Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 25
Implicit Methods for Parabolic PDEs

Very good morning, in the last lecture we had a introduction about finite decimals approximations to parabolic PDE's, and also we have discussed implicit method. So, that means considering the time levels to compute the values at a particular time level, we need the data the past time level. Further the values at the next time level are obtained explicitly in terms of the past time levels. So, today let us see about implicit nature of a finite difference approximation for a parabolic PDEs. Still we are considering the heat conduction equation. So, that is the reason I kept the title as implicit method for parabolic PDEs.

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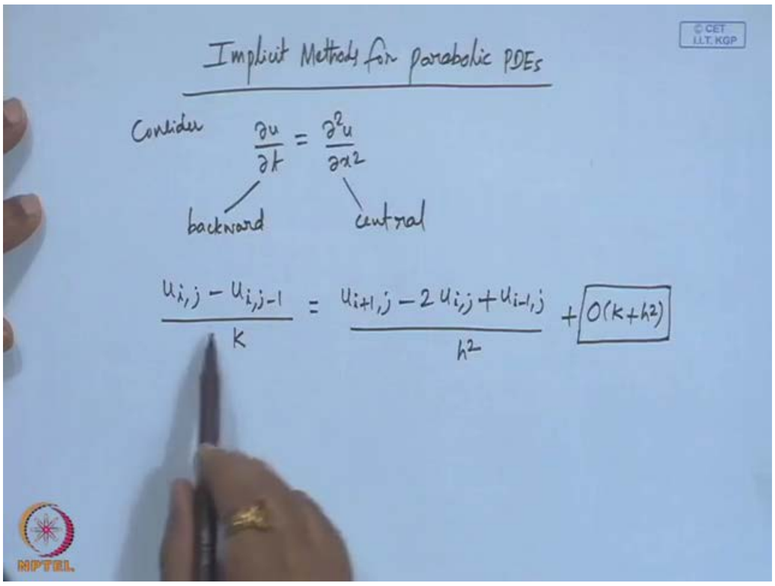

So, we consider the heat conduction equation. Now, in order to obtain in the last we have for a explicit, we made central and then this is forward. Now, what we do is backward for this and central for this. So, accordingly we obtain, now when you approximate backward, there is non zero leading coefficient starts k, when you approximate central this starts at. So, I am not talking about this right now, because we learn about this in detail in the coming lectures.

So, now even though we have written see for example, this is backward right j j minus 1, but for a convenience what we do for convenience, let us write down the same formula at j plus 1 in a. because this formula contains j and j minus 1, but it is a general notion that we represent using j plus 1 and j. So, it is just a convenience.

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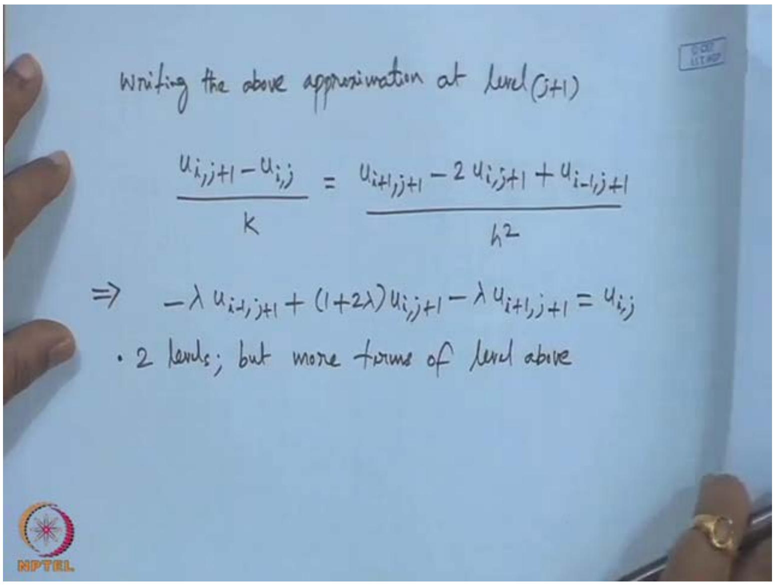

So, let us write down this at level j plus 1. So, writing the above approximation at level j plus 1 we get. So, that means j j plus 1 that means j minus 1 will be j. You can see now you can see two levels involved j plus 1 and j. So, this can be simplified, so lambda is our grid parameter. So, left hand side you have terms at level j plus 1 right hand side we have terms at level j. So, this is some sense two levels involved, but more terms of level above. So, this is an important remark, more terms of level above than the level below.

