Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 22
Introduction to First order PDE

Hello, so far we have discussed numerical solutions of ordinary differential equations. Let us start discussing about partial differential equations. So, as you could recall solution of ODE will be a curve, so it represents a curve. So, before we proceed to numerical solutions of PDE, let us review a bit on the definitions, and then classifications, and maybe if possible little bit of theoretical sense.

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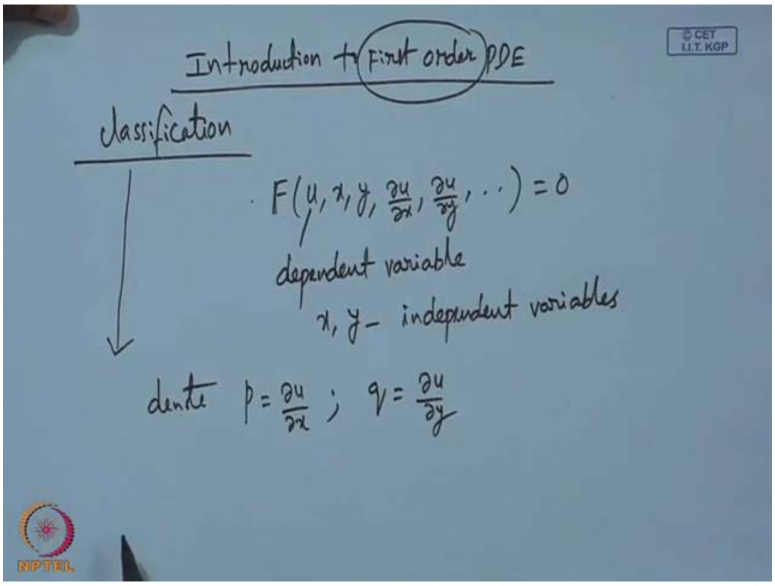

So, let us start introduction to partial differential equations. So, I would like to stick to first order and in general any functional relation; for example, u x y dou u by dou x, so this is called a PDE where, u is dependent variable x and y are independent variables. So, however depending on the form of F, one can classify so for example, let us denote this is a standard notation.

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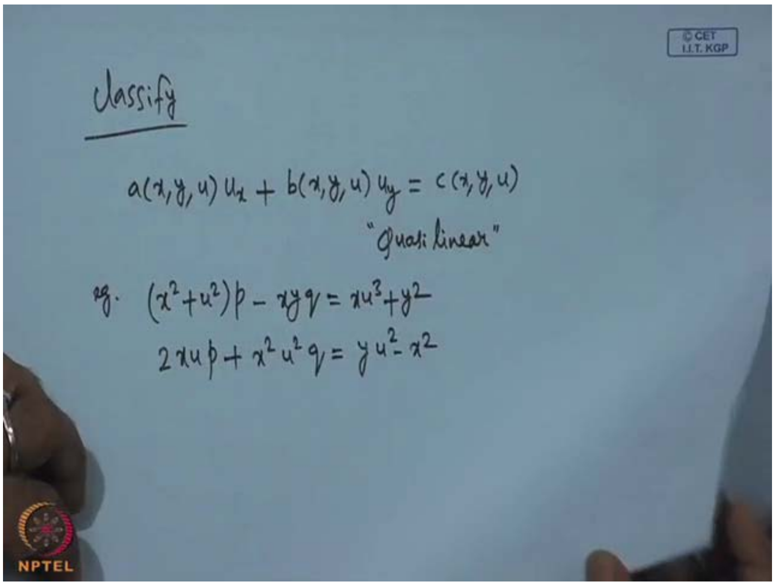

Now, with respect to the standard notation, so we would like to classify, consider the form and this is partial derivative with respect to x, partial derivative with respect to y. So, if a PDE first order of course, because only the first derivatives are involved, if it is in this form remember that means with respect to u x and u y, it is linear. However, the coefficients can be of this form, it can depend on x y and also u of course, the right hand side. So, such a classification is called Quasi linear so examples so I will switch over to p u x is p q, so these are examples of Quasi linear.

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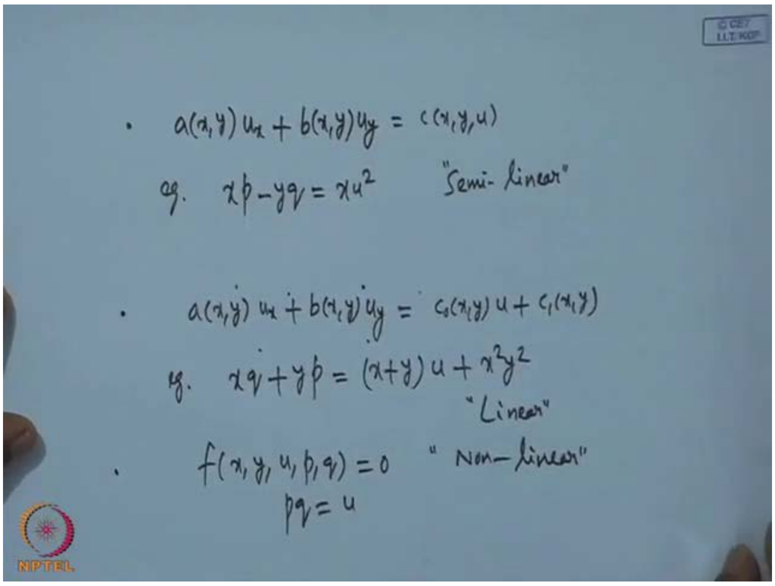

