonded. Now, for a specific method this could happen independent of any conditions. However for a particular method, this could happen subject to some conditions. So, the remarks, this could happen unconditionally in the entire domain of a domain of definition or conditionally within a range. So, will when we come to a specific method we talk about this.

Then, the next consistency given a PDE L u equals to f approximated by say L i j u equals to f, then the finite difference scheme l i j u equals to f is consistent with the PDE. If L psi L i j psi goes to 0 as delta x delta t goes to 0 for psi smooth enough. So, that mean, let us have a understanding given a PDE L u equal to f approximated by this, then the finite difference scheme l i j u equal to f is consistent with the PDE if l psi minus l i j psi goes to 0 as this parameter goes to 0 for psi smooth enough.

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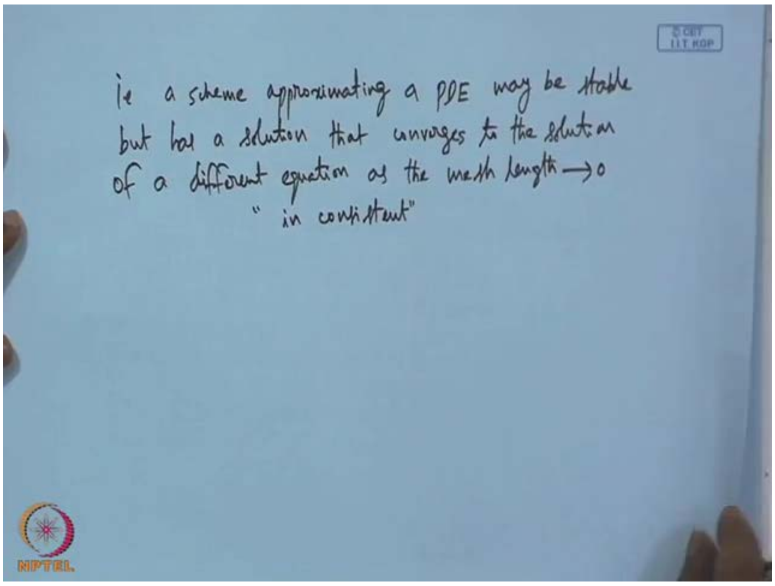

So now what could go wrong, what could go wrong? So, that means a scheme approximating a PDE may be stable, but has a solution that converges to the solution of a different PDE different equation. So, this is inconsistent that means a scheme approximating a PDE may be stable, but the solution converges to some other some other PDE as the mesh length goes to 0. So, this is inconsistency.

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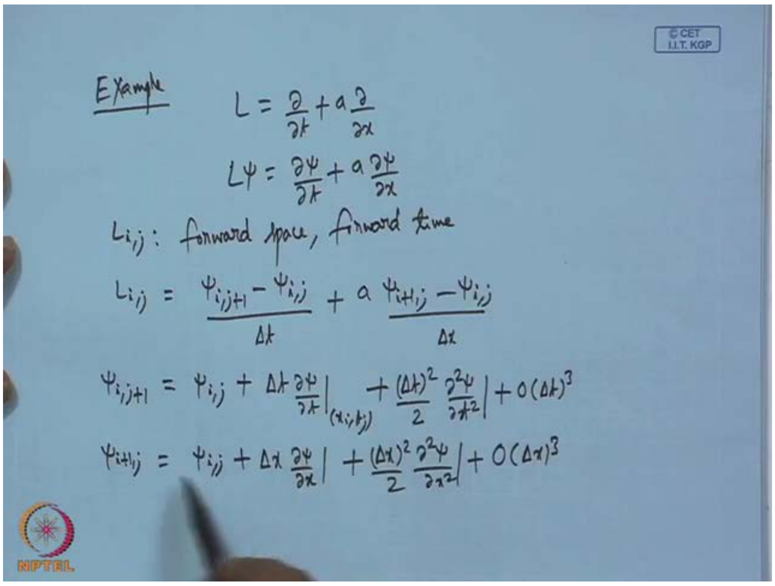

So, let us consider an example. So, let us say our operator is, first order for the time being then L psi is, then L i j say forward space forward time. So, accordingly we get L i j to be. So, we get psi j plus 1 j by delta t plus constant times forward space . So, this is our L i j, right now in order to compute the error we have to expand in Taylor series. So, let us expand psi i j plus 1. So, this is evaluated at t squared again evaluated, at this point plus. So, similarly this we have done number of times. So, now if we substitute then L i j psi is L i j psi becomes we can just straight away substitute.