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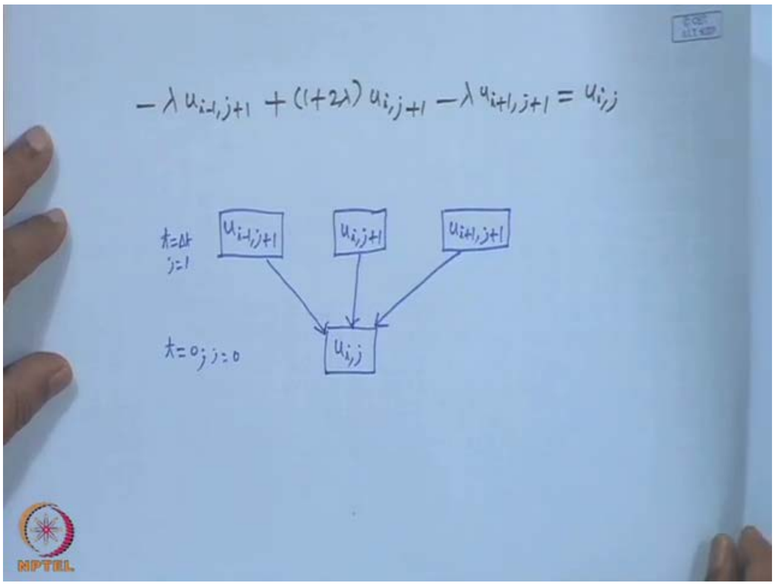

So this is an important remark. So, this shows the implicit nature of the scheme, so how this is going to give us the implicit nature. So, let us have a look, the formula we have which means suppose this is u i j. So, this is in contrast to the explicit scheme which we have derived in the last lecture, you can see more terms at level j plus 1 compared to the level below. So, how this is going to bring up the implicit nature we will see. So, this corresponds to say T equals to 0 and this is nothing but j equals to 0. Then when T is delta T j is 1.

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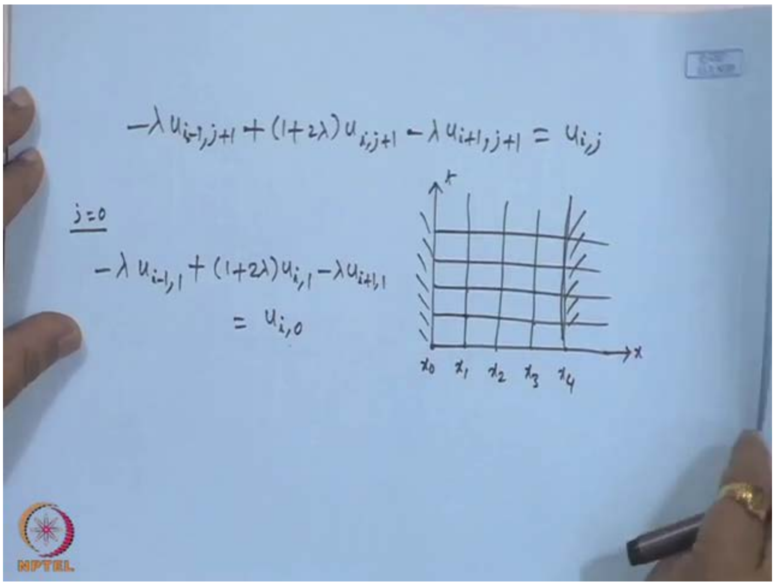

So, let us run through it with a with a specific case then see how the implicit nature comes into picture. So, we need the equation again this is the equation u i minus 1. Now, for example, let us say this is t, so if you see let us say these are the boundaries. So, this is x 0 this is x 1 x 2 x 3 and x 4, see this is our boundary and this is our boundary. So, essentially we need to solve at the internal grid points these are the grid internal grid points because these boundary points we know the solution. Now, let us run the solution at consider j equals to 0. So, what we get 1 1 1.

So, this is our equation if you consider j is 0. So, that means in this expression we know only u i 0 values because these are at higher time level. Now, still we have to realize the implicit nature. Now, we start the what will be the first value of i for which we can run this equation see because of this term if you put i equals to 0. So, you get u minus 1 which is outside therefore, the first value of i for which we run the equation is i equals to 1.

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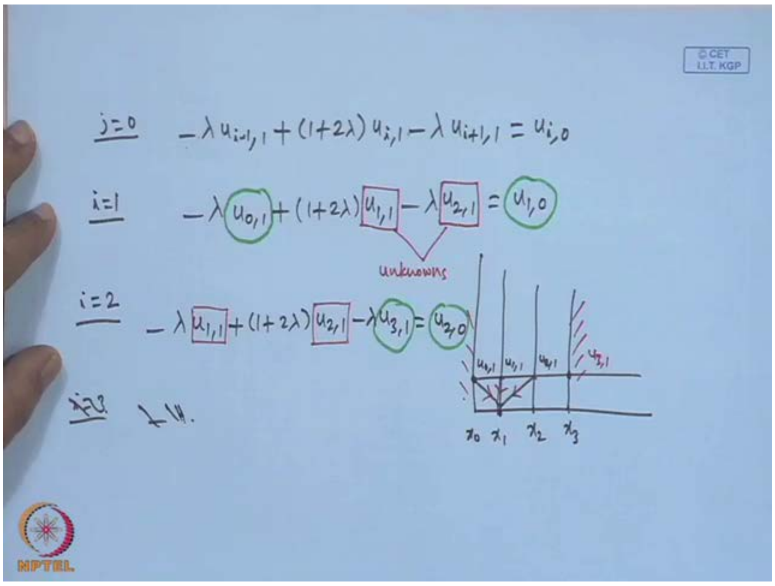