So, now suppose equation is of this type so that means a and b are independent of the dependent variable u. Example see right hand side you can have non-linear term, so linearity with respect to only the first derivatives. So, this is semi linear for example, if it is of this form, so this is linear further this is non-linear because we are allowing any combinations, so this is non-linear.

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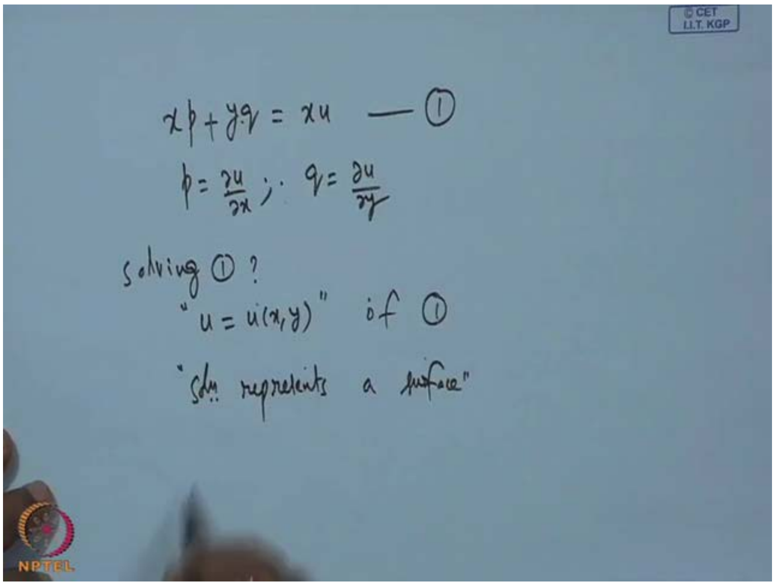

So, as I mentioned suppose you consider something like that, so what is our notation? So, what do you mean by solving 1 that means we would like to obtain a solution of 1 and then we have solved 1. So, having classified first order PDEs it would be worth discussing in brief how do we approach the solution. And first of all what is the solution?

As I mentioned in case of ordinary differential equations, the solution represents a curve so one can guess in case of partial differential equations. Since we have one dependent variable and two independent variables, so definitely as you vary with respect to this, one would expect solution represents a surface. So, let us try to understand little bit on the solution.

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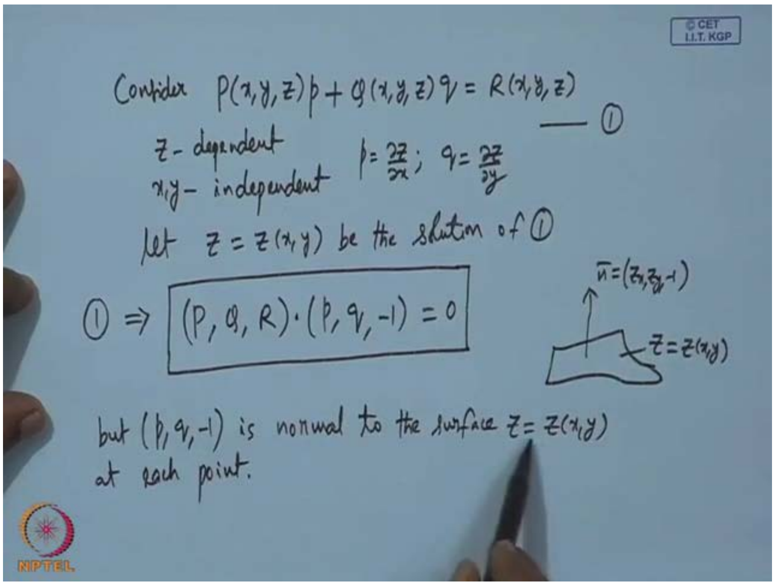

So, we can play with the notation as long as we understand so I am introducing z, so this is our dependent variable P Q. So, this is a quasi linear and z is dependent, x y is independent. Now, let this be the solution of 1 and one can be represented as follows, so these are all functions P Q where, P we can so P Q R dot p q minus 1 equals to 0. So, it can be put it in this form 1 look at carefully, so if you bring it minus R, so I have taken P Q R there and here small p q and minus 1. So, exactly one can be represented in this form, but remember suppose this is some z equals to z of x y, this is surface. So, then Z bar the normal is z x z y minus 1 but p q minus 1 is normal to the surface z at each point.

