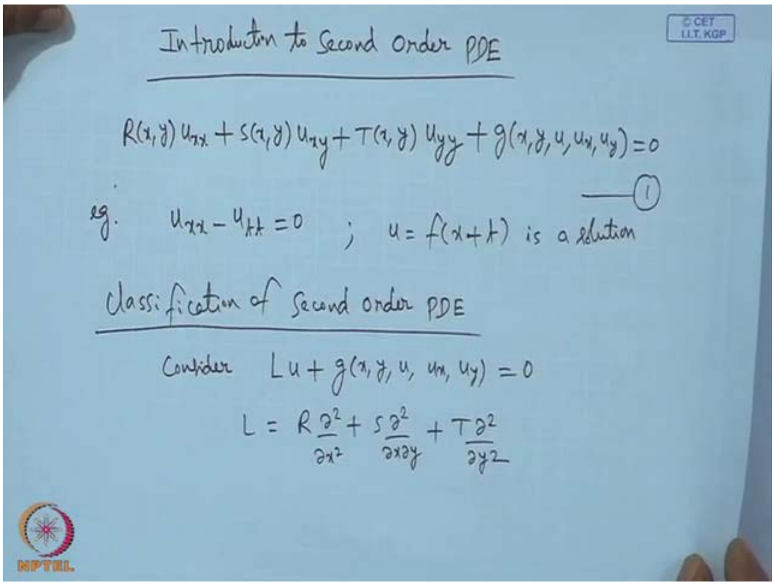
Numerical Solutions of Ordinary and Partial Differential Equations

Prof. G.P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 23
Introduction to Second Order PDE

Hello welcome back. In the last class we have discussed briefly about first order PDE, a bit of classification followed by Lagrange's method. How to get analytical solution and we tried to understand a bit of characteristics, what do you mean by characteristic curves? So, the solution of PDE represents a surface. Now, let us proceed to the second order bit of analytical touch. So, then we proceed to the numerical methods.

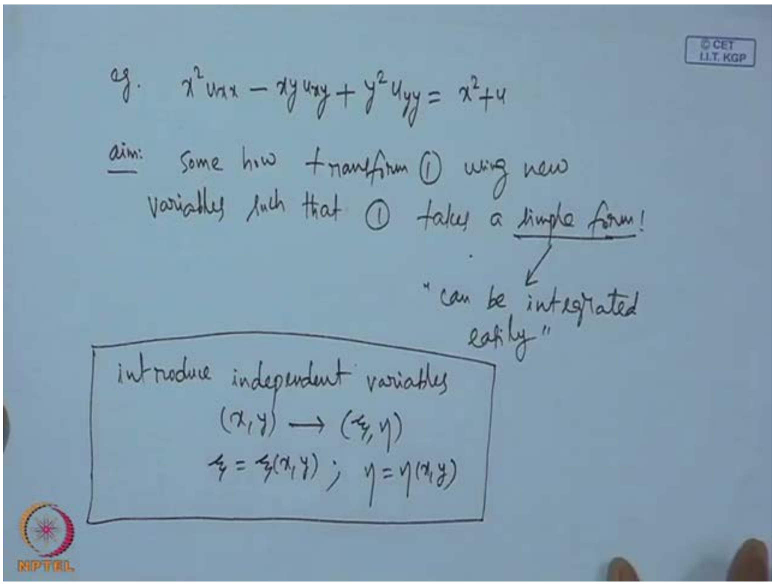
(Refer Slide Time: 01:01)



So, let us start with second order. So, more general represents, so u is our dependent variable and x y are independent variables. R, S, T are continuous functions of x and y, and g is this form. So, definitely this can be considered as semi linear. So, what do you mean by solution of this solution, of this means any u which is u of x y must satisfy 1. So, for example, consider so for this u equals to f of x plus t is a solution, because it satisfies this equation.

So, now we would like to classify classification of second order PDE. So, let us consider 1 in the operator form, where l is given by, so I would like to draw your attentions, some books put a 2 3, but i am considering just as instead of 2 as so accordingly things would change a bit. Now, what do you mean by classification and then how do we get the solution once we classify. See, if you look at this as such as such from the way it appears, it would be very difficult to guess what could be the solution, okay?

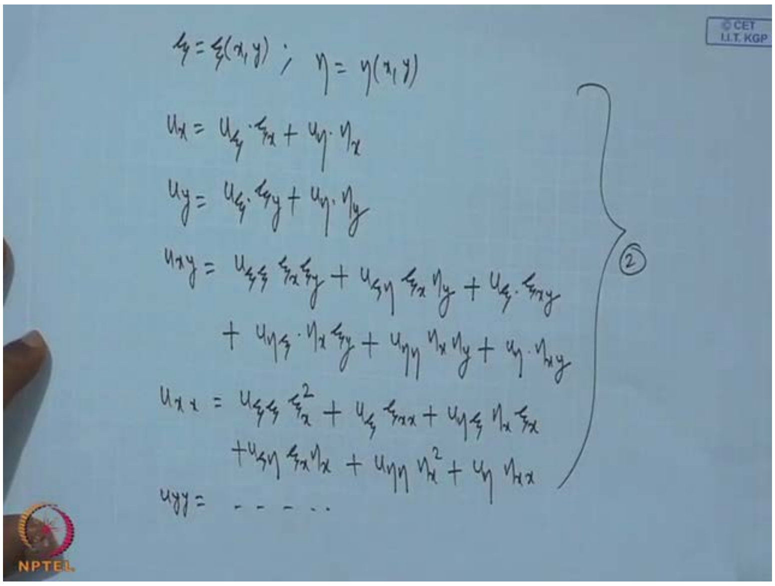
(Refer Slide Time: 04:19)



See, let us for example, suppose something like this, so it is very difficult to guess what could be the solution, somehow if we can simplify. See, look at the combinations you have u x x, u x y, u y y, so what would be our aim? Our aim is somehow transform 1 using new variables, such that 1 takes a simple form.