So, let us run this equation for i equals to 1. So, the equation we have for j equals to 0. Now, if you run equation for i equal to 1. So, what is this doing so u 0 1. So, this is x 0, x 1, x 2, x 3 say. So, u 0 1 will be this, then u 1 1, u 2 1. So, this is connecting. So, that means out of four quantities you know only this value, so these two are the unknowns involved in this equation. So, what are the unknowns, so these are the unknowns. So, now let us run the equation at i equals to 2. So, then we get here what are the unknowns and assuming this is boundary in this case. So, this is u 3 1, this should be known quantity. Now i 3, if you if you run i 1, i 2, i 3 is on the boundary, this will introduce fictitious values so we do not go for i equals to 3 right.

So, what happened in this case we get one equation where you have two unknowns in second equation we have two unknowns and this is unknown. So, let us give some other color. So, this is known quantity this is known quantity and this should be known this should be known, so two equations and two unknowns. So, you can see the implicit nature, the implicit nature of this scheme is visible because it involves two time levels, one is j and j plus 1 of course, in a explicit method also the same. However in the explicit method you have one quantity at level above and you have more quantities at level below to the right hand side.

So, for each value of a time step you get explicitly the values at each nodal point. However, in the implicit what happened we have seen we get a system of equations. So, how do we get systems of equations because we have more quantities at level above than level below. So, this brings a implicit nature of the scheme. So, let us see in detail. So, I would like to explain with more in detail.

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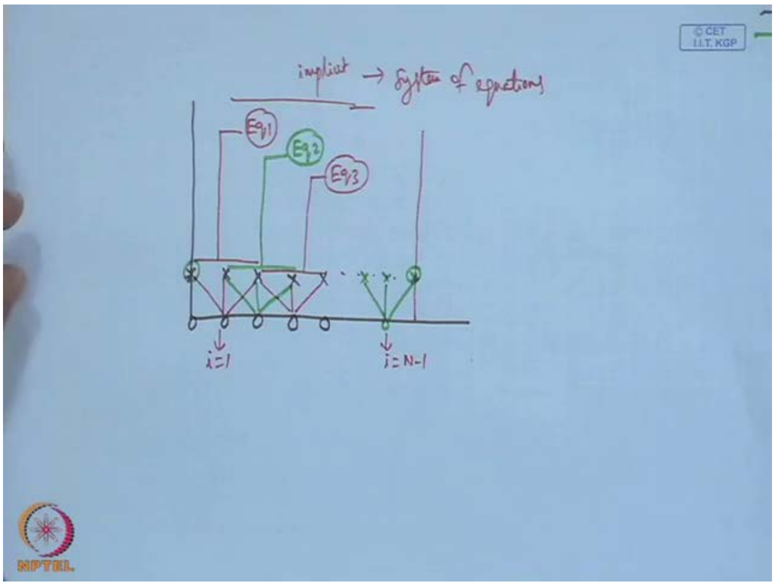

Suppose these are the boundary points, first time when you run these are the three points connected to this point and we get equation one. So, next time when we run the equation. this is another boundary point. So, then this is another grid point let us say. So, then next time, then what will happen for these, you get equation 2, then next time these three points are connected and you get equation three. So, we march past along the nodal points this. So, we get in equation however one boundary point is removed two at end points. You have two unknowns this and this whereas, in the second equation we have all three unknowns then we do like this.

So, if we do continue like this, at the boundary point what we will have this and this so they will be connected with. So, here this is a known. So, at the equations where we run corresponding to corresponding to i equals to 1 and corresponding to i equals to n minus 1, we get one known quantity here and one known quantity. So, the remaining so this forms implicit and this is nothing but system of equations. So, let us see with an example.

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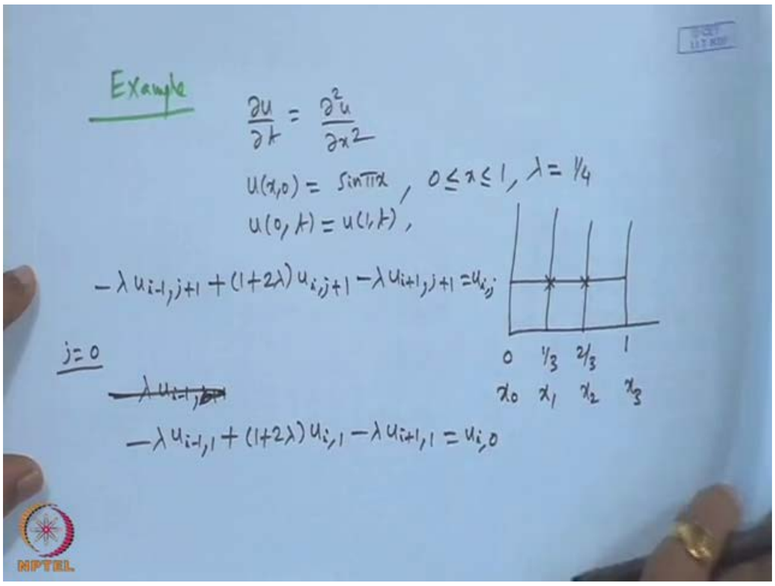