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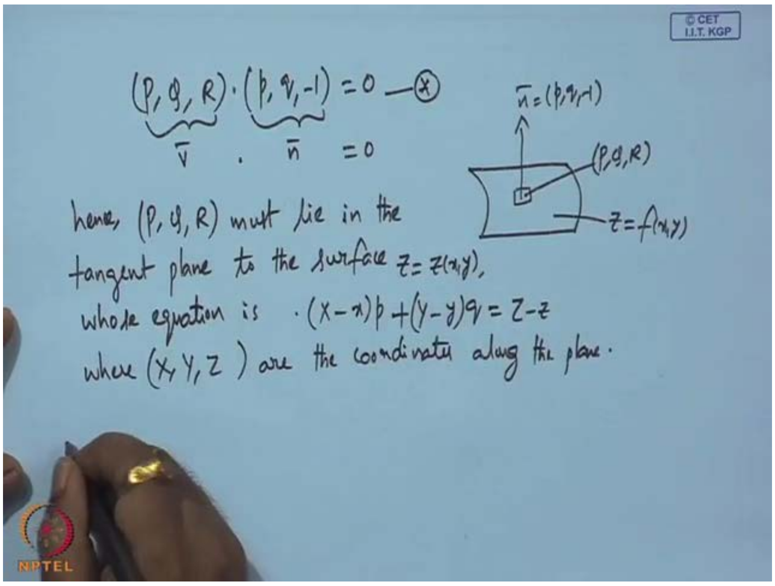

So, this is normal to at each point, but what is our relation? Our relation is P Q R dotted with, so this is normal to your surface and if you call this is a vector v bar. So, we have v dot n 0 therefore, what is the inference, our inference is v bar must be the tangent plane to the surface is not. So, suppose this is z equals to f x y the surface, so this is our normal therefore, P Q R must be in the tangent plane, so what is our inference? Hence P Q R must lie in the tangent plane to the surface, whose equation is where these are the coordinates along the plane. So, what is this star forcing? Star forcing is that all the surfaces satisfying star are such that at each point, they are tangential to the vector.

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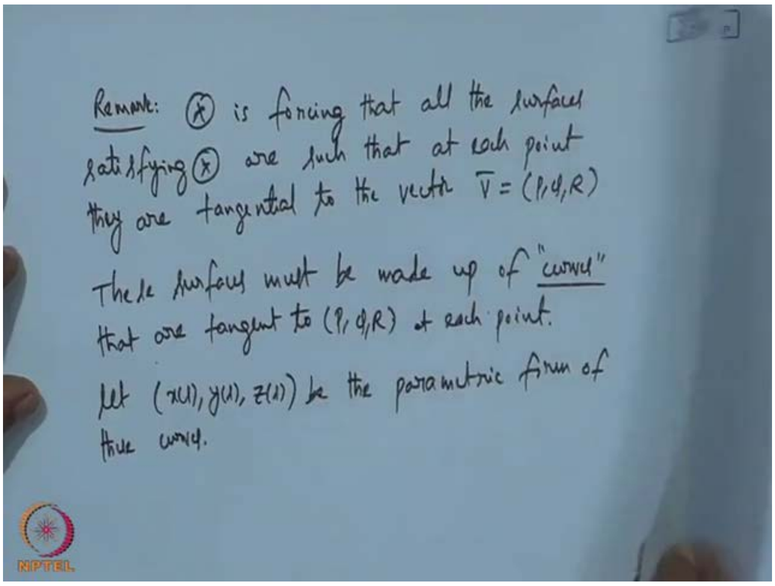

So, what is the remark? The remark star is forcing that all the surfaces satisfying star are such that at each point they are tangential to the vector, which means what is happening here. See if you can visualize a solution of a PDE as a surface so what is happening, the star what is that relation, the vector v bar which is consists of the coefficients p q r, that says that it is along the tangent plane and you have a surface like this and normal is like this.

So, hence the vector p q r must lie in the tangent plane, so if you take all surfaces containing this p q r, so they have this special property. What is that property? They must be coming from star that means all this z s must be solutions of that particular PDE, so let us visualize this clearly. So therefore, these surfaces must be made up of curves that are tangent to P Q R at each point that is what I just explained. So, let this be the parametric form of this curves suppose this is the parametric form of this curves.

