Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 21
Tutorial – III

Hello, so we have discussed a some techniques, how to solve two point boundary value problems. For example, finite difference methods, both linear and non-linear, and then also shooting methods. So, let us have some problems using these techniques, so especially we try to solve some problems explaining the concepts of finite differences.

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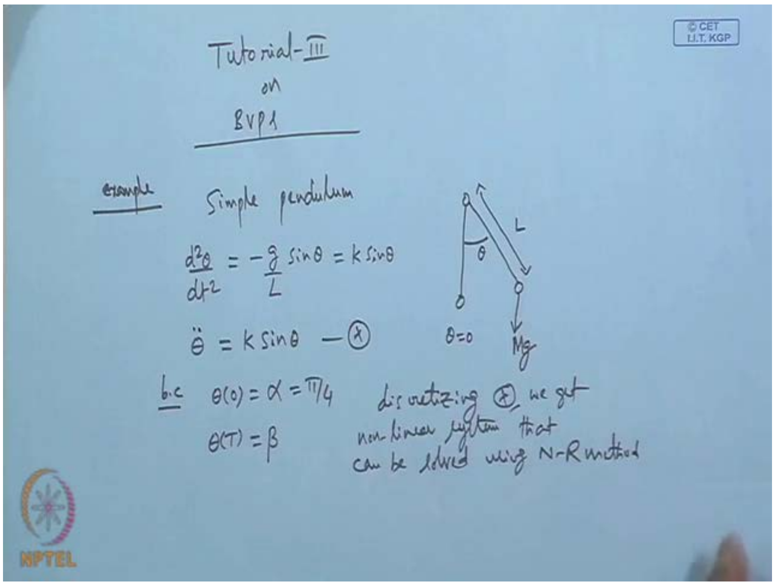

So, let us do that on BVP's, see we have been solving a lot of examples, but can we get some real life example or practical example, where finite differences could be used, so in particular non-linear, so simple pendulum so I can consider this example. So, you have a simple pendulum, so this weight is M g, so this is the initial configuration when theta 0 and this length is L let us say, and this is theta.

So, I do not want to explain the physics behind it, because you can derive balancing the forces one obtain d square theta by d t square equals minus g by L sin theta. So, this one can call k sin theta, so as B v p if you would like to define, this is k, so this is where the non-linearity in the in the dependent variable, T is a independent variable. And the boundary conditions theta of 0 is alpha, so say some pi by 4 then say theta of at any time at time T 0, let us say we have released at an angle, so then extend some other time b. So, this we want to give some other time it, it must be beta, so this can be defined as a boundary value problem. And if you discretize, we end up with discretizing star we get non-linear system that can be solved using N-R method so this is one example I would like to mention.

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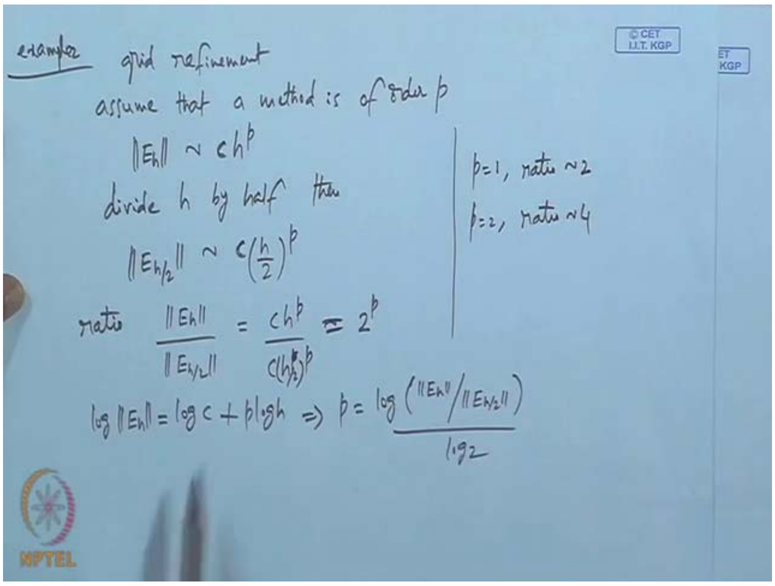

Then grid refinements, so another is grid refinement assume that a method is of order p. So, then we have seen the error with a specific h is bounded and it is behaves like this. suppose, we divide h by half then what would happen this with h by 2 c times h by 2 power p assuming the same constant. So, then the ratio this goes like h by 2 power p. So, this is this is 2 power p, so this ratio is 2 power p. Moreover this is log c plus p log h then we can solve for p in terms of the ratio. Now, if p is one 1 ratio is 2, p is 2 ratio is 4, so this given idea on if we refine the grid how the error and then compare to the earlier h, the ratio of errors how they behave. So, accordingly without to recalculating once can get some estimates.