So, the same example which we took in last lecture. So, 0 1 this is 1 by 3 and this is 2 by 3. So, this is x 0, x 1, x 2, x 4 now we want to run the scheme. So, the unknown points the internal grid points are this two. So, let us consider the formula right. So, consider the case where we have lambda equals to 1 by 4, now j equals to 0, so this is 1. So, this we are going to run i equals to 1 and i equals to 2.

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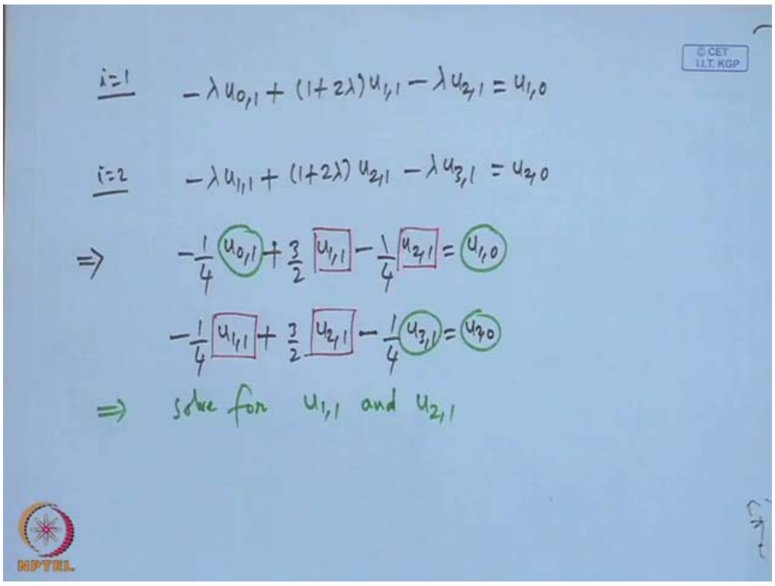

So, corresponding to i equal to 1, we get corresponding to i equals to 2, now if we use the values of lambda the system becomes, so lambda is 1 by 4. Now, let us identify the unknowns so these are the unknowns and so these are the known quantities. So, then solve for u 1 1 and u 2 1. So, this is can be done very easily by substituting the corresponding values. So, we have seen both explicit and implicit methods for the parabolic PDE. So, the explicit and implicit nature is coming because of the discretization corresponding to the time levels.

So, for example, if you do forward we have seen forward with respect to time, we have seen we obtained explicit, whereas if you use backward we obtain implicit. Now, I have been postponing to address one of the issues what is that the issue is, when you approximate a given PDE with suitable finite differences scheme, you are throwing away terms from certain order right.

So, what is this and then how do we conclude that whatever we have thrown is up to this order and what is this called. So, definitely you are true this is called local truncation error because when you approximate any particular scheme. For example, forward a time when you have done it what is happening, the difference and then the non 0 leading coefficient. If you consider that is multiple of k that is delta t.

So, that means it is a first order whereas, if we use the second order derivative approximation with central you are getting leading non 0 coefficient is a order of h square that is order of delta x square. So, that means this is a second order with respect to space. Now, combining this you have order of k plus h square. So, that means the error term is of this order. So, our next aim is to study about local truncation error in detail. So, let us talk about this.

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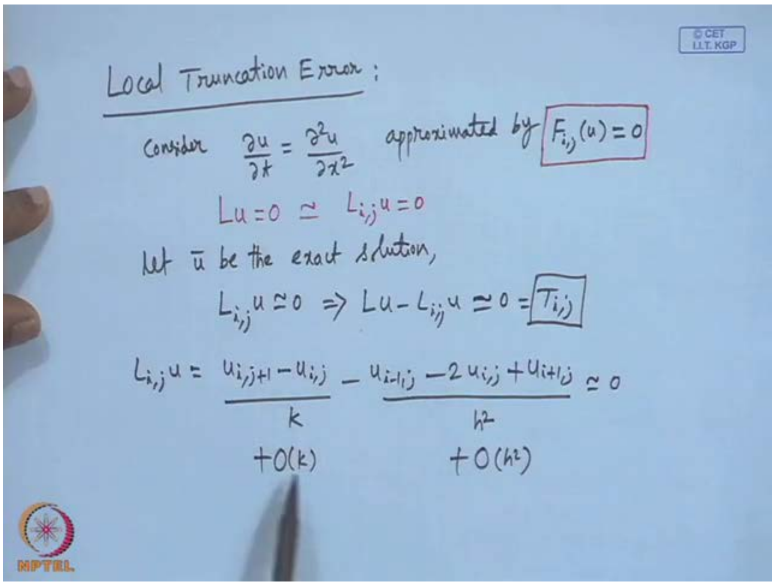