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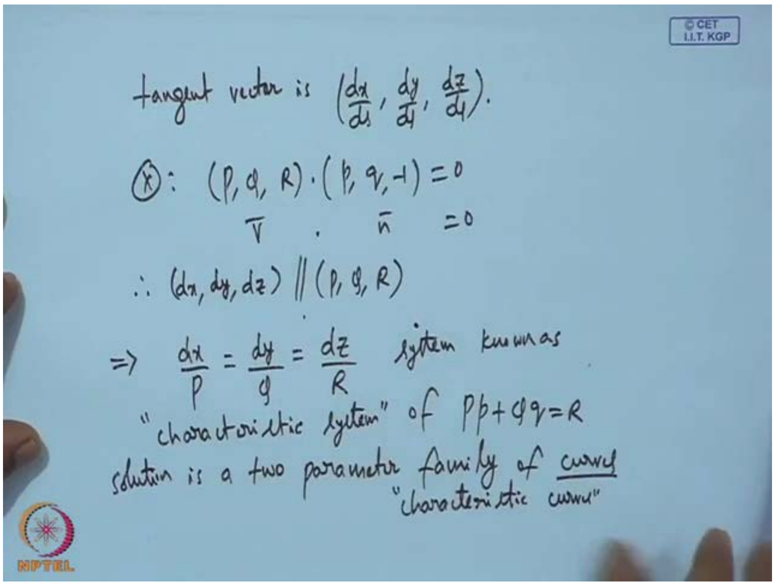

So, then the tangent vector is given by where s is the parameter arc length, so this is the tangent vector. Now, from star what is our star p q r, so this is n bar, this is v bar, this is 0, so we have noticed that v bar is along the tangent plane therefore, this is the tangent vector therefore, d x d y and d z is parallel to P, Q, R. So, this implies d x by P, d y by Q, d z by R, so this represents a system known as characteristic system of our PDE.

And what will be the solution of this, obviously solution of this will represent if you take this two and solve you may get one and take this two, depending on that however you get two parameter family of curves. So, solution is a two parameter family of curves and these curves are called characteristic curves because depending on the character of this curves family of curves the solution is obtained.

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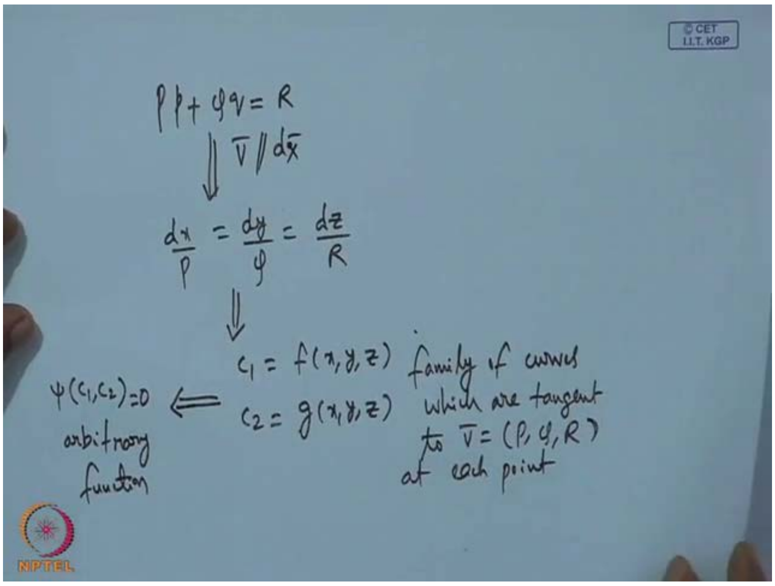

So, let us briefly discuss we have arrived v bar is parallel to d x bar, so this implies d x by P, d y by Q, d z by R. So, this is a first order ODE system, this implies we get two parameter families of curves, which are tangent to v bar at each point. So, these are family of curves, which are tangent to P, Q, R each point. Then once we have this we get a functional relation, you can express h functional relation, so as arbitrary function. So, let us see how we arrive at the solution, so let us take simplest examples to start with.

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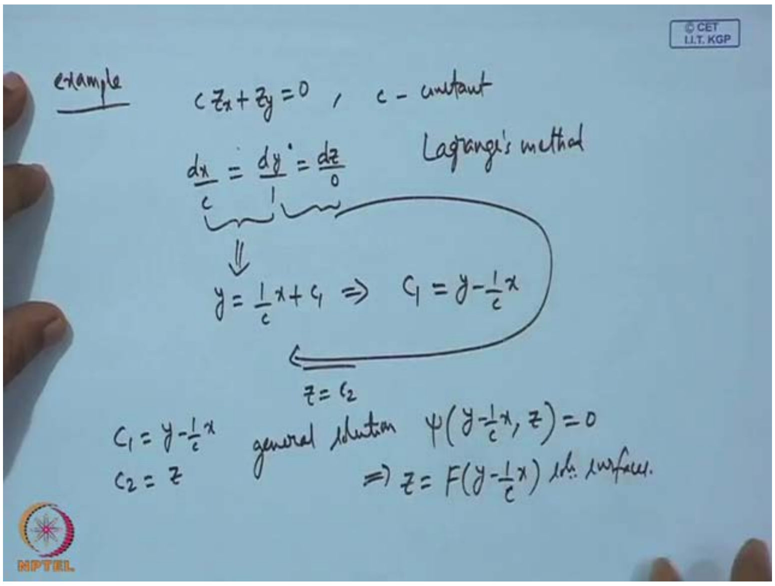

So, z is a dependent variable x and y are independent and c is a constant. Now, the characteristic equation is d x by c, d y by 1 and d z by 0. Suppose, I solve this two, so these two implies say y equals to 1 by c x plus c 1 then suppose this two implies other two implies z equals to c 2. So, we have c 1 equals to y minus 1 by c x, then the general solution is psi of or otherwise z equals to some function of we can write. So, this is a solution surfaces so this is very standard technique in literature called Lagrange's method of characteristics.