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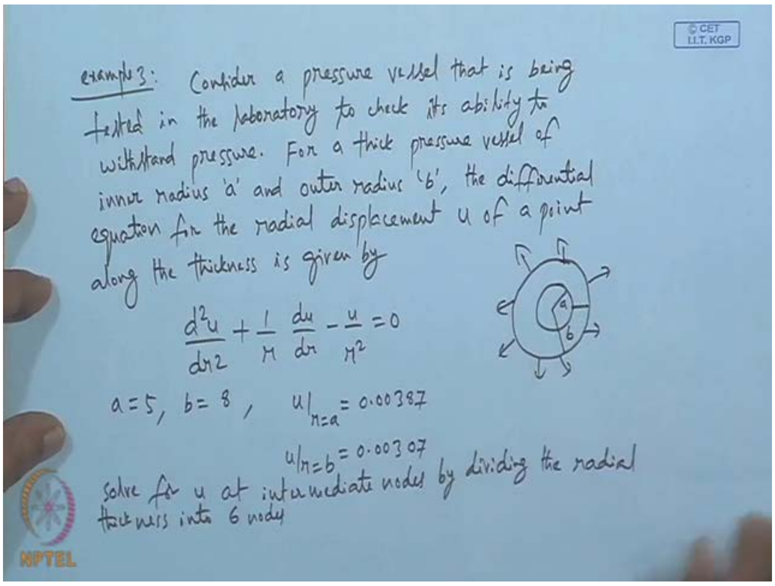

So far, whatever examples we have discussed or a while delivering lecture as well many of the examples, many times in the books they are given in terms of cartesion coordinates. So, let us look at a slightly a different problem. So, I would like to write down completely, so that we literally in a better position. Consider a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. So, there is a vessel so in this case we are going to define it, it is circular and it is being tested to check its ability to withstand pressure.

For a thick vessel of inner radius a and outer radius b the differential. So, you can consider a some tube, so inner radius a outer radius b there is a tube and we are testing the pressure. So, the differential equation for the radial displacement u of a point, along the thickness is given by is given by so that means. So, the radial displacement along the thickness, so it just a function of r, right? Radically, because we are testing for the pressure so the radial displacement so it is a tube kind of thing.

So, it is done by this equation and let us say a is some 5 units b is 8 units and given u at r equals to a is some value, and u at r equals to b, solve for u at intermediate nodes by dividing the radial thickness, radial thickness into 6 nodes. Suppose, this a this a kind of a application what I would like to mention is its need not to be every time we are in a Cartesian frame work, so this time we are in a radial frame work now so this a to b this distance, we have to cut it into six nodes.

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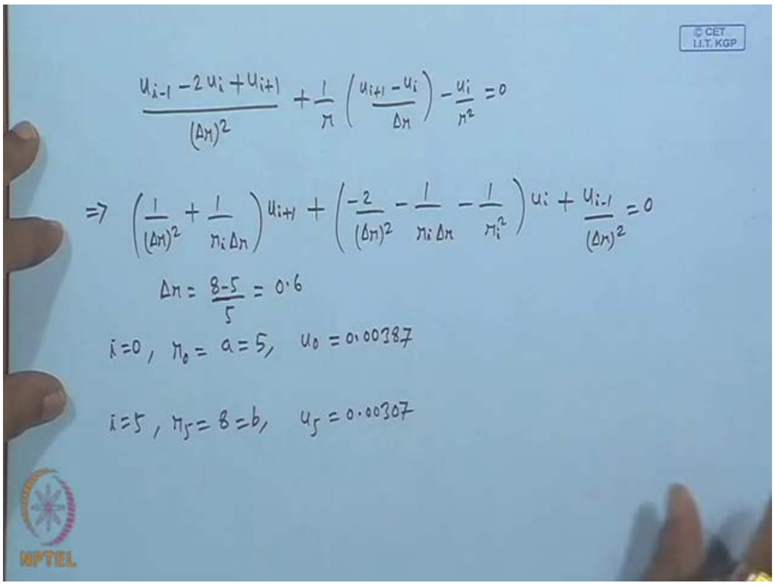

So, let us discretize, so discretizing we get so instead of h, since it is a radial coordinate the increment delta r square. So, this is u i by r square, now this can be written by collecting the coefficients and delta r b minus a by 5, 6 nodes of 5 intervals. So, this will be 0.6 i equals to 0, we have r 0 equals to a equals to 5 and the corresponding u 0 is then i equals 5, r 5 is 8 and u 5 is…

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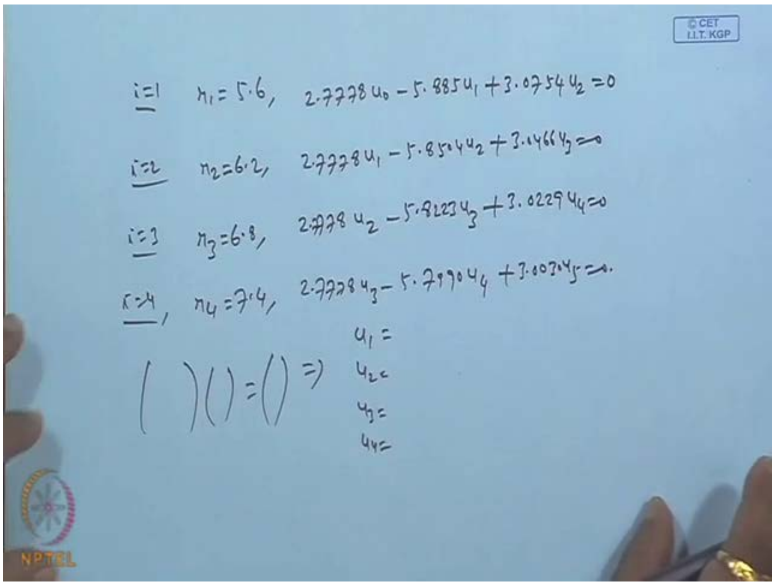

Now, intermediate values we write i equals to 1 r 1 is 5.6, then we simplify the coefficients and write a system. So, i equals to 4, so we can solve or this a tri diagonal system so we can obtain u 1, u 2, u 3, u 4. So, this is some sort of application where we do not have always a radial coordinate. So, we have we do, do not have a Cartesian coordinate we have a radial coordinate.