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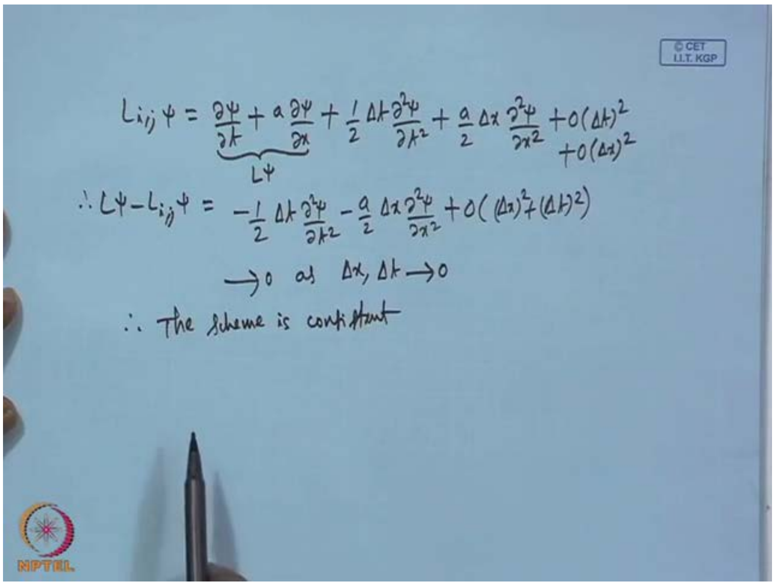

For example, we can see this sitting here. So, this get cancel. So, first term here delta t also cancels. So, first term is dou psi dou t and here this term cancel with this delta x cancels with this. So, first term is a times, this is just an algebra. So, if you do that we do that we get you can verify some series of notation you could have written simply one we go therefore, L psi minus L i j psi equals to. So, this will be which goes to 0, therefore, scheme is consistent. So, the scheme is consistent now we should talk more about consistency, stability, convergence, what is a interplay among these three properties.

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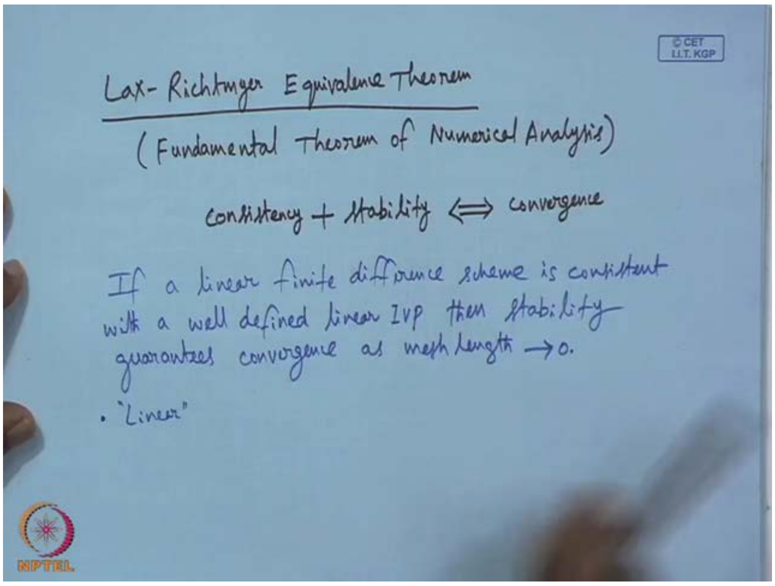

So, there is an interesting theorem in numerical analysis which talks about this. However there are some restrictions in order to apply this theorem. So let us see which ensures a what. So, the theorem is Lax Richtmyer Equivalence Theorem. So, this also called fundamental theorem of numerical analysis. So, before putting the statement the net gist of this consistency plus stability both sides I am putting consistency, stability ensures convergence, convergence ensures consistency and stability, this is the gist of the this theorem. However as I mentioned there are some restrictions.