Now, when we say what could be the simple form simple form such that this can be integrated easily. This in a crude sense, so how do we do it, so what is our aim? Somehow transform one using new variables, such that 1 takes a simple form. So, let us try to introduce independent variables x y to zeta eta such that zeta is function of x y. Now, our aim is to simplify the equation in terms of these new variables, okay?

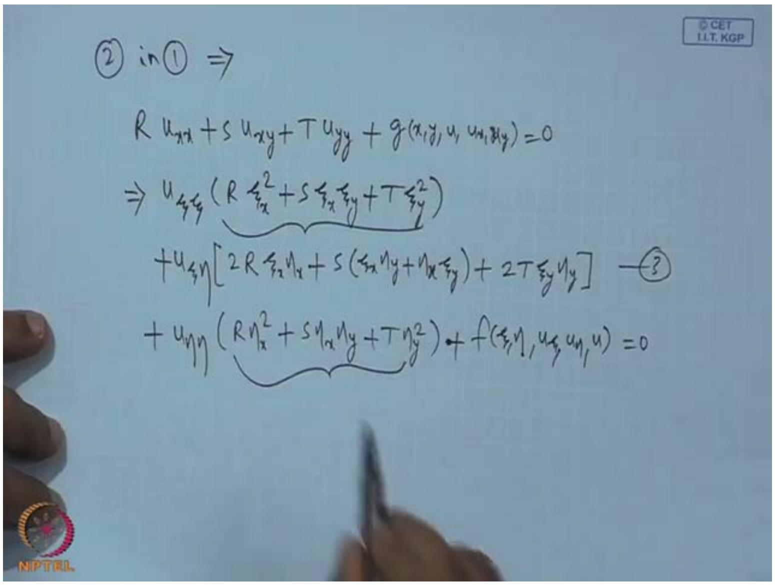
(Refer Slide Time: 07:09)



So, how do we proceed? So, zeta is zeta of x y, eta is eta of x y, now u x, u zeta, zeta x, so we are using chain rule. Similarly, u y and the mixed dose, say u x y. So, this with respect to y, so we have here u g zeta 1, zeta x is present and we are differentiating with respect to y. So, we get this plus u zeta eta plus keeping this derivative of this with respect to y, then this term u eta zeta eta x zeta y plus u eta eta eta x eta y plus u eta eta x y.

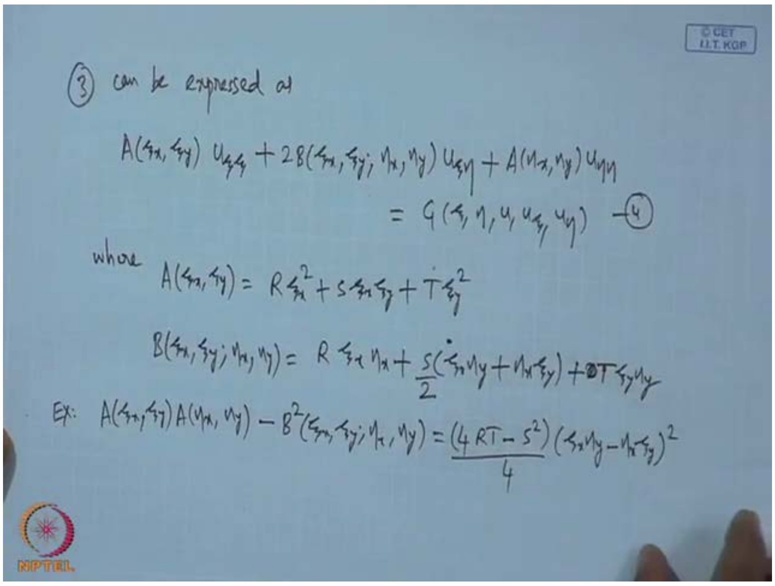
So, we similarly u x x, so for example, I can do then u y y, I can leave it so this u zeta zeta zeta x square plus. So, we make mistake there is a chance, so first term u zeta zeta and this we get a zeta term plus u zeta eta, then keeping u zeta fixed. Similarly, u eta zeta plus so I will say so these are some just 2, okay?

(Refer Slide Time: 10:22)



So, I am leaving u y y as exercise that means, u y y. So, 2 in one reduces u, so R u x T, so this only the left hand side I am considering. So, it is a bit of algebra, so this say this plus g u y equals to 0 implies this plus some function, we get equals to 0. Now, this I will say 3, so this can be put it in a simpler form, you identify this structure, try to identify this.

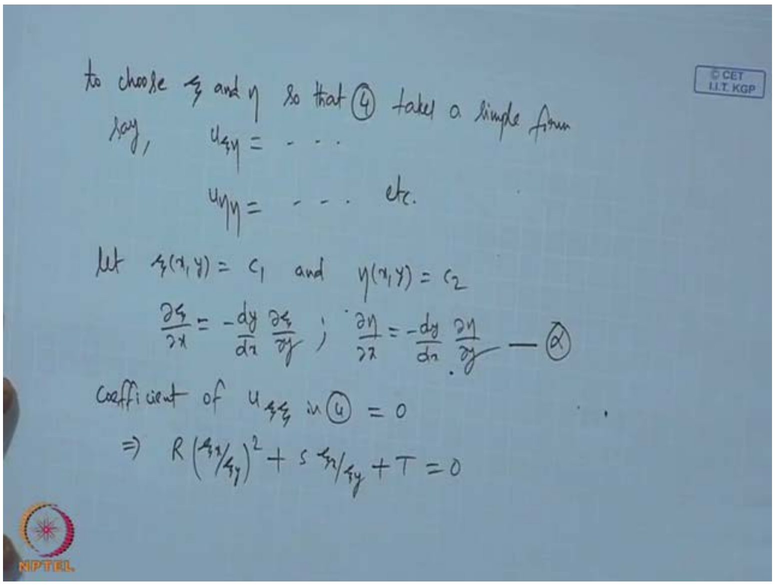
(Refer Slide Time: 12:41)