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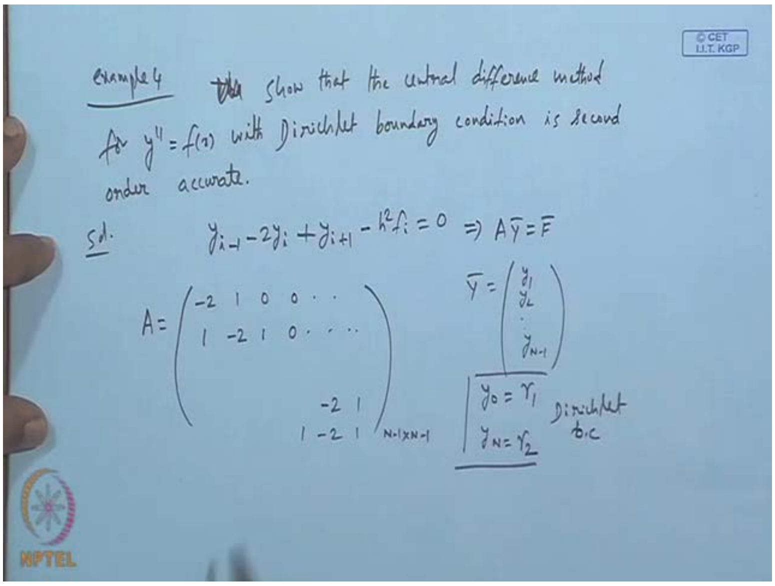

Show that the central difference method for y double equals to f of x with Dirichlet boundary condition is second order accurate. So, that means show that the central difference method used for this with Dirichlet boundary condition is second order accurate. So, that means we have to show that the error is second order. So, if we use central difference, we get this then if you put it in the matrix form, the matrix A will be, so then here. So, this is our A so this can be put it in the form where, so this matrix N minus 1 cross N minus 1, then we have y 0 is some gamma one y n gamma 2. These are the Dirichlet boundary conditions because no derivatives are involved.

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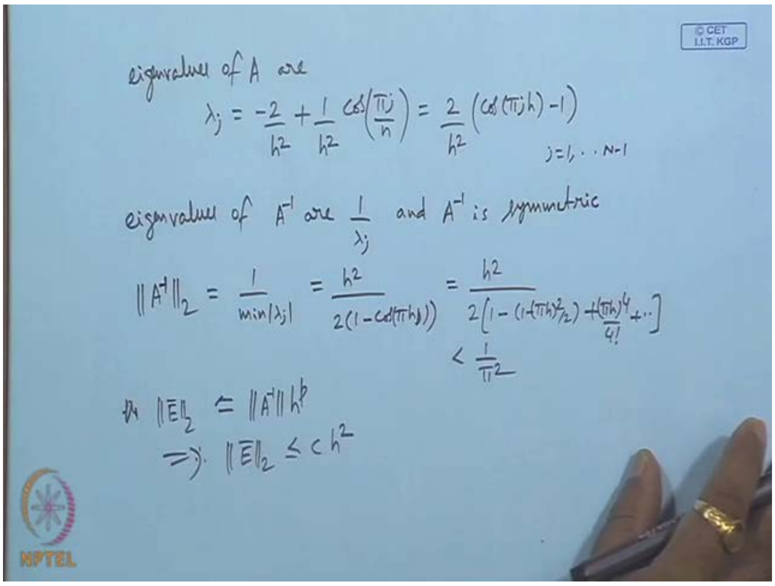

Now, so the Eigen values of A are lambda j so this is so these are the Eigen values. Then the Eigen values of A inverse are 1 over lambda j and A inverse is symmetric. Therefore, the 2 norm I have given in one of the lectures, so you may refer for the error to be bounded there is a condition I have stated a condition. The A inverse the norm of A inverse must be bounded, so this with 2 norm it will be this, so this h square, so this can be shown to be phi h phi h square by 2 by phi h 4 by 4 factorial.

So, minimum we take it so this is phi h, so this, this can be shown to be bounded, therefore the error with 2 norm this is actually this is we have obtained times h square second order accurate this is 1 p is the method. So, in this case so it is not therefore we have this so in this case this is this and we are getting h square, so the total this employees is less then or equals to some c times h square. So, therefore it is second order accurate. So, one can check accordingly the accuracy of the methods.