So, let us put down formally if a linear finite difference scheme is consistent with a well defined linear IVP that is a initial value problem. Then stability guarantees convergence as mesh length goes to 0. So, this says if a linear finite difference scheme is consistent with a well defined linear IVP, then stability guarantees convergence as mesh length goes to 0. So, method that is consistent stable convergences guaranteed but, in general it is both ways however the main restriction is the remark is linear.

So, this is quite important because in general people look for non-linear because several of the challenging problems involve non-linear stuff. However this is restricted to linear problems. Moreover consistency stability ensuring convergence is trivial in some sense however the theorem assures the converse convergence implies consistency and stability as well.

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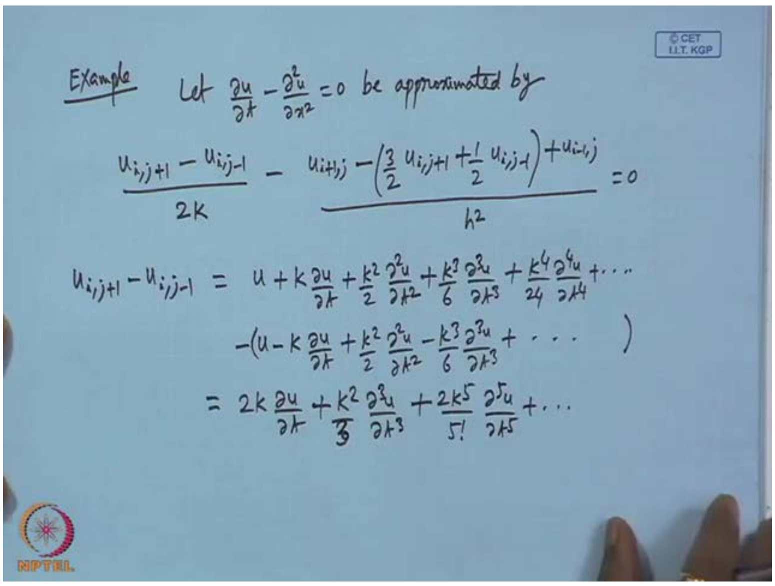

So, let us proceed to a little complicated example the second order. So, that we get some idea. Let this be approximated from delta t I am using k. So, you may be thinking every time the same example so for a change we have gone to a different example you can see. So, this looks little heavy let us see how we proceed. So, the story is the same, we have to expand in Taylor series and collect the coefficients etcetera.

So, let us consider term by terms. So, this is one term and this and this one term and this is another term for me. So, u i j plus 1, this if you consider u plus k dou u dou t. So, I am dropping the evaluation at x i t j, it is understood minus u. So, this become u, u get cancelled, you get 2 k right then this k square by 2 with a negative sign this get cancelled. So, the next term by 3, here actually. So, there is this should be 3 so then the next term.

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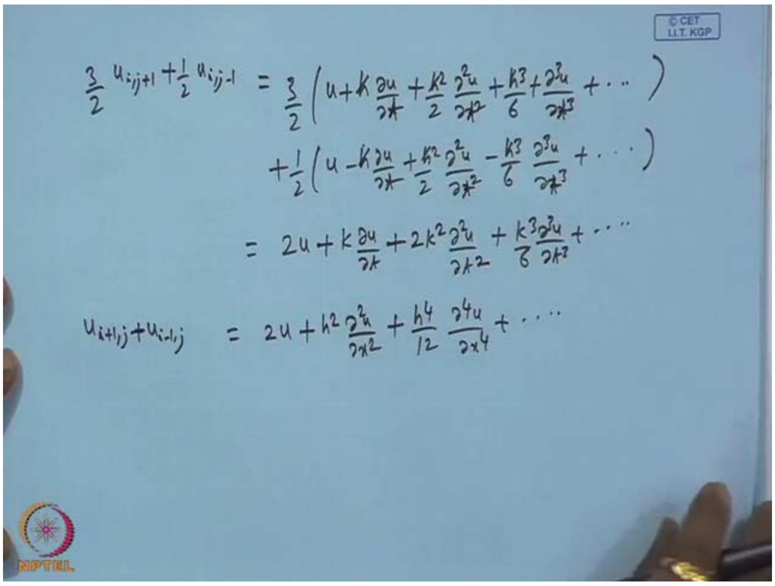

