Numerical Solution of Ordinary and Partial differential Equations
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Lecture - 2
Single - Step Method for IVPs

Good morning. So, the second lecture is single step methods for initial value problems. So, we have learnt formally, what is an initial value problem? Now, we would like to learn single step methods which are numerical methods to solve initial value problem.

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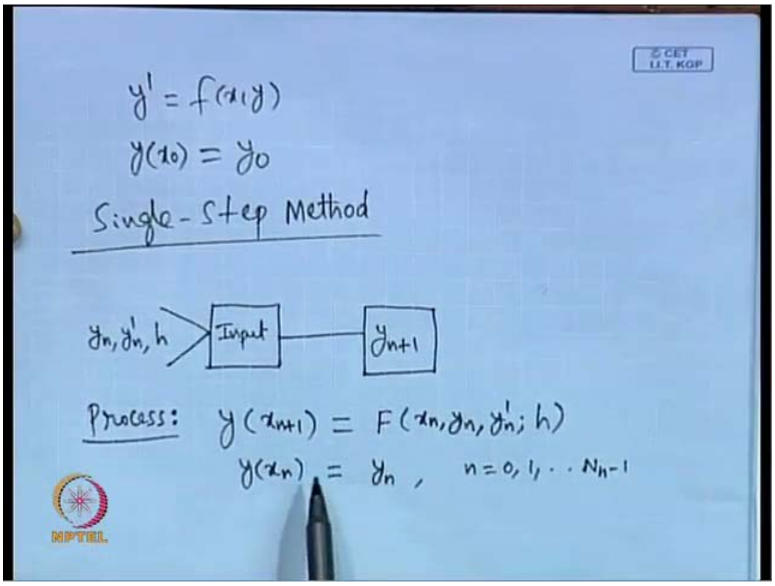

So, let us consider an IVP as follows y dash equals to f of x y, y of x 0 is y 0. Now, we would like to define single step method, single step method. So, what is this? Basically the numerical methods are the algorithms in some sense, they are the algorithms to get values at discrete points. So, your single step method looks like this, we have input and we get output. So, what is input? We are giving y n, y n prime, and the step size. Then we get y n pulse 1.

Now, what is reason why are we are calling this is single step method. So, let us define the process. So, then we will understand y of x n plus 1 equals f of x n, y n, y n prime h and y of x n equals y n, n equals to 0, 1 say some N h minus 1. So, why we are calling this is a single step method. If you see carefully, there are two indices n plus 1 and n involved. So, the right hand side expects values at n, and left hand side we are obtaining at n plus 1 that means the process is demanding only one past value. Only one past value that means at x n, then it is generating y of x n plus 1. So, this is the motivation to say that this process is a single step method. Why it is single step, now you can correlate y of n plus 1 is in terms of some y of n. So, only one past value is required. So therefore, it is single step method.

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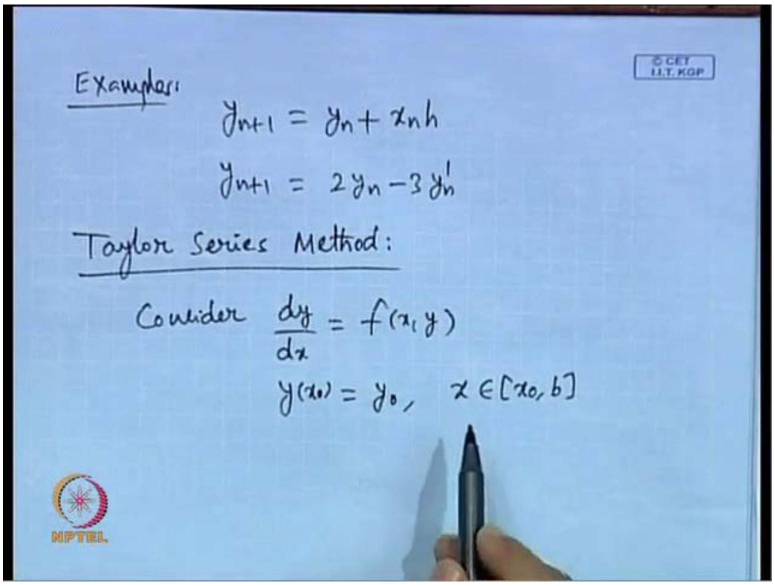

So, example say y n plus 1 is y n plus x n h. Suppose there is a method like this. So, this is single step method because it is asking only x n and y n, it is generating y n plus 1. Suppose, say 2 y n minus 3 y n prime. So, this is also single step method because it is asking only information at 1 pass 2 valves that is at x n. So, y n of x n y n prime is y dash of x n. So, we get y n plus 1. So, this are the single step method.