So, 3 can be expressed as A times this plus 2 B of plus, how did we do this? Look, so this I am putting it as A of and this is same operator because R S T on this. So, the operator is same just here zeta x, zeta y, eta x, eta y. So, this times B, so it can be put it in this form where and B. So, this is R S by 2, I would use, so 2 I would like to remove T, okay?

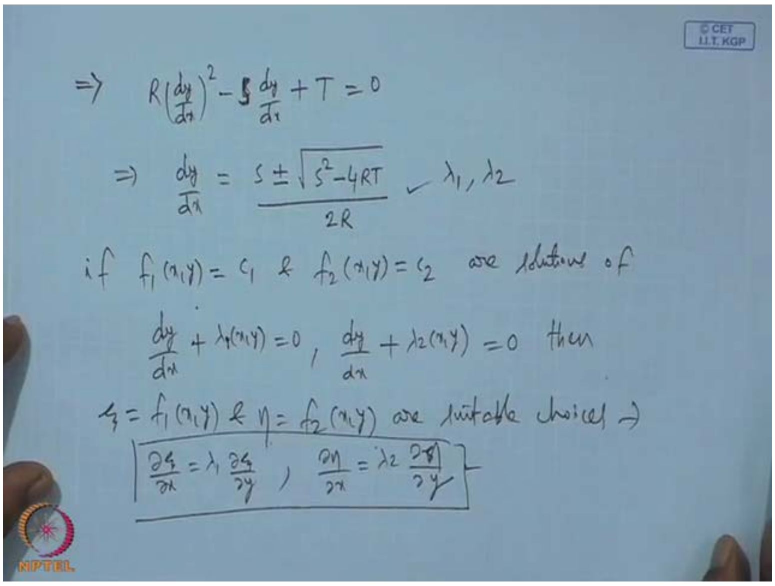
So, now an exercise A, on this A, on this minus B square is 4 R T minus S square. So, this exercise I would expect that this can be done. Now, what is our aim? See, look at this 4, so in the new variables our equation reduce to this. Now, the question is can we choose zeta and eta such that this takes simple form, what kind of simple form? For example, can we get zeta and eta such that for example, this vanishes, this vanishes so that we get u zeta eta or for example, this, this vanishes so that this form etcetera, okay?

(Refer Slide Time: 16:56)



So, what was our to choose zeta and eta so that 4 takes a simple form? What is this simple form? Say, equal something etcetera, so this is our aim, so this is our aim. So, let us try do that, so let zeta of C 1 and eta of x y equals to C 2. So, then we have and so if this is the case we have this, this we can verify if just by chain rule, just by chain rule. Then from these 2, say this is some alpha, alpha coefficient of u zeta. In 4, what is our 4? This is our 4. So, coefficient of u zeta, zeta is this. So, this will be 0 implies R S plus T equals to 0, please follow coefficient of this, which is this. So, I am equating to with 0, so I have normalised by zeta y. So, then it will be this, but from this we have d y by d x, this ratio is minus d y by d x.

(Refer Slide Time: 20:33)

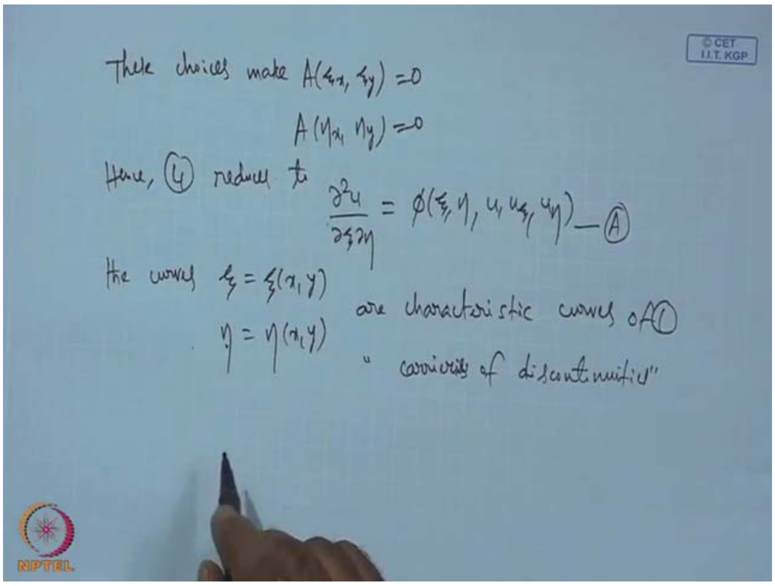


So, this implies we have R minus b, sorry minus S T equals to 0, this implies d y by d x equals to S plus or minus root S square minus 4 R T by 2 R. So, this is the scenario d y by d x is this, so that means if the coefficient of this in 4, if this is 0, so this will be 0, then d y by d x is this, right?

So, in some sense if f of x y equals C 1 and f 2 of x y equals to C 2 are solutions of d y by d x plus lambda 1 x y equals to 0, d y by d x plus lambda 2 x y equals to 0, then zeta equals to f 1 of x y eta equals to f 2 of x y are suitable choices, such that eta y. See, we are forcing the coefficient of I will explain, so coefficient of this is this. So, we expect that this must reduce to a simple form which one. So for example, coefficient of this we want to be killed so that we get a nice form and it can be integrated very easily, okay?