So, while discussing numerical solutions of CDE, we talked about local truncation error. Now, again the equation we consider, still the same. Now, let us say this has been approximated by f i j u equals to 0. So, that means this is the approximated operator. So, this is our original operator and this is the approximated operator. So, in some sense l u equals to 0 has been approximated by l i j u equals to 0. So, I have given f i j u, now let us say u bar be the exact solution. So, then what do you expect, exact solution satisfies this, but when you substitute exact solution in the discretized version the approximated version, we are not sure whether u this will be exactly 0.

So, the remark is l i j u is approximately 0. So, that means this implies l u minus l i j u should be approximately 0, which we call T i j. So, this is our local truncation error. So, let us see what will happen, how do we compute. So, we are going to compute. Our l i j u is, consider the explicit scheme. So, we are considering the explicit scheme, now this is approximately 0. So, whatever I have mentioned we are going to realize that what is that, earlier we were mentioning this is order of k and this is order of h square. Now, we are trying to realize that indeed the error which is involved in this approximation is in this order.

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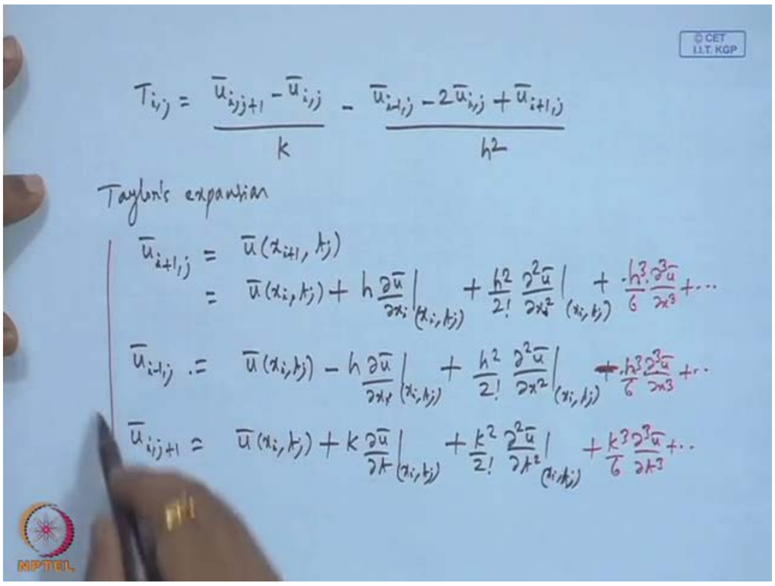

So, how do we do that by Taylor's expansion. So, consider error is nothing but you replace by the true in the discretized version. So, that is what I am going to do you take the discretized version and replace the true value. So, I have considered the discretized version and replaced by the true value that is u bar. Therefore, this should be the error. Now, we go for expansion as follows, u bar i plus 1 j is equal to this is nothing but u bar of xi plus 1 T j. So, if we expand I am using h notation instead of delta x.

So, h of course, this is evaluated at plus h square by 2 factorial this is evaluated at T j. Similarly, this is not component this we can keep it as a x actually indeed this you can leave it as x i or x because we are writing this notation. So, u i plus 1 we have computed i minus 1 and we have a with j plus 1. So, better we write this term as well. So, this is with k and the derivative is with T.

So, I am not writing the third term we can write it because we may use it later on. So, I hope you can see so this is by 6 and here with a minus sign and here. So, now all this we substitute in this. So, let us substitute then we get T i j is equal to, so let us write down the scheme so that we do not make any mistake. So, this is the scheme now we have to substitute the Taylor series expansion which we have done previously.

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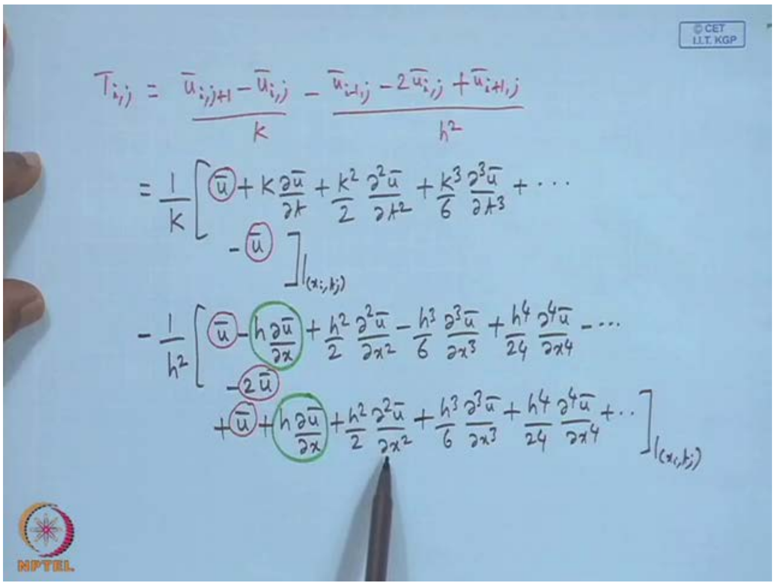