Now, we would like to learn a particular method to solve I V Ps and this particular method is a single step method. So, what is that it is very popularly known method Taylor series method. So, we consider the I V P d y by d x equal to f of x y, y of x 0 equals to y 0 x belongs to x naught b. Now, when we say Taylor series, definitely we are going to use Taylor series expansion of the solution about some point, but before we do that it will be better to know the assumptions and at which Taylor series expansion possible.

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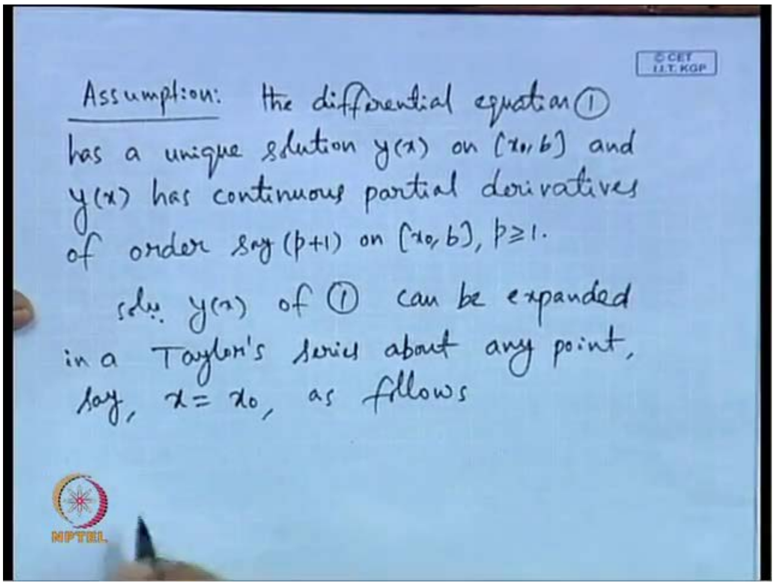

So we would like to write down assumption. So, what is assumption, the differential equation which equation say differential equation this is 1. So, the differential equation 1 has a unique solution y of x on x 0 b and y of x has continuous partial derivatives of order say p plus 1 on. So, this is assumption what is that the differential equation 1 has a unique solution y of x on x 0 b. Essentially this is existence the uniqueness theorem and y of x has continuous partial derivatives of order say p plus 1 on the central. So, then the solution y of x of 1, can be expanded in a Taylor series about any point say x equal to x not as follows.

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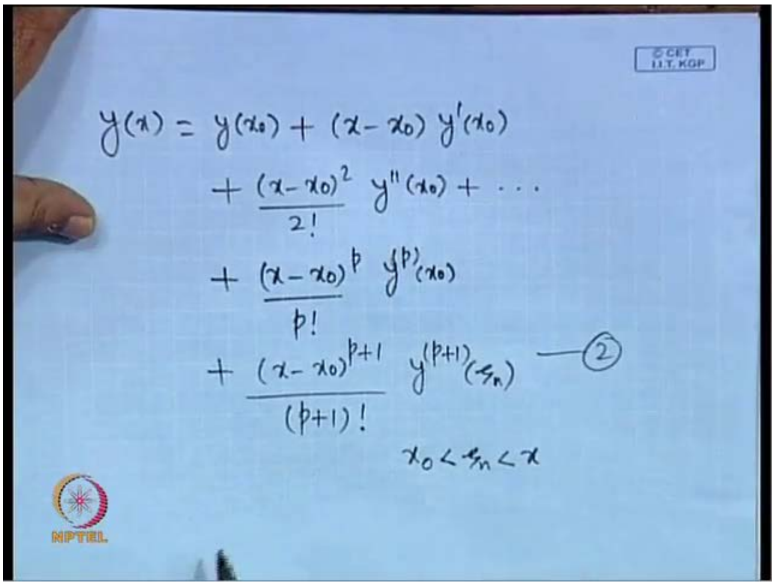

We will write down Taylor series about x equal to x naught. So, it is given by y prime of x naught plus factorial 2 y double of x not plus, so on plus, p so I switched over from this notation to this notation. Please try to follow y 1 y 2 and then p is. So, it is a customary to put it in a parenthesis, this plus where this is since we are expanding around x 0 within this. So, let us call 2. So, in case we are expanding around x n then zeta varies x n to x.