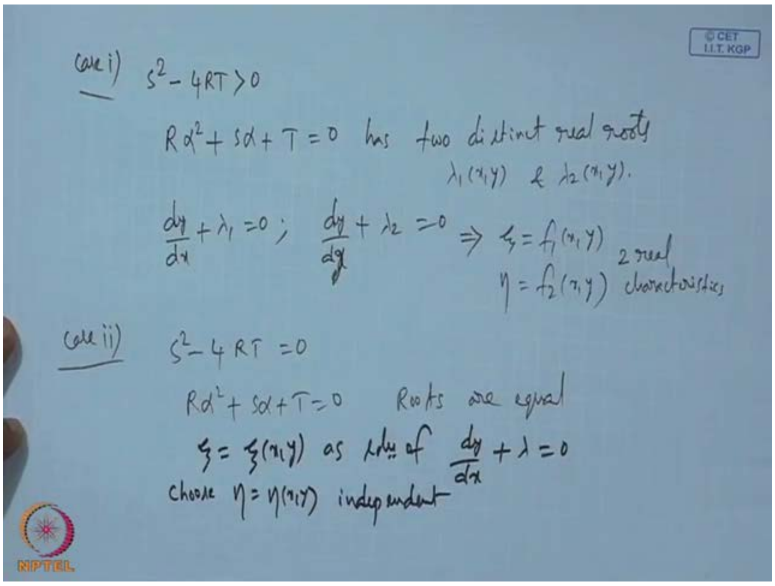
So, in order to that so for example, we have considered these are constant surfaces then from this we get this relation. Now, if you force coefficient of this is 0 exactly, we need this quadratic and on solving we get d y by d x. So, this is say lambda 1, lambda 2, so essentially that is what we get d y by d x plus lambda 1 is 0. d y by d x plus lambda 2 is equal to 0. So, then zeta equals to f 1 and eta equal to f 2 are suitable choices such that, see only this happens then the coefficient will become 0, only this happens the coefficient will be 0 and only this happens the coefficient of u eta eta will be 0, okay?

(Refer Slide Time: 25:15)



So, these choices make so these choices will make this coefficients 0. Hence, 4 reduces to some phi of, so this we can easily integrate so the curves, so these are called characteristic curves of 1. So, these are nothing, but carriers of discontinuities because how did we along this the coefficient is getting 0, that is how we have determined. So, these are called characteristic curves. So, now this depends on how many characteristic curves exist depends on this quantity. So, various possible cases we can discuss. So, for example, if this is positive then we will have two distinct and if it is 0 then we have only 1.

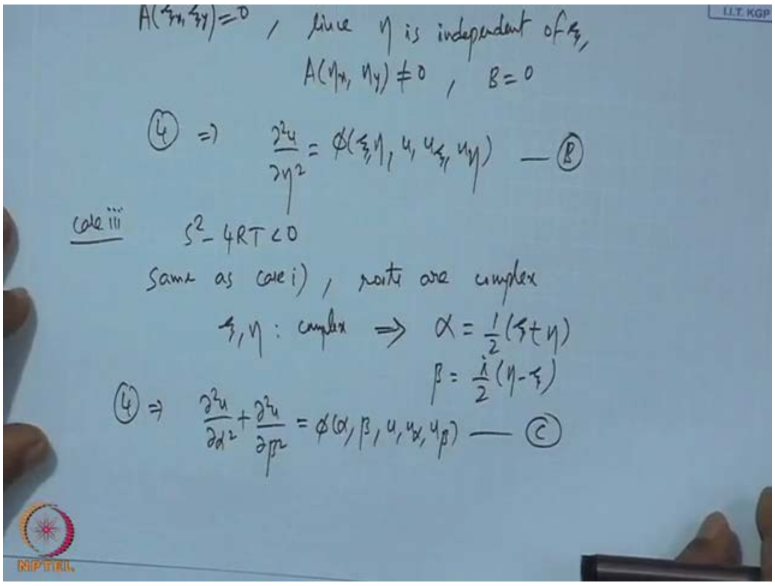
(Refer Slide Time: 27:39)



If it is negative then we have imaginary, right? So, we are going to discuss so case 1, S square minus four R T greater than 0. So, if this is the case then R alpha square plus S alpha plus T equals to 0 has two distinct real roots, lambda 1 and lambda 2. Therefore, we get d y by d x plus lambda 1 0, sorry d x plus lambda 2 0 and on solving we get zeta equals to f 1 of x y eta equals to f 2 of x y, okay?

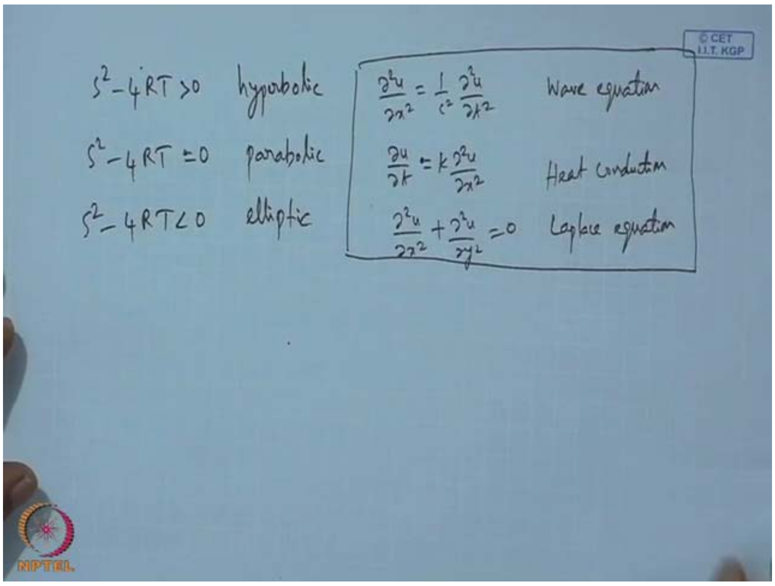
Then case 2, S square minus 4 R T equals to 0, then roots are equal. So, that means in this case 2, real characteristics so roots are equal. So, then we have only zeta of x y which is as solution of d y by d x plus lambda equals to 0, so which means we have 1 here and then we have to choose eta equals to eta x y independent. So, we have to choose eta equals to so if this is a case, okay?