Therefore, this reduces to, I keep 1 over. Then j plus 1 this is u bar. So, I am not writing evaluated at x i T j that is understood please, then the next term will be k then u bar i j. So, this is we write it minus u bar, so this because there is no expansion u bar. So, I have already mentioned this is a evaluation at x i T j, if you want here you can write down for all the terms. Now, minus 1 over h square u i minus 1 j, this will be u bar minus h plus h square by 2 minus. So, it is a little lengthy algebra, but we enjoy at the end of it. So, this is plus I am going one more term. So, then minus 2 u bar, then the next term plus u bar plus h, everything evaluated at.

So, now let us see what is going to happen. So, we have this term cancels with this term then we have this cancelled. Then if we look at here these are the only terms which with respect to time derivatives. So, nothing from here may get cancelled, the first observation first look, but later on when we use the equation there may be some tricks can be done. So, but before we do that let us see the straight forward terms. So, these are straight forward, then this term goes away with this term.

So, the first term which we are going to get see k gets over, then this is the first term we are going to get. From here you see h square by 2, so h square by 2. So, we get h square dou square u bar by dou x square, then h square also get cancelled. So, the first term which we are going to get is dou u bar by dou T minus dou square u bar by dou x square. So, this is the first term, let us write down.

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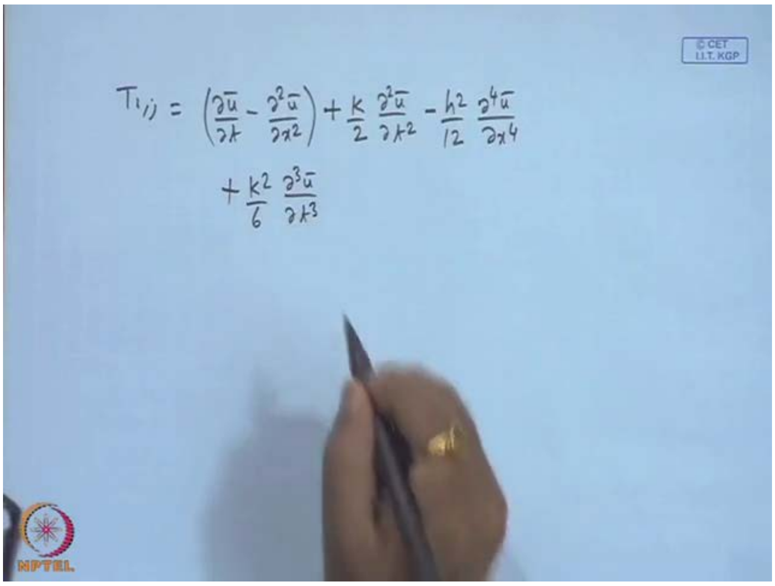

So, T i j reduces to of course, evaluated at x i T j, then to have a remark this is done. So, now what is the next term as I mentioned time derivative is only in this. So, may be let us put down the next term this one. So, I am putting a tick. So, k square by 2 dou square u bar by dou T square. So, let us write down the next term plus. So, it is not k square because there is 1 k there so it is k by 2.

So, this is done then these terms again get cancelled cube terms. So, they get cancelled, the next term we have in hand. So, this is this term with a negative sign and h square goes off. So, I am putting a tick. So, next term we are going to put down is these two. So, h square goes off and this is twice of it. So, the next term will be minus h square by 12. Now, next term k goes off, so I am going to put down this. So, the next term I am going to put then what would happen to the h 5 coefficient, here h 5 would get cancelled.

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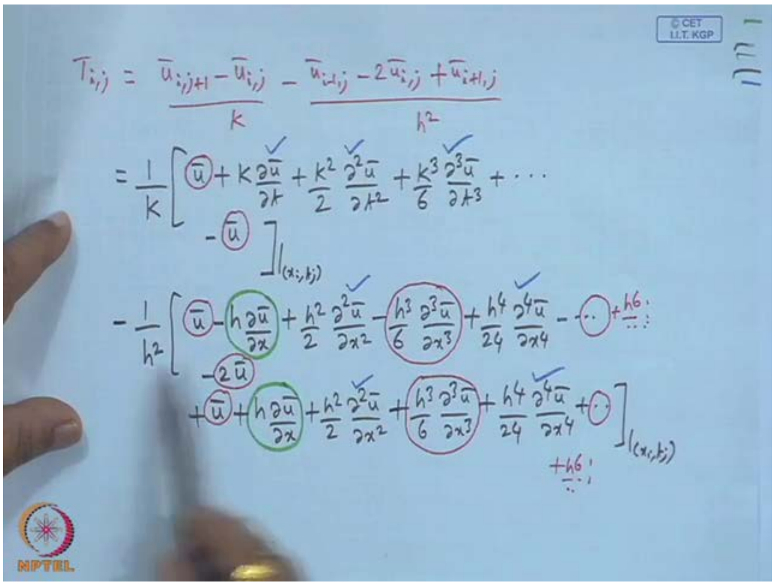