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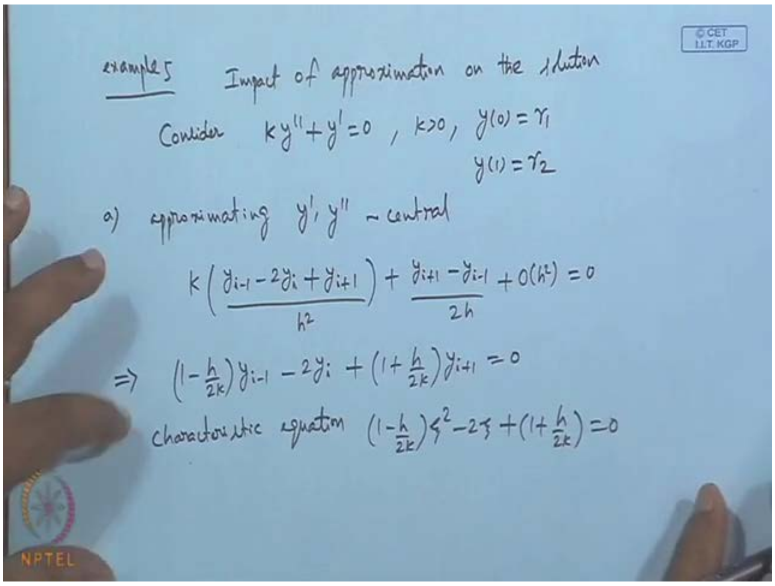

So, this example will give impact of approximation on the solution consider, so this can be thought as one dimensional defection equation. So, in the context of the convention defection the k has a specific meaning any dropping that this can be thought of application to convection defection problem. Now, we would like to discuss impact of the approximation of the solution right. So, case a approximating with the derivatives using central, so if you do that so this can be put it in the form, so we get the following difference equation then the characteristic equation is given by this.

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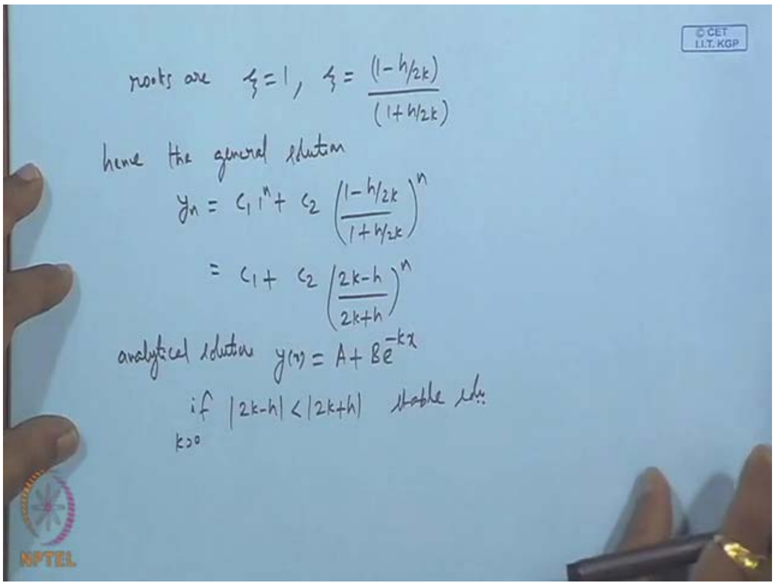

The roots, so roots are this hence the general solution is given by y n is c 1 1 power n c 2. So, this is the general solution, so we need to obtain the conditions first ability because what is the use, we are trying to discuss impact of the approximation on the solution. So, for the central we obtain this, and this is a solution. So, this is c 1 plus c 2, 2 k minus h and analytical solution. That means, the exact solution so if you look at this, so this, this is a disturbance and this how it is growing, so we have to discuss the growth of the disturbance so if mod is less than, than we get stable solution. Otherwise this would grow, if mod 2 k minus h is greater than 2 k plus h. Of course, so this is one condition.

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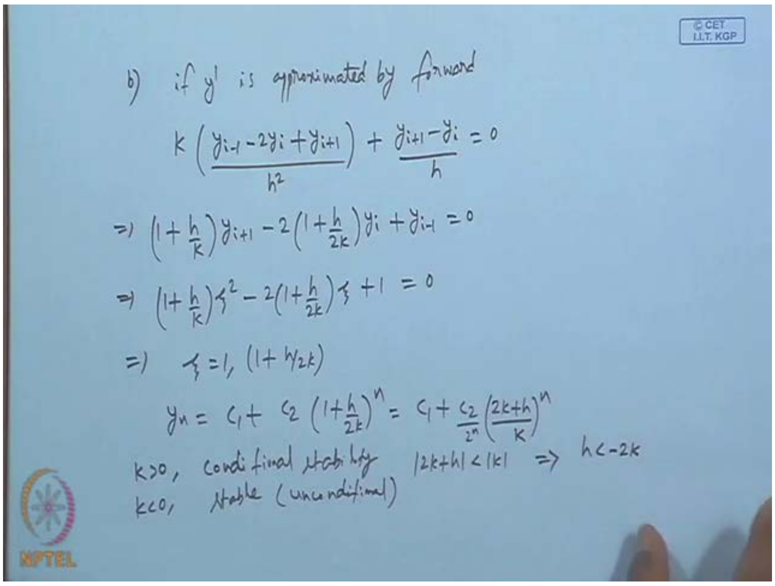

Now, let us look at b if y dash is approximated by forward then of course, y double prime usual central. So, then this implies, so the characteristic equation and the roots are therefore, the general solution is this is the general solution. So, this is now k greater than 0. So, let us check the roots of this equation, so I think these are the correct, so these are the roots and then this is a solution.