(Refer Slide Time: 30:43)



So, in this case what would happen is we have a of zeta x, zeta y 0? And since eta is independent of zeta is non zero and b is 0 in this case, so the equation 4 reduces to so this is case 2. Then case 3, S square minus 4 R T less than 0. So, this is more or less same as case 1, but roots are complex. So, we have to roots are complex, so we cannot work with zeta and eta. So, here in this case these are complex, hence we define alpha equals to half of zeta, beta equals to and try to change the entire equation in terms of alpha and beta. So, then 4 reduces to, so what is the representation corresponding to each of the cases.

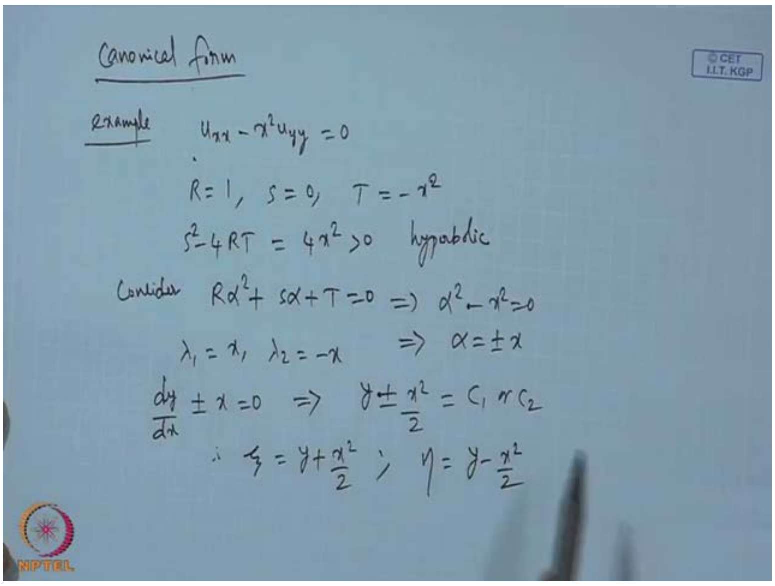
(Refer Slide Time: 32:59)



S square minus 4 R T greater than 0, then the equation is called hyperbolic. Then S square minus 4 R T equals to 0 parabolic elliptic. So, that is because when the transformed equation in the new variables almost represents a chronic. So, these words are derived keeping the transformed equation in terms of a chronic. So, examples so this is Laplace equation, you can verify S square minus 4 R T will be less than 0.

So, it is elliptic and so this is heat conduction equation. Of course, one dimensional, this is parabolic and this is one dimensional wave equation, which is hyperbolic. So, these are I am not giving the derivations of these equations, but these are most physically discussed examples with respect to second order PDE, okay?

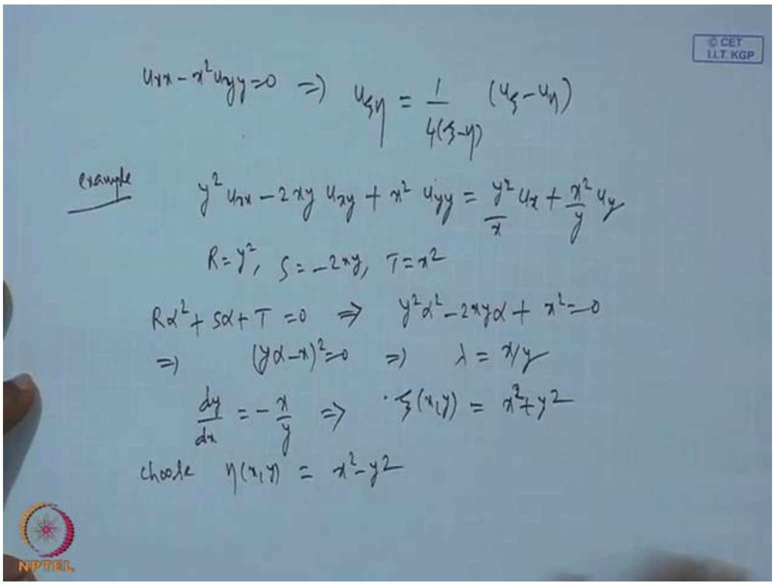
(Refer Slide Time: 35:21)



So, we have not seen the advantage of reducing it to new variables. So, let us see that so that is called canonical form, which one using the new variable and reducing it to canonical form. So, let us see the example, suppose this is our equation so here R is 1, S is 0, T is minus x square. So, S square minus 4 R T is 4 x square greater than 0. So, it is hyperbolic everywhere, now consider so this implies alpha square.