So, if you consider this is just a labor alternate signs. So, ultimately we get this is j plus 1, all should be t there apologies this is 2 u and all this k. So, then we are left with 2 more terms this and this. So, that also we can compute so many times we can guess. So, there will be u and u that cancels and here with a plus h and minus sign and here with a minus h so it will be 2 times. So, we get there is a this is a plus sign then u u that will be 2 u then plus h minus h they get cancelled, plus h square term becomes twice. So, h square by 2 is twice then cubic terms cancels.

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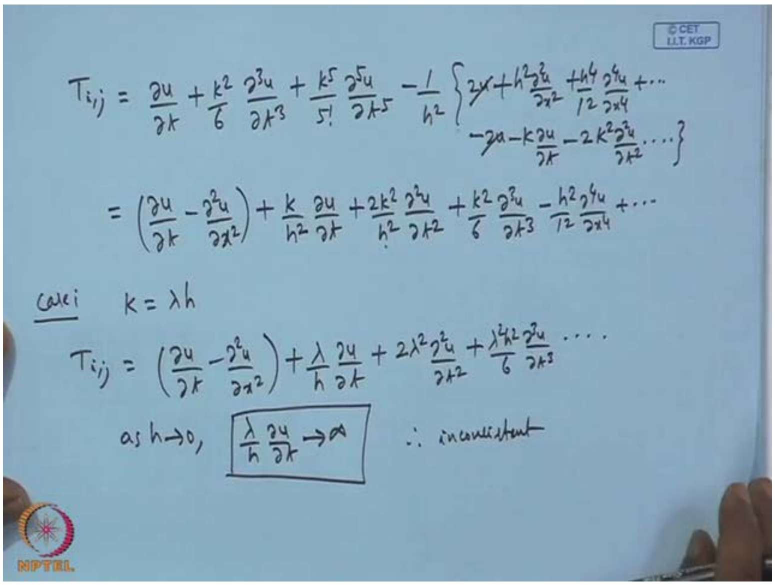

Now, using all this t i j become minus 1 over h square. So, I am giving only few terms. So, this is an important exercise so even though there is some labor involved better to do this. So, this clubbing we can see this get cancelled. So, h square cancels you get dou u by dou t minus this term is what I put. Then the next term I am going to put is this one, then the next term is this, then I am putting this, then we have this term.

Now, case one see even though we clubbed these terms because this is our actual equation, but unfortunately there is one more term with a first derivative first derivative with respect to t right. So, we have to analyze carefully suppose k is lambda h. So, lambda is our grid parameter say k is lambda h. So, then t i j reduces to plus k is lambda h. So, you have lambda by h plus 2 lambda square 2 lambda square k is lambda h. So, you have you have one term. So, let me check k is lambda h. So, you have this then etcetera. Now, as h goes to 0. So, this is trouble maker because as h goes to 0 this is not getting observed in this therefore, inconsistent therefore it is inconsistent.

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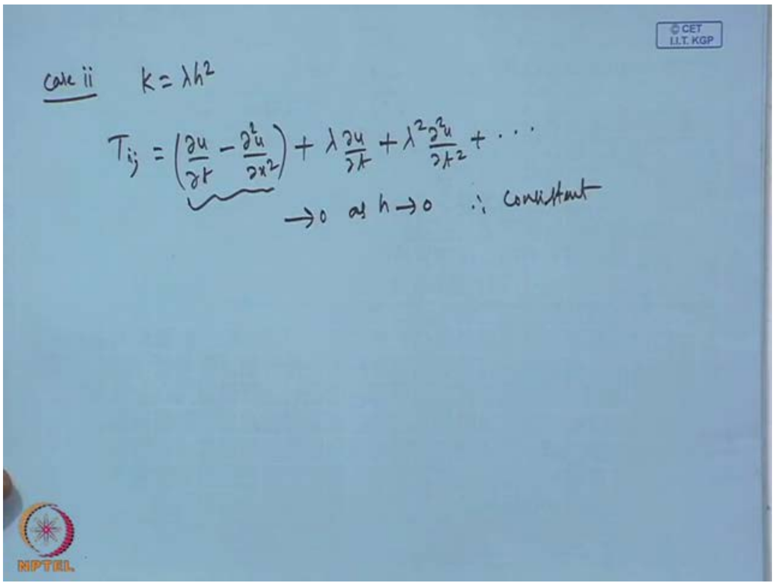