So, the next term should have been h 5 so that get cancelled. So, what is the next term. So, the next term here h 6 and here also h 6. So, h 6 by something that get cancelled h square cancelled. So, h 4 remain with the corresponding coefficient twice of it. So, let me put down that.

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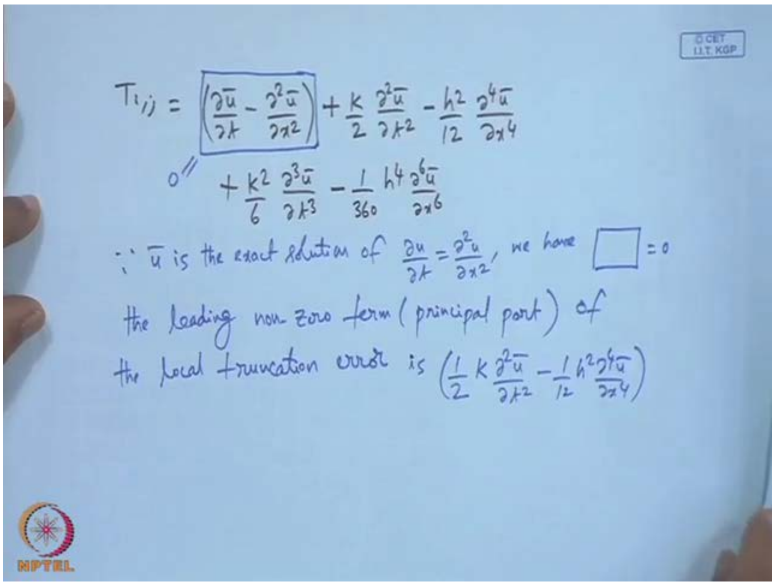

So, next term minus, we have written terms up to this T 4, the next term is T 4. So, I did not write down the T 4. So, this is what now let us observe closely what is this combination having observed that u bar is exact solution u bar is exact solution. So, what is this, since u bar is the exact solution of we have this box equal to 0. Obviously therefore, what will happen the leading non zero term, the same is called principal part of the local truncation error is leading non zero, because this becomes 0. Now, leading is coming from here. So, this becomes, a leading term, let us observe this carefully.

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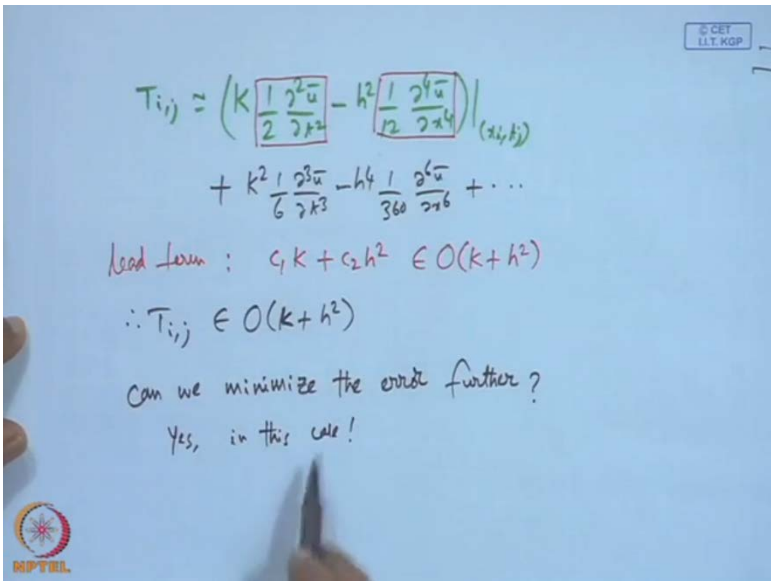

So, T i j reduce to, so I am putting k there half minus h square here, evaluated at x i T j then we have remaining terms. Now this quantity when we evaluated this they reduce to some specific value. So, the leading lead term becomes some c 1 k plus c 2 h square, assuming this is c 1 and c 2. So, this is nothing but is the local truncation error which we have seen. So, since this is the leading order we say therefore, T i j is this is what we have realized. So, this is what we have been writing when we have discretized. Now, having obtained this error, definitely our aim is to reduce the error, to minimize the error as much as you can.