So, k greater than 0, if k greater than 0 then we get conditional stability, so what should be the condition if k is greater than 0, we should get see essentially we should obtain the condition mod 2 k plus h must be less than mod k. So, then we get a stable solutions so this is a region we should try to get, so this implies we may get minus 2 k then k less than 0, we get stable unconditional and k is less than 0, so k is much smaller than 1, it is are the infeasible because when grows. So, this is the impact of the differences approximation.

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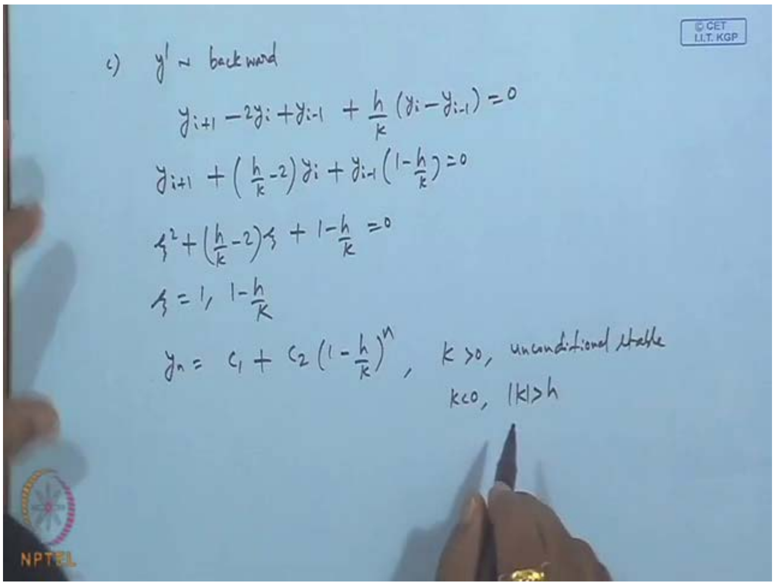

C y dash these approximated by backward, so this will be the characteristic equation is and the roots are, therefore the solution general solution is c 1 plus c 2 1 minus h by k power n. Therefore, k greater than 0 so this will be unconditional stable k less than 0 we must get the condition. If k is less than 0 so then we must get k has to be less than h modulus of k, so this is the conditional stability.

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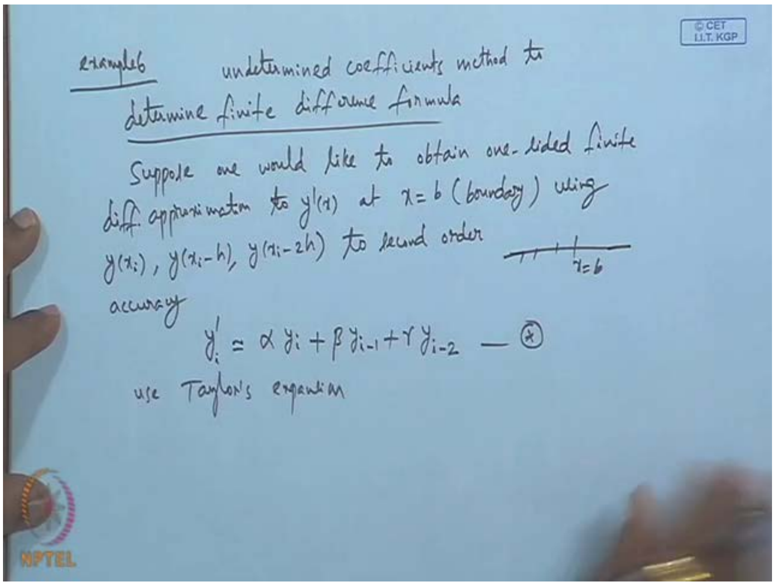

So, let us proceed further with few more examples undetermined coefficients method to determined finite difference formula. Suppose, one would like to obtain one sided finite difference approximation to y dashed x at say x equal to b which is the boundary using. So, one sided means see this is x equals b, so one sided means does not involve one of the sides so either this side or this side.

So, using in this case to second order accuracy, so what is our aim? To obtain y dashed x approximate up to second order in terms of one sided all terms to the left hand left side so that means y I prime is alpha y i beta y i minus 1 plus gamma y i minus 2, it must be of this form. So, we have to now determine undetermined coefficients so we have to determine the coefficients such that this approximation is second order. So, what do we do? Use Taylor's expansion.

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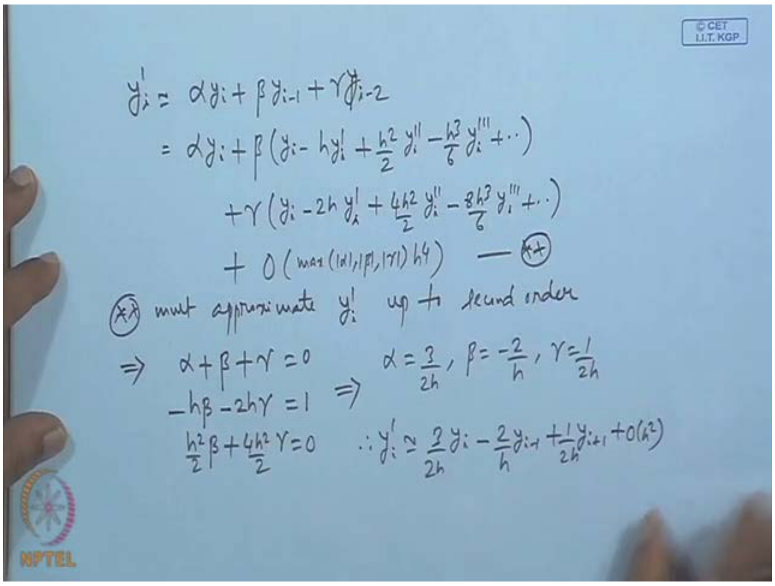