Suppose case 2 k is lambda h square then t i j, this goes to 0 as h goes to 0. So, therefore consistent therefore, in this case the method is consistent it is agreeing with the original PDE. So, this test is very much important if you come across any finite difference scheme approximating a given PDE, we do this test for consistency. Now, having done consistency check let us see whether we can discuss about stability of a method.

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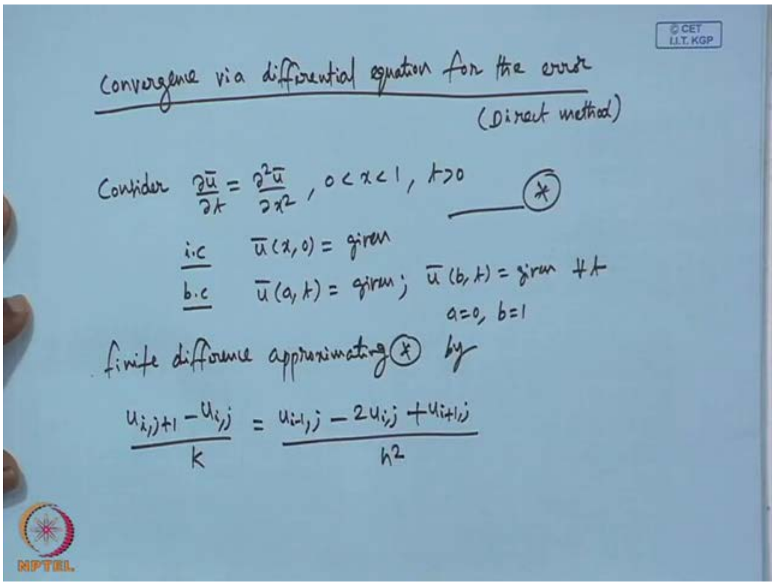

Stability and convergence, so convergence via differential equation for the error, so this is called direct method. So, consider so I am putting a bar just to make sure that this is exact then initial condition see u bar is given, then boundary condition u bar some at is given u bar b t is given. So, in this case a equals to 0 b equals to 1. So, this is for all t now consider finite difference approximating star by u i. So, now we would like to talk about convergence right. So, what would happen at each grid point there will be some error.

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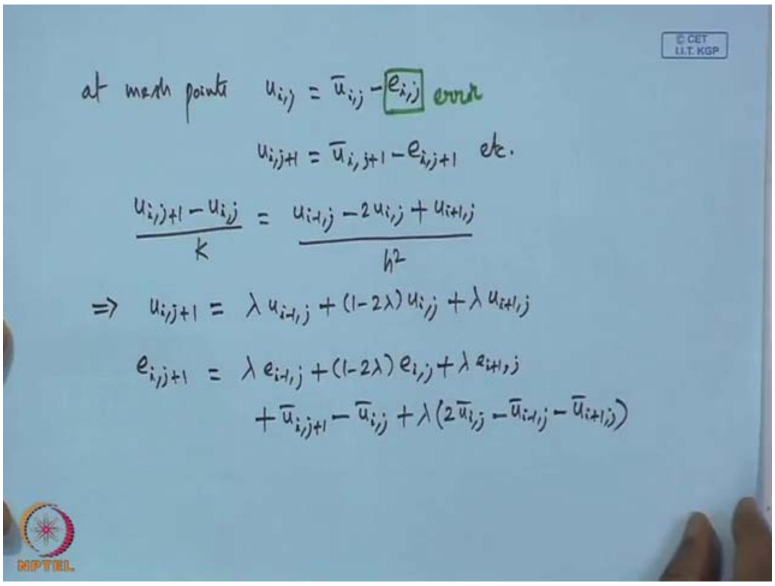