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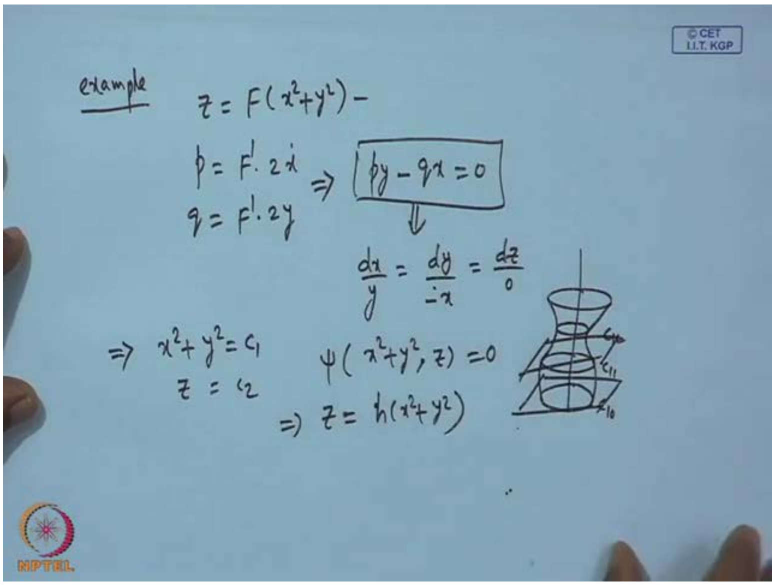

So, let us look at another example, see this time let us represent this is our surface. How do we form a PDE? So, we would like to form a PDE corresponding which governs this surface that means solution of whose PDE is the given surface. So, you differentiate z x is p then F dash with respect to x is 2 x, then with respect to y this implies p y minus q x is 0, this is our governing equation. Now, let us go back to this how do we do it, from here d x by y, d y by minus x, d z by 0.

So, from these two we get x square plus y square c 1 and from this we get z equals to c 2, therefore the function is this one. So, the characteristics are family of curves, so in this case what these families of curves are. Look in this case if you keep on varying c, c is suppose c 1 some c 1 0, c 1 1, c 1 2, so this is for each c you are getting curves and you are getting z equal to constant c 2. So, these are the planes z equal to constant, so you get family of curves and the surface generated by this represents the solution. So, what we may get something like this is the solution of all this PDE.

So, we have also discussed how to obtain PDE, so we have just given a first order PDE how do we solve. Of course, Quasi linear via Lagrange's method it is a hard way to do a non-linear stuff, but we will just have some analytical rigour and then we proceed further to numerical, which is the aim of the current lectures. Sometimes what happens you find the solution, but subject to some initial data, so this is also an interesting as far as first order PDEs are concerned, so let us briefly discuss.

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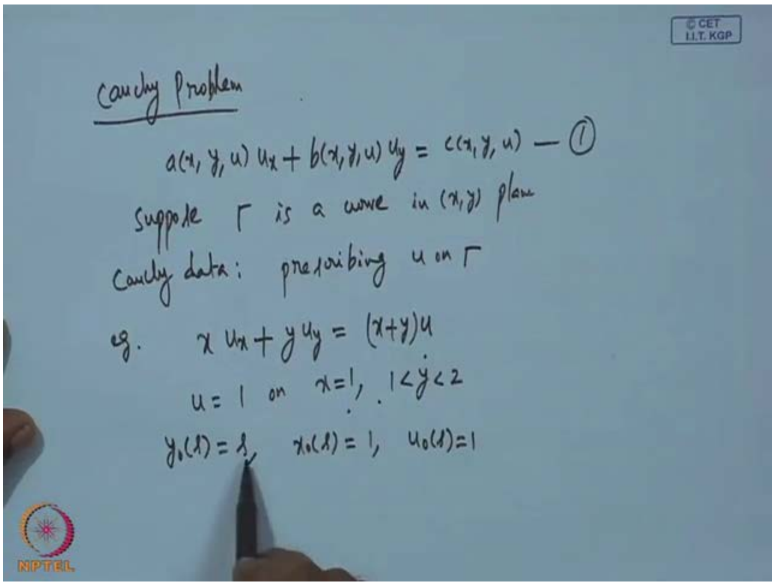
,, 

So, this is called Cauchy problem, so Cauchy problem is a x, y, u then u x plus b x, y, u, u y equal to c x, y, u. So, it is just for an understanding need not be p it is not based on the notation, it is based on our understanding. Suppose gamma is a curve in x y plane then what is Cauchy data, Cauchy data is prescribing u on gamma. So for example, now u is 1 on x equals to 1, so that means in parametric notation say y 0 of s is s, x 0 of s is 1, u 0 of s is 1. So, that means we expect not only to get the solution, but the solution which passes through the initial curve.