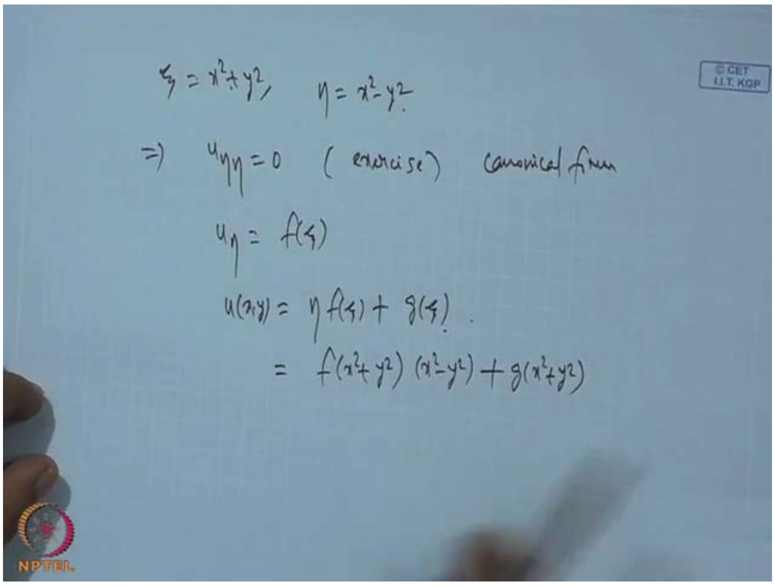
So, let the roots lambda 1 is x and lambda 2 is minus x. So, d y by d x plus or minus x equals to 0, so this implies y plus or minus x square by 2 is C 1 or C 2. Therefore, zeta equals y plus x square by 2 and eta is y minus x square by 2. That means, for this choice also eta and zeta only this equation gets reduced to a specific form. So, that is the idea, okay?

(Refer Slide Time: 37:25)



So, if you use the chain rule and try to reduce, this gets reduced to so which is a simpler than the original, okay? So, let us consider another example looks big, but let us simplify. So, we get so this implies, so lambda equals to x by y, so d y by d x equals to minus x by y, so zeta of x y 1 characteristic. So, since roots are equal what was our earlier discussion, choose eta which is independent. So, let us choose then if we use zeta, this eta, this then our equation reduces.

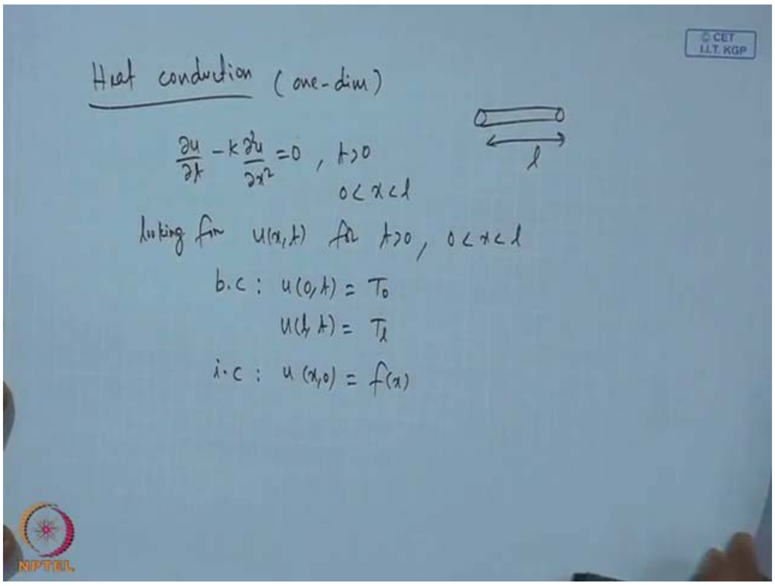
(Refer Slide Time: 39:34)



So, this is exercise please using chain rule, then what is advantage? So, this is our canonical form and hence if you integrate and further if you integrate which is nothing, but so any combination of these functions will be solution of the original equation. So, this is reduction to canonical form and once it is reduced to canonical form solving. So, this is using a change of variables and reducing to canonical form so that it is easily integrable.

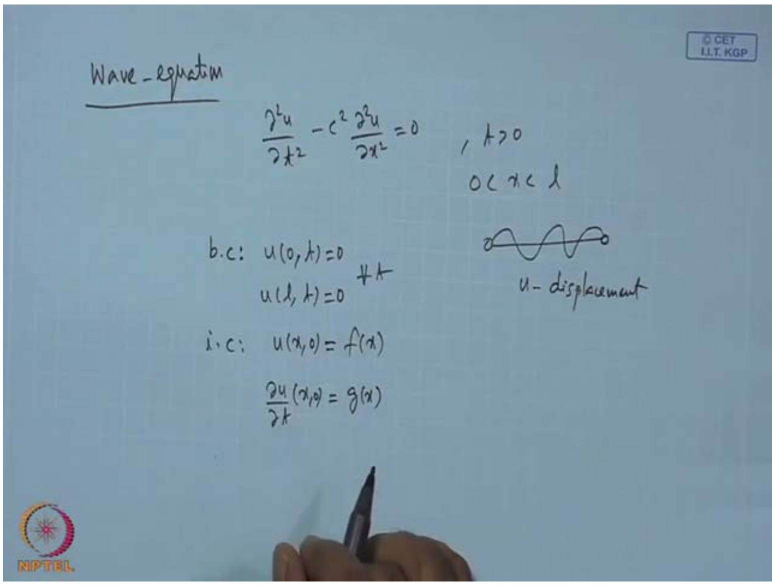
So, now let us look at physically at each set of equations, each of the categories. For example, elliptic hyperbolic and parabolic as a physical problem, let us define that means together with the boundary and initial conditions so that we are making our setup to proceed to numerical competitions. So, let us define the complete set of problem, okay?