So, at mesh points, this is a discretized and say this is exact, they differ by this is the error then following this answers. Now, consider our scheme substitute this in this scheme we get. So, left hand side u i j plus 1 minus u i j, so you can convert this into equivalent form. So, that will be u i j plus 1 equal to lambda u i minus 1 j plus 1 minus 2 lambda, this is our standard form. So, then let us substitute these answers. So, then what will happen we get here u i j plus 1 minus.

So, we retain error to the left hand side we get lambda, then plus the exact solution we get this. Now, our aim is to get an equation for the error and then estimate the error and see whether we can make it bounded whether conditionally or unconditionally. So, that we talk about the convergence and stability.

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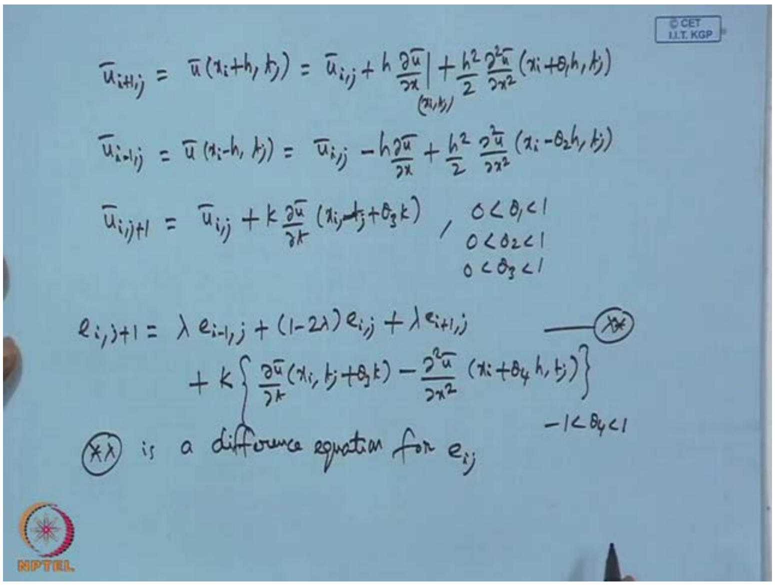

So, now using Taylor series which is a only first term I am writing. So, this I will put it as a error term x i plus some theta 1 h. So, this is the error term this is evaluated at x i t j. So, then this is Taylor's remainder term then say some theta 3 k. So, h is increment for a space and k i is increment for time. So, where this is standard Taylor's remainder conditions, so then using this in our error equation plus k times, so where from these 2 we get this. So, this is double star is a difference equation for e i j. So, this is the difference equation for e i j.

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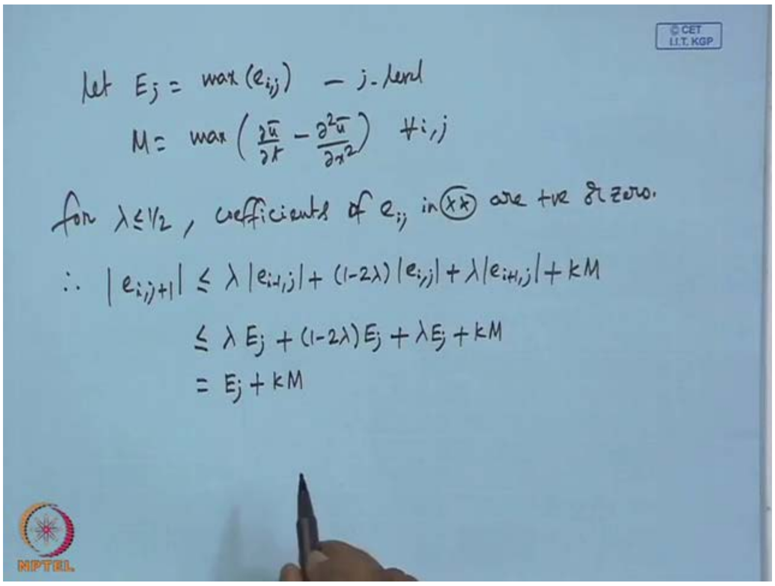