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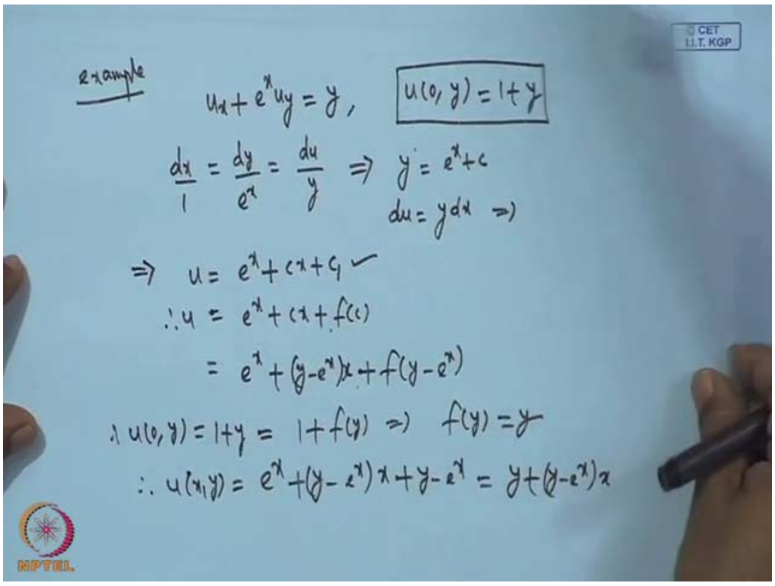

So for example, this is a kind of initial condition, so let us see d x by 1, d y by e power x, d u by y. Now, we can solve two of them in any combinations, so we get then d u so this is solving these two then we solve d u is y d x, so this implies u equals to and total. So, we can represent sum function of c because we need to eliminate, so this implies what we get d u is y d x, so y is this so we got this. Therefore, u is by eliminating so this is y minus e x into c into x plus f of y minus e x, but our requirement follow this therefore, u of 0 y, so this will be 1 plus f of y this implies f of y is y.

Therefore, u of x y is obtained as; this is satisfying some initial data so this is a typical Cauchy type problem. So, far we have it is a with respect to first order PDE, but still we have not got exact idea of what are the characteristic curves and then how these solution is related to that. So, let us try to understand with respect to some examples.

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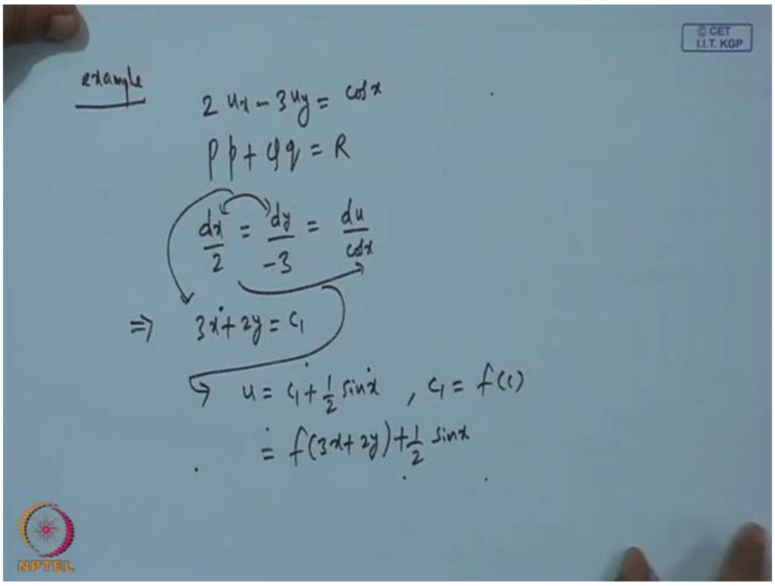

So, this is first order it is very elementary because with the constant coefficients, so this is of the form what we have started that is P p plus Q q, so p and q are constant coefficients, so we can adopt the method. So, using Lagrange's method, now from this two we get then let us consider these two. So, these two we got this, so from this two u equals to c 1 plus half sin x, so c 1 equals to f of c will be solution, so u is 3 x plus 2 y plus half sin x this is our solution. So, let us try to understand little bit.