Now, this is standard Taylor series expansion about x equal to x naught. Now, definitely when we use Taylor series expansion to compute a solutions at the particular point we have lot of restriction, what are the restrictions. The first restriction is how long we consider, how long we take terms into account like 5 terms 10 terms 20 terms. So, that means we have to consider the number of terms in that account.

Naturally one would guess easily that the accuracy of your method as direct consequence in terms of the number of accounts. So, let us see the last terms we have written in a very specific form because we have not written in a terms of a x 0 rather we have put it as zeta n, where zeta n lies within some interval. Now, this last term has a particular name and then it has a vital role.

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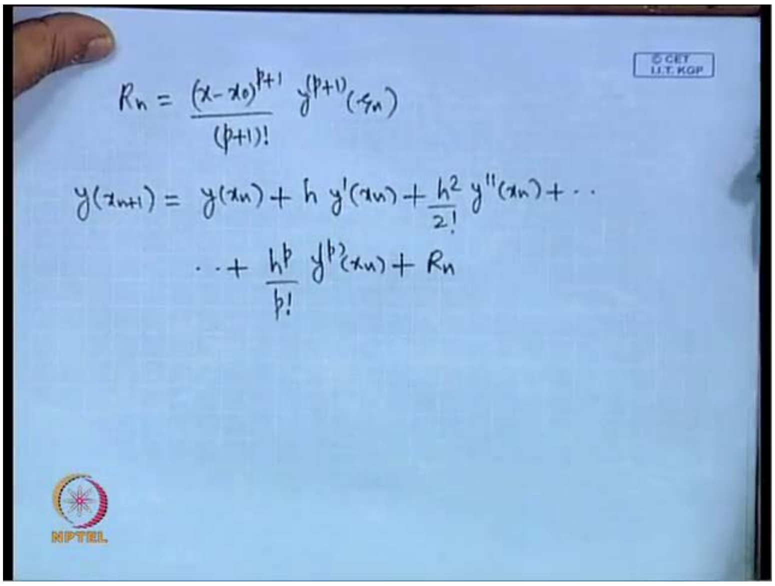

So, let us see what is that? So, this is reminded term R n. So, this the reminded term. So, if we generalize y of x n plus 1 y of x n and x h y dash of x n p plus R n. So, what is notation we are adopting now. Earlier we have written if you recall, we have written x minus x 0 x minus x 0 so on.

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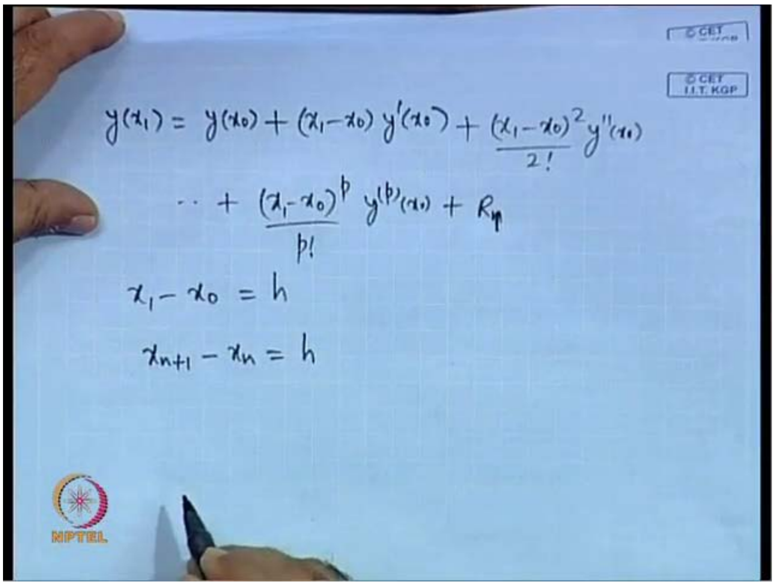

Now, if you need at a particular point x 1 y of x 1 is y of x 0 x 1 minus x 0 y dash x 0 plus x 1 minus x 0 square by factorial 2 y double of x 0 and x 1 minus x 0 for p, p factorial y p of x 0 plus the reminder. So, strictly speaking this should be R 1 because we are expanding at. So, what is x 1 minus x 0, this is h the step size therefore, x n plus 1 minus x n this is our h. So, what I just have written is at x n plus 1 the Taylor series expansion at x n plus 1.