Now, let e j is max of e i j. So, that is at j th level and m is max of then, if you carefully observe. So, we have 1 minus 2 lambda therefore, for lambda less than or equals to half coefficients of e i j in double star are positive or 0. This is an important remark therefore so for this we need positive or 0 because we need an estimate. So, therefore for this range coefficients positive or 0 therefore, mod of this is less than or equals to lambda mod e i minus 1 j plus 1 minus 2 lambda j plus lambda plus k m, how we are getting carefully observe. So, let me repeat we have considered the answers this is a approximated with exact differ by an error.

At each grid point we play with the index, then this is our difference scheme. Then you substitute these answers in the difference scheme we get this. Then we expand in Taylor series and substituted back. So, we get this now this is a difference equation for e i j. Now, let at j th level the maximum error because at each grid point the error is different. So, the maximum is E j and correspondingly you call this. Now, we are estimating the bound. So, this is we called already m therefore, we get this E j. So, this is right now we obtained.

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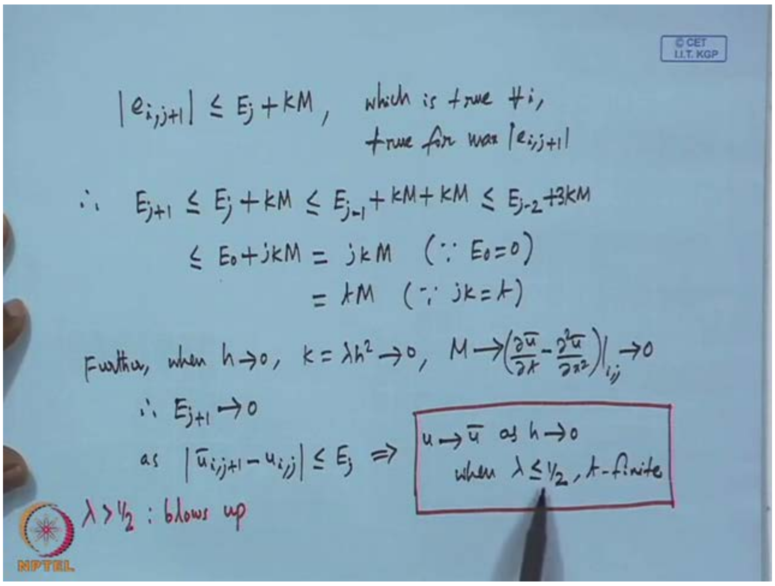

This is less than or equals to E j plus KM and this which is true for every i hence true for max. Therefore so, max is E j plus 1 are you getting. So, this is true for every I therefore, true for max. So, which is nothing but capital E j plus one, so this is E j plus K M, now you iterate. So, this if we do here 2 times we get 3 times so, on if you do. Now, what will be this any idea E 0 error at t equal to 0. So, there is no error and j k is t, since this then this is t M. So, further when h goes to 0, K is lambda h square goes to 0, M goes to at i j. This also goes to 0 therefore, M goes to 0 therefore, E j plus 1 goes to 0.

That means any E j, as E j this implies u goes to u bar as h goes to 0 when lambda is less than or equals to half and t finite. So, this is our net conclusion u is agreeing with the exact only when lambda is less than or equal to half. So, the remark is lambda greater than half the error blows up you can verify. So, in this case, this is conditionally that means the error is not growing, therefore it is a stable only in this range. So, this is called conditional stability.

Suppose, you do not get any condition and then we say it is unconditionally stable right. So, we estimate the bounds the domain in which the method is stable by using this error answers. So, that means we formulate the corresponding differential equation corresponding to a given finite difference scheme. Then we find estimates and we find the domain within which the method is stable. So, we discuss more, there are other methods to talk about stability. So, we discuss in the coming lectures until then.

Thank you.