So, by doing this is alpha y i beta y i minus 1 gamma y i minus 2, so this alpha y i beta y i minus h y i prime is h square by 2 y i double plus gamma y i minus 2 h this is y i minus 2. So, 2 h plus, plus order of maximum of mod alpha mod beta mod gamma h power 4, so the remaining. Now, double star must approximate y i prime up to second order so up to second order this must approximate.

So, let us force the coefficients up to second order so from here so we need up to second order approximation. So, y i is alpha plus beta plus gamma is 0 and y i prime right hand side minus beta h, then here minus 2 h gamma and right hand side 1 then squared here beta h square by 2 and here h square by 2 gamma and here there is nothing. Now, on solving we get alpha is 2 by h beta is minus 2 by h gamma is this therefore, we get the approximation 3 by 2 h y i i minus 1.

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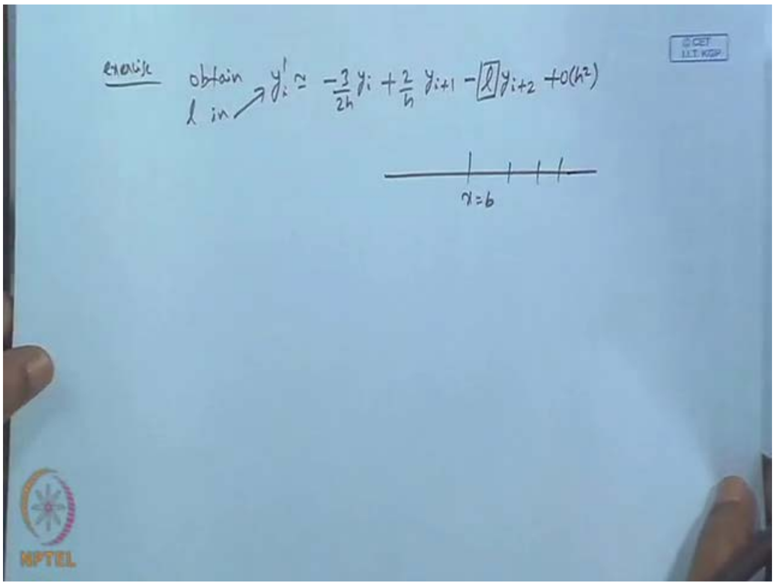

So, an exercise for you obtain, so obtain l in so that means you have to determine this, so you can see in this case. Suppose, this is x, so this is one sided, but the right hand side this considered, so this is very similar to what we have done.

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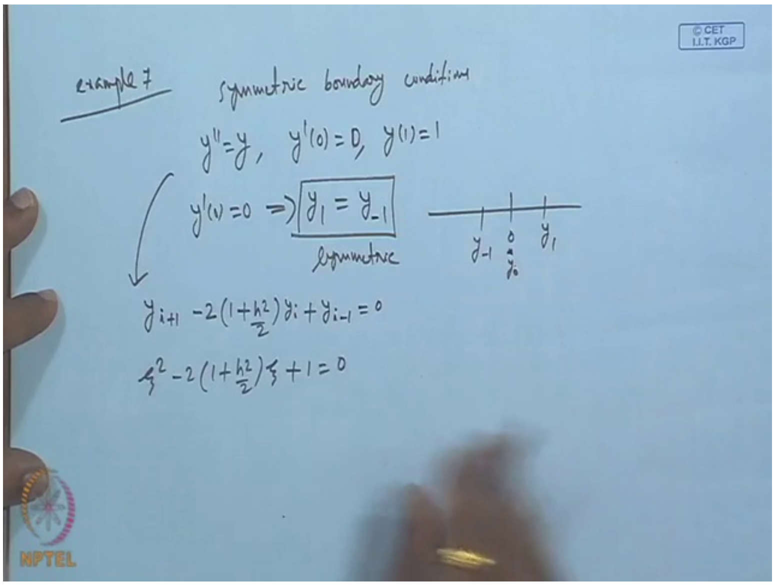

Now, it is example seven let us talk about symmetric boundary conditions, suppose we have the following problem y dashed of 0 is one of course. So, if we discretize suppose the condition is this. Suppose if we discretize y dashed of 0 is 0 this implies y 1 equals. So, this is posing symmetry, so here it is y 0 now this is symmetric case, now if we discretize and the characteristic equation and the roots are…