So, here in this case we have obtained non 0 leading term. Then we have computed the estimate on that and we declare that the local truncation error is of this order, but however we are interested in minimizing the error as much as you can. So, this is what you should look for in general for any numerical method in general. So, now in this case is there a possibility to reduce the error further. So, let us observe closely and see whether there is a possibility.

So, if you observe closely, if this term is 0 somehow so then we can realize the next non 0 term will be this, so where you have a k square and h 4. So, that means you are going to minimize. So, the question is can we minimize the error further. So, can we minimize the error further. So, in this case at least the answer is yes in this case how let us see.

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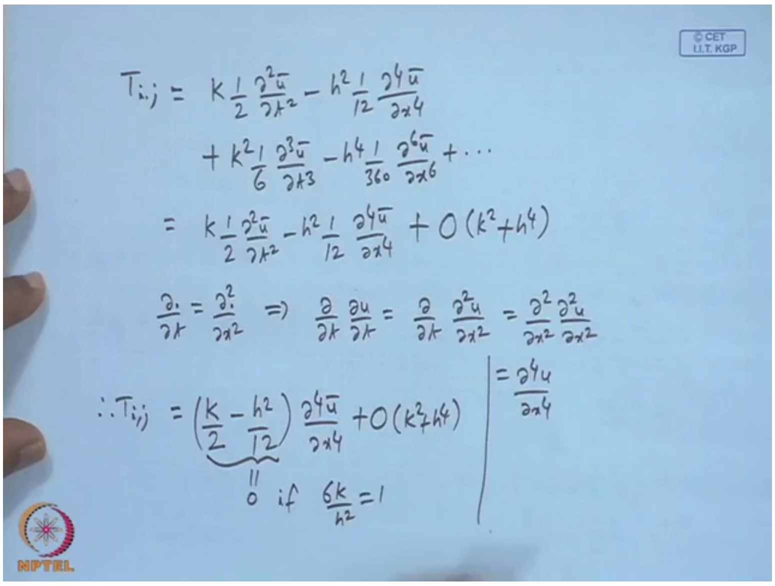

So, if you consider our T i j, so this is k times half we have this right. So, this is particular term is order of k square h 4. So, this you can write plus order of, so you can write. Now, let us play a trick, the trick is we have dou by dou T equal to this is from our equation right, hope you are this is just our equation with dot is u. Now, consider this is nothing but see 1 dou by dou T is square of it, right hand side already you have. So, this term is equal to this term right therefore, Y i j becomes k by 2 minus h square by 12, you can plus order.

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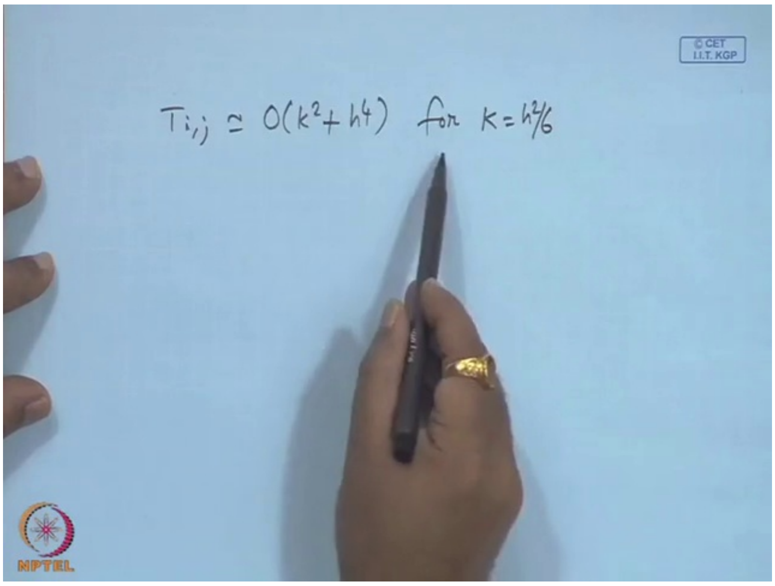

Hope you are able to follow right, this is the trick we did then this term I am writing I have taken substituted, for this term I have substituted this and taken common. This is the next non 0 term. So, now force this to be 0. So, this is 0 if 6 k by h square equal to 1. So, in other words T i j is of order k square plus h 4, for k equals to h square by 6. That means if you choose your discretization such that k is h square by 6, then we improve because the next non 0 it has gone 1 step further right.

So, the error is getting minimized therefore, there is an improvement. So, when you discretize, you should make sure that this choice of a step size is chosen such that the error is minimized. Now, this depends on the method, this depends on the corresponding approximation and for sometimes we may not be able to get exactly this kind of scenario. So, there you have to use your intuition and then come up with a some other technique.

So, what we have learnt so far, is we have learnt explicit implicit methods for parabolic PDE in particular we have considered the heat conduction equation, how to compute the local truncation error, estimate on the local truncation error. Now, there are serious concerns about these approximations. So, these concerns we will discuss in the coming lectures until then bye.