(Refer Slide Time: 41:24)



So, let us start with heat conduction. So, in case of one dimensions conduction in so of some length, so I am not giving the derivation as I mentioned this is our equation, K is the constant of this is the thermal conductivity coefficient and we are discussing within a specified length. Now, this requires what are the boundary conditions and see we are looking for u of x t for t greater than 0 within this range. So, definitely we need a initial condition and then we need the boundary conditions. So, boundary conditions we need u of say, let us say this 0 and l, some t 0 u of 1 t. In this case l t some t l and we need an initial condition u of x 0 is any initial profile. So, this is heat conduction.

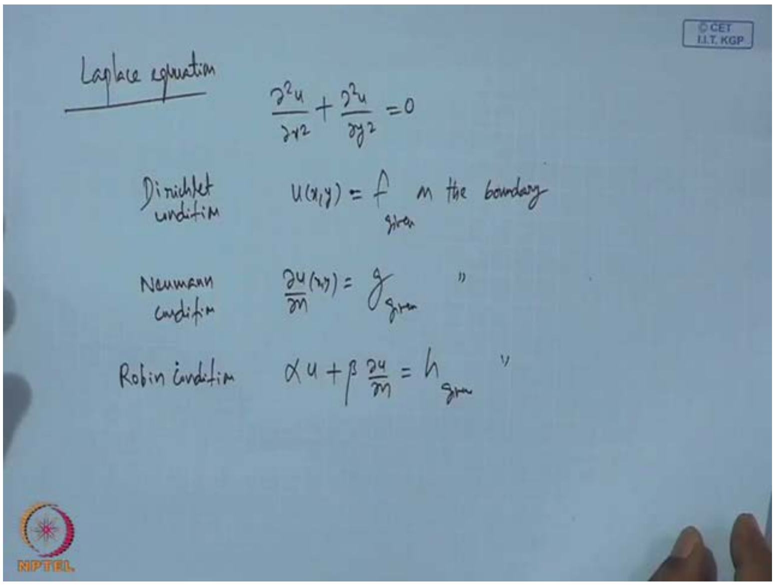
(Refer Slide Time: 43:13)



So, remember I am putting down only one dimensional cases. So, in this case the solution represents the temperature for a given time at any point, so that is the solution. Now, in this case wave equation so again it depends for t and x, so which means both ends are fixed. So, what this represents? This equation governs the displacement of a string in this case, which is fixed at both ends. So, how it is getting displaced?

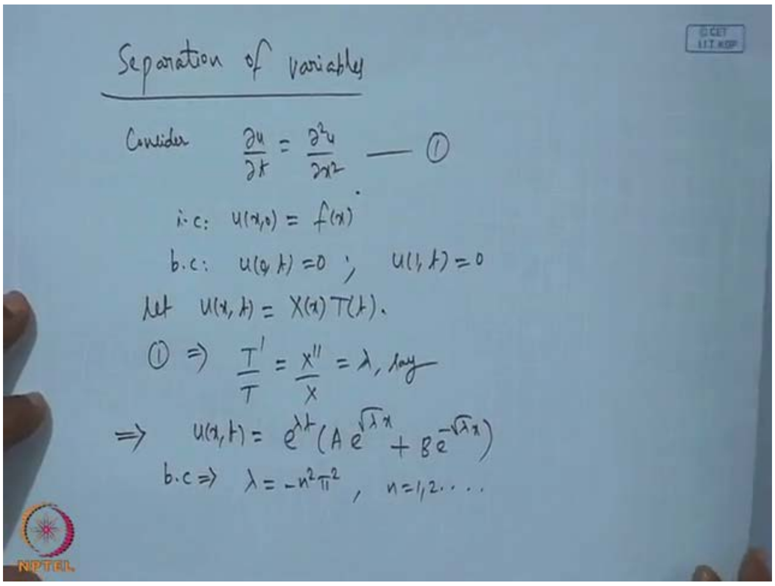
So, as a set of physical problem we need boundary conditions, so what are the boundary conditions? Since, both ends are fixed so u represents displacement in this case the earlier problem, u represent temperature in this case, u represents displacement. So, the boundary conditions since both ends are fixed we have no other option for all t then we need initial conditions. Remember, this is problem on displacement and we have two time derivatives. So, we require initial displacement and yes velocity, so this is the complete physical problem, okay?

(Refer Slide Time: 45:42)



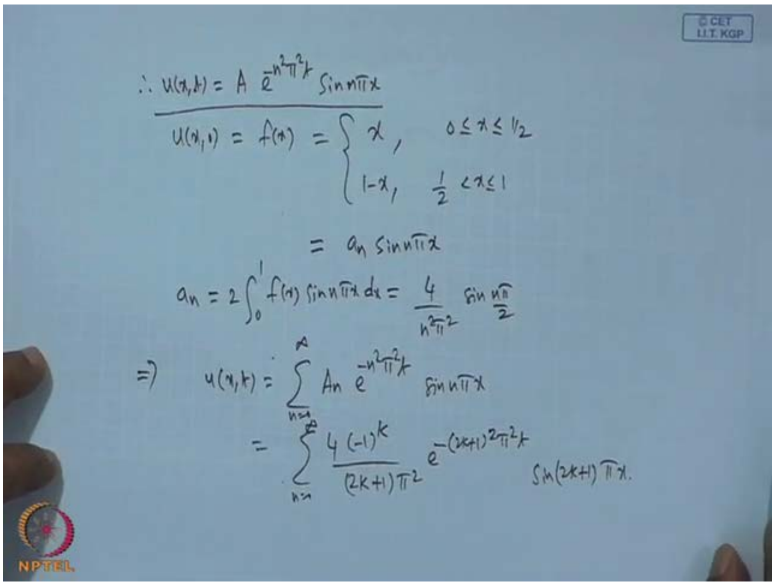
So, now Laplace equation, so this is in two space dimensions. Now, I would like to mention here if u of x y equals to some f on the boundary. So, this is called Dirichlet condition, then Neumann condition if the normal to the boundary is prescribed. Of course, on the boundary then robin condition a scalar times u plus another scalar times normal. So, this is prescribed h given all are given, h given on the boundary. So, this is robin condition. So, before we go for numerics, let us see for the simple cases, can we get analytical solutions? So, one since the equations are linear, so we would like to attempt separation of variables.