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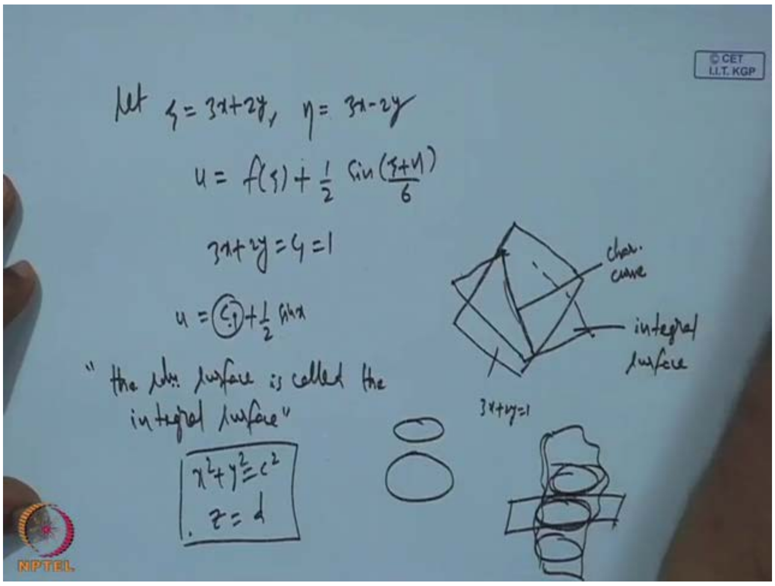

Let, zeta equals 3 x plus 2 y, eta is 3 x minus 2 y then the solution becomes u equals to f of zeta plus half sign zeta plus eta by 6. So, what is the scenario geometrically, you can see suppose this is 1, then we have u equals to c 1 plus half sin x. Now, for some particular value, so this is a intersection, this is the back background. So, you have two planes intersecting, so this is a curve where it is getting intersected and this is integral surface, what is that, the solution surface is called the integral surface.

So, these integral surfaces and then these are the planes, which we got form the characteristic equation and they intersect along a curve and this curve is called the characteristic curve. So, if you can recall so take some irregular cylinder, suppose these are the circles due to which the surface is formed. If this plane cuts, so this is the circle, so this surface is formed as a result of family z equals to d. So, similar thing is happening here, so these are the characteristic curves. So another example, the same one I have discussed.

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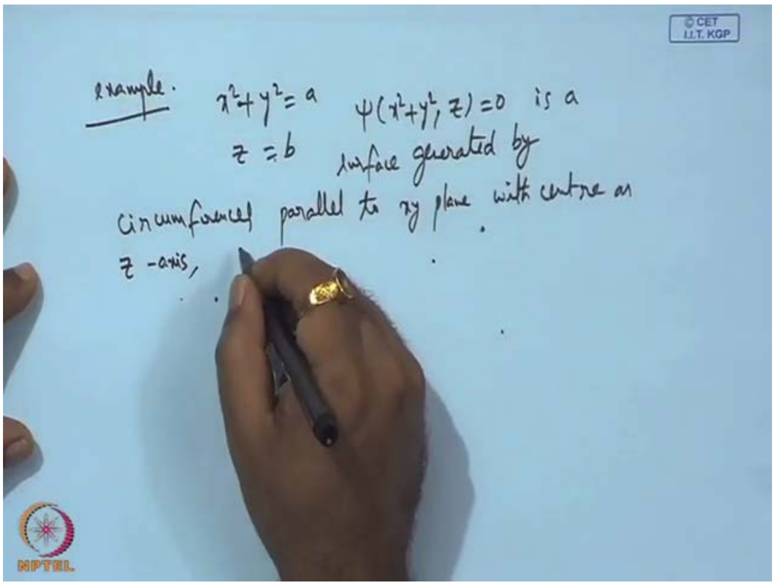

So, physically this gives a lot of insight psi of x square plus y square z equals 0 is a surface generated by circumferences parallel to x y plane with centre on z axis. So, this is a surface of revolution, so for example, these are the characteristic curves then this will be our solution so this is the physical interpretation. So, I discussed first order, this is a simple case where either constant coefficients or if you have variable coefficients how to solve using Lagrange's method. Of course, the constant coefficients and then even the variable coefficients that we have seen, they are very minimal effort is required.

However, in particular cases in order to use Lagrange's method really, we have to spend a lot of time in order to come up with what could be the characteristic curves, so it depends on the complexity of the PDE. So, before we proceed to second order let us look at some little bit of complicated cases.

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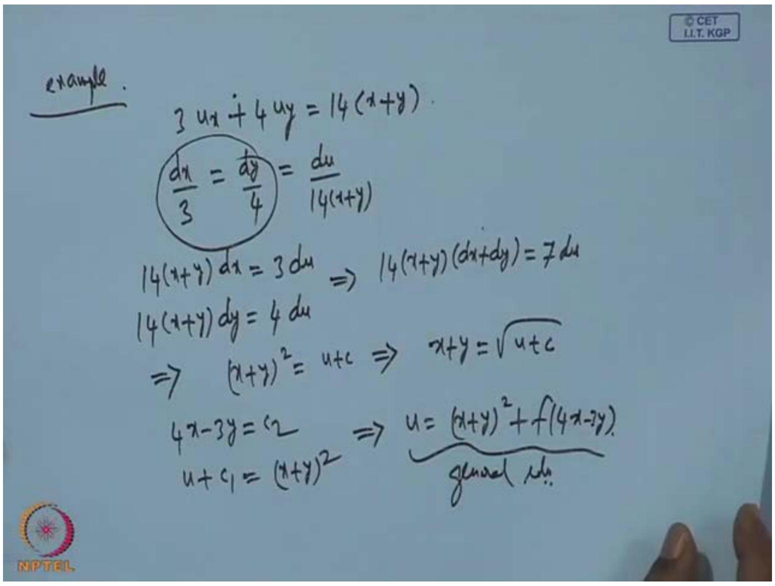