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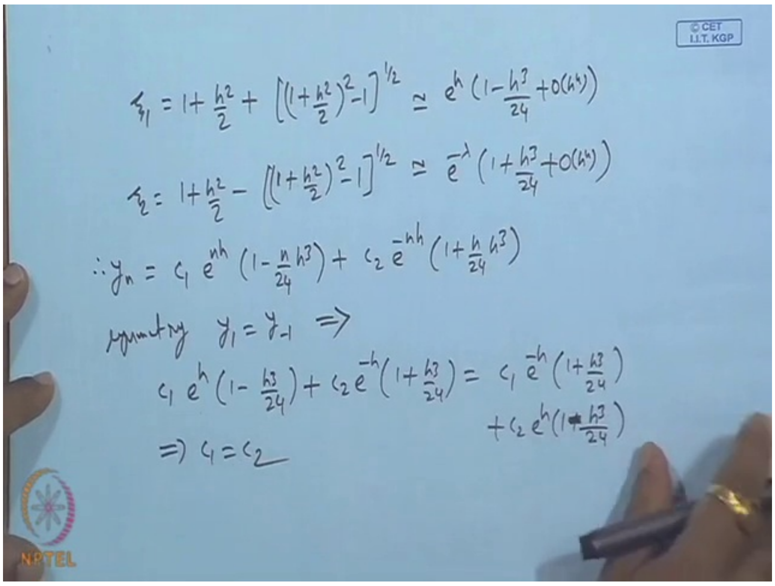

And this is so one can approximate this by series expansion, therefore the general solution is so this is the general solution now the symmetry y 1 is y minus 1. So, this implies c 1 e h y 1 minus 1. So, this is minus here so this implies c 1 equals to c 2, so that means when we have symmetry boundary condition.

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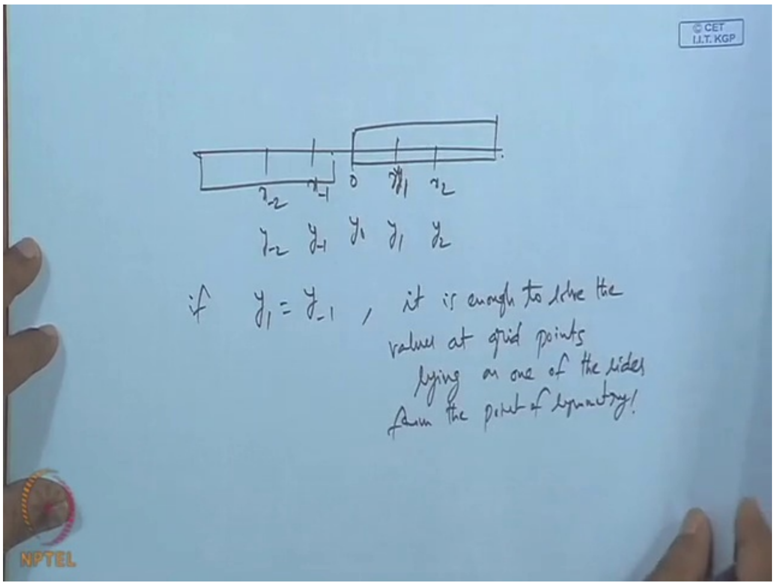

Suppose, this is 0 so this is x 1 x minus 1 x 2 x minus 2 so the correspondingly, so if we have symmetry conditions then it is enough to solve the values at grid points, lying on one of the sides from the point of symmetric. So, for example, we can solve here or we can solve here.

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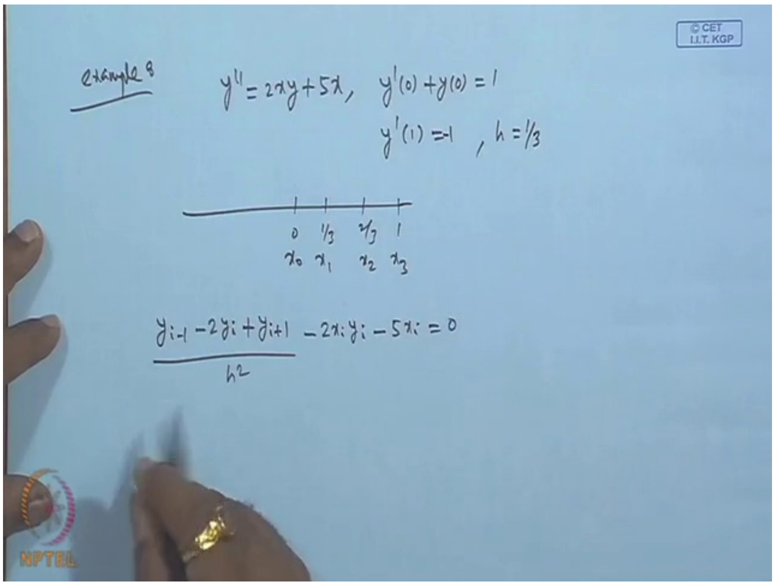

So let us look at derivative boundary conditions. So, minus 1 and the h is 1 by 3, so usual discretization, now usual discretization.