(Refer Slide Time: 47:30)



So, consider so k I have taken as 1, now we have to define the complete problem. So, initial condition and boundary condition, so since we would like to try a separation of variable solution, let so then 1 implies we get lambda. Say, so by substituting this in 1, we get this so independently we solve. So, then applying boundary condition this implies u of x t, okay?

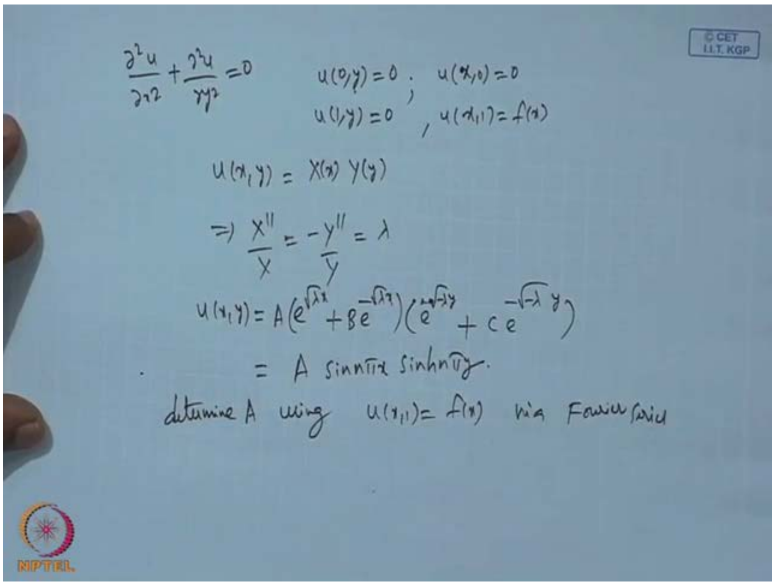
(Refer Slide Time: 49:45)



So, then boundary condition implies lambda values are determined, therefore this is the solution. Now, we are still left with initial condition, suppose so we are left with f of x. Suppose, f of x is x and 1 minus x, this range then how do we determine a. So, what we do is we depend on the Fourier series of f. So, let the Fourier series of f be this where a n.

So, then we can employ the initial conditions because see look at it we need f in this form, u in this form, but the initial profile is given in this form. So, we depend on the Fourier series and we employ the initial condition so that we determine the coefficient a n. Therefore, u x t is given by which is so we get the solution. So, initial condition is applied depending on the relying on the Fourier series of f, okay?

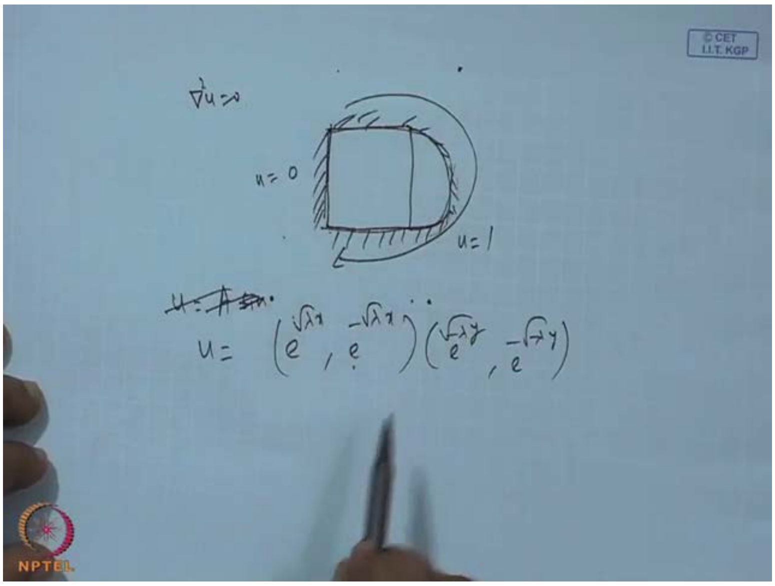
(Refer Slide Time: 52:48)



So similarly, let us consider let us consider Laplace equation, now we need 2 conditions on x 2, conditions on y. So, the entire boundary is prescribed so we follow the same procedure. So, this implies therefore and again depending on the form of f we determine a determine a using of course, via now where is the numerics, where is the necessity of numerics?

See, we have got the solution very nicely and then where is the question of numerics? The question of numerics arise, for example if your boundary is not regular. So, using separation of variables we get it, but unfortunately to get the arbitrary coefficients, if your boundary is not regular then we cannot really substitute and get it, get the exact arbitrary coefficients, okay?

(Refer Slide Time: 55:23)



So for example, we are solving Laplace equation and the boundary is something suppose on the just now we have solved on a square. Suppose, there is an extension given, so this is our new boundary. So, suppose from here to here u is 1 and here u is 0, so how do we solve because our solution we cannot represent using one single, because our solution.

So, it consists of like this e power root this terms product with, so this line can be represented using the cartesian coordinates, but there is no single equation. So, we will end up with difficulties, so in order to solve such irregular geometries even though equation is linear. In order to solve such irregular geometries we have to go for numerics. So, with this motivation in coming lectures we start numerical methods for partial differential equations, until then good bye.