So, this can be a better one to solve at home, so consider this. So, from this we get these are the possible approaches, somebody may try like this, somebody may try like this. So, I am just trying to explain what the different cases are; somebody may try this two to come up with. Then this combination u plus c 1 equals to x plus y square, which we got so these two together u is x plus y square plus f of 4 x minus 3 y. So, this is the general solution, strictly speaking still this is not complex enough.

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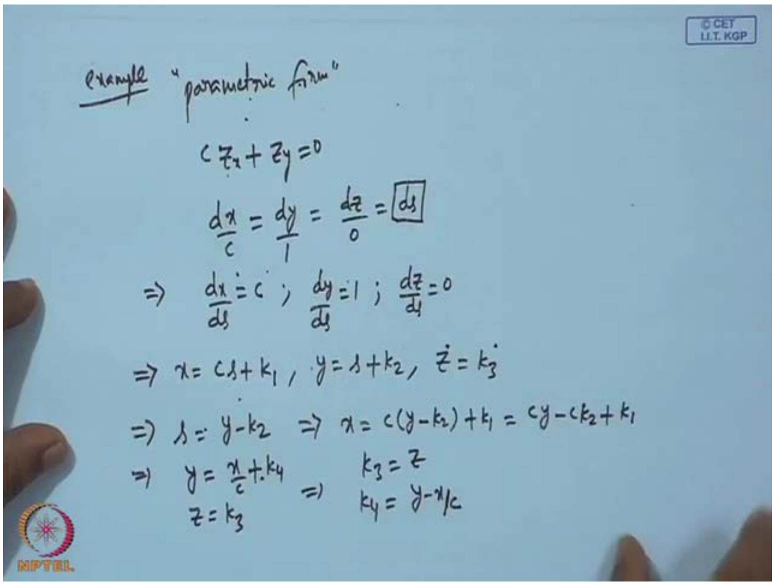

So, there is slightly different approach using parametric form, so consider the simplest example that we have considered then for this the characteristic equations the auxiliary equations. Now, we are introducing this parameter, so this implies, from this s equals to y minus k 2, so what is our aim? We get the characteristic curves only when eliminate the parameter s, so this implies x is this, so this implies y is this, so this implies k 3 is z some k 4.

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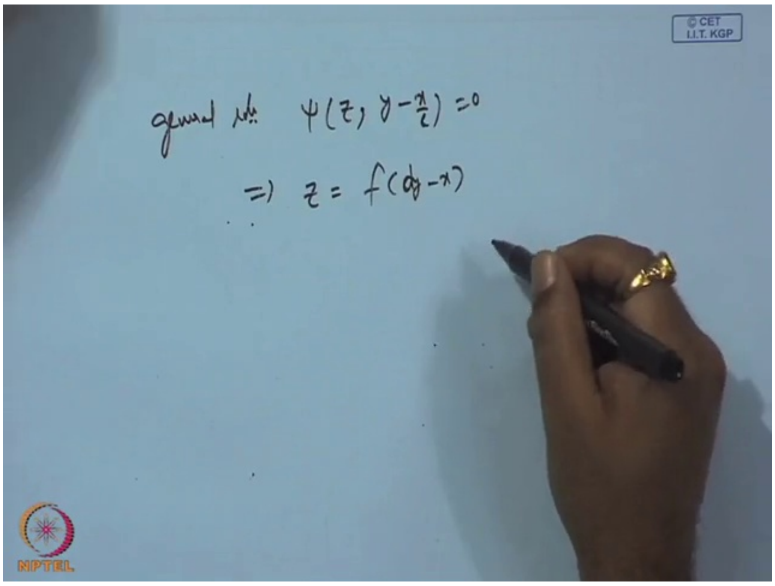

So, the general solution psi of z, so this implies z equals to some other d. So, these are various ways of solving first order PDE, so this is Lagrange's method still, but however depending on the combinations, we try to get the solution. So, within the framework of Lagrange's one can attempt the solution using variable coefficients or constant coefficients. The bottom line is one should understand that the once you write down the auxiliary equations you get two family of curves and by eliminating the arbitrary constants we generate the surface, which represents the integral surface.

So, all that I would like to inform you is you must understand that solution of a PDE represents a surface, which we call integral surface. And further classification is possible and then classification of integrals is also very much possible, but since this course is restricted to numerical treatment, we differ those things and we proceed to second order classification then we proceed to numerical methods of these partial differential equations. So, in the next lecture we will discuss about second order classification until then bye.