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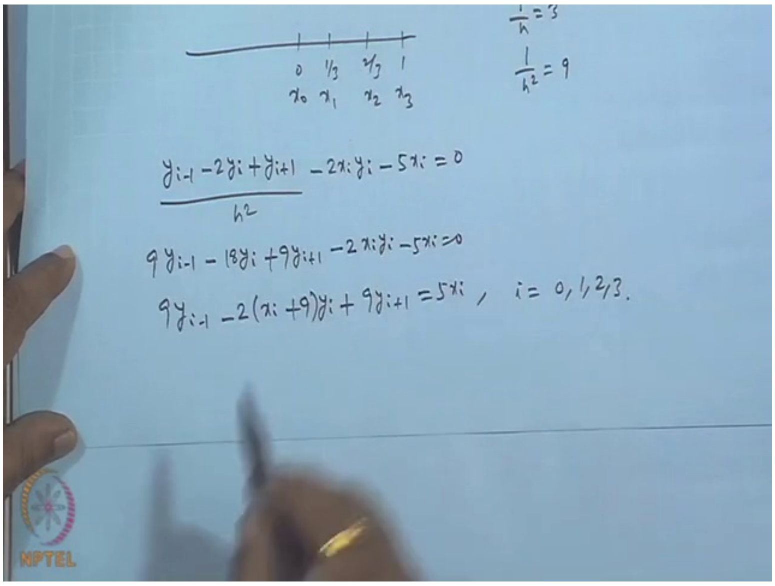

So, we have so 1 over h is 3 so 1 over h square is 9, so we have 9 minus 18 y i. So, we have minus x i plus 9 y, so this must be done from i 0, 1, 2, 3. The reason is we have derivative foundry conditions.

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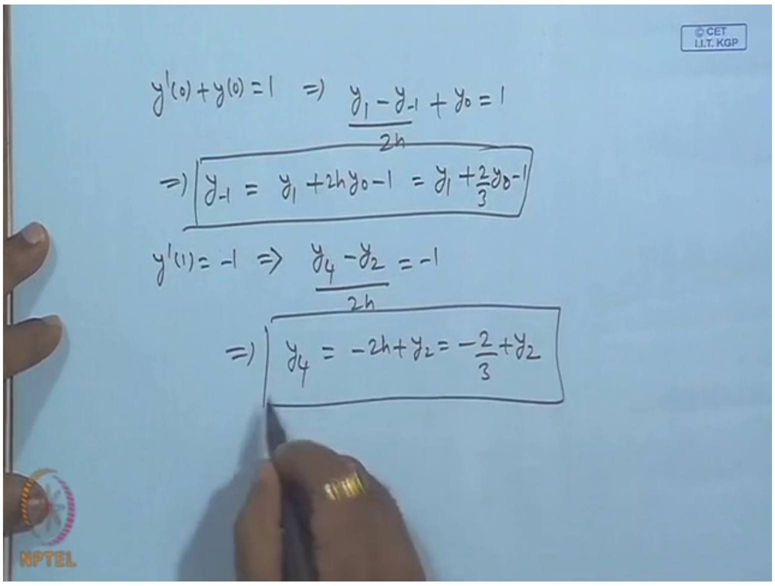

So this will be y 1 minus 1 by 2 h so y 0 is 1, so this employees y minus 1 equals y minus 1 equals y 1. So, this y 1 plus 2 by 3 or 0, and the other we have y dash 1 is minus 1, so this employees y 4 y 2 by 2 h minus 1. So, this employees y 4 is so we have 2 fictions values then we run the equation 0, 1, 2, 3. So, we get 4 so how many unknowns so 2 fictions value will come, so they will be eliminated so 4 equations in 4 unknowns so if you run this for example.

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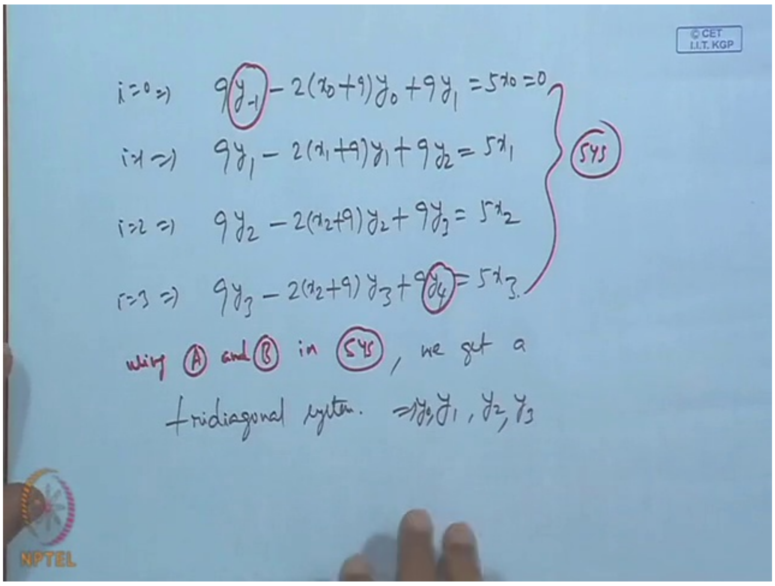

i 0 employees 9 y minus 1 x 0 is 0 y 0 y 1, so we have then y 1 minus 2 x 1 plus 9. So, where the fictions values and with already we have computed. So, we substitute, so using A and B in sys, say this is sys we get we get a tri diagonal system. So, one can solve so these examples I am sure, I would place you little comfort, because we have covered variety of issues redefinement and boundary conditions error calculation, and several aspects, for example how to obtain a different scheme like undetermined coefficients. So, this teach you definitely little comfortable, so that you can attempt any problem covering final difference methods. So, let us look forward for moving further until then bye.