So, this is a Taylor series expansion at x n plus 1 y of x n plus h y dash perfection plus x square by 2 factorial this and reminder terms right. Now, who is going to supply us as the higher order derivative, you see suppose somebody who like to use Taylor series expansion up to to 5 terms. So, then 5 terms means 1 2 3 4 will be h cube 5 will be h 4. That means derivative of y 5 are required. So, who is going to give us these the initial value problem which is going to give us.

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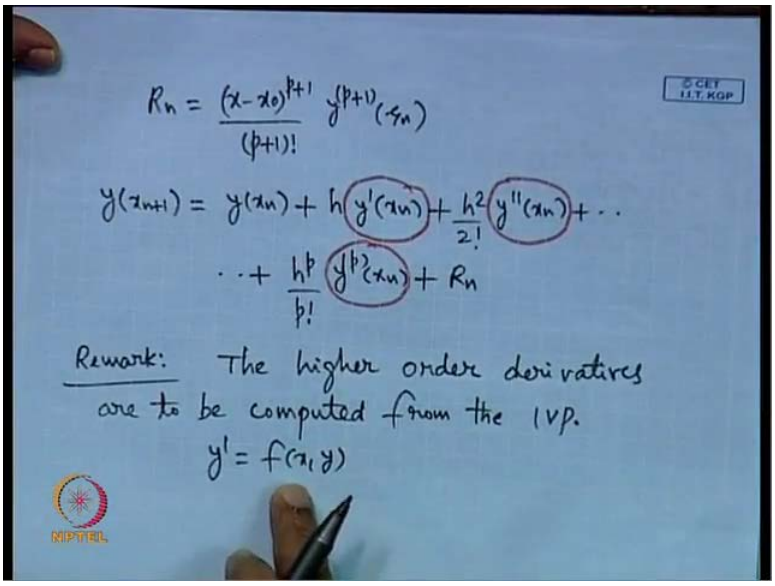

So, the remark here is the higher order derivatives. So, the higher order derivatives should be there the higher order derivatives are to be computed from the I V P, but how do we compute. If you look it your I V P is y dash equals to f of x y. So, we require high order derivatives means under assumptions what is assumption see, higher orders for example, y 2 will be first derivatives of f right. So, y 3 will be second derivatives of f. So, we required higher order derivatives of y which means we required higher order derivatives of f.

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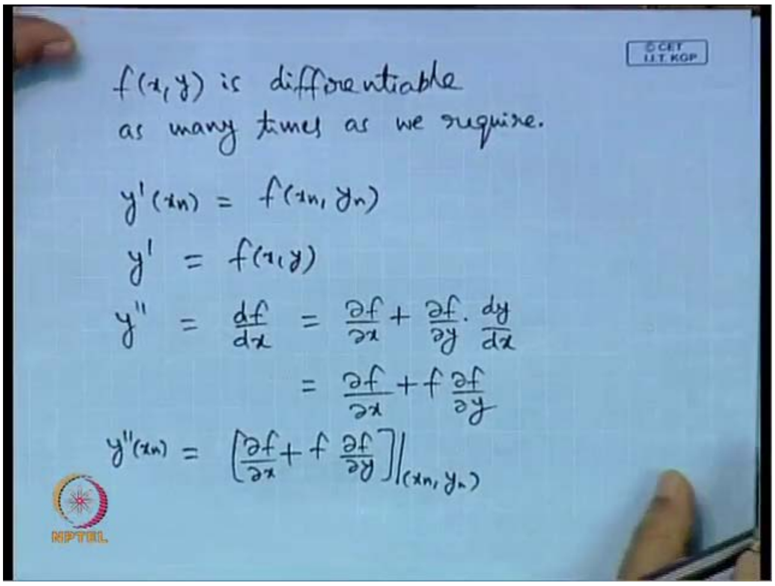

Now, to have that of what is assumption, the assumption is f of x y is differentiable as many times as we require. So, these are the two assumptions. The first assumptions is the higher order derivatives have to become computed from the initial problem and may this not assumption this is a the process, but for that what is assumptions required f of x y is differentiable as many times as we require. Now, let us try to compute. So, y dash of x n is f of x n y right now we need. So, in shorthand notation this is now we required y double. So, this is d f by d x.

So, this now we compute using chain rule, tau f by tau x plus tau f by tau y into d y by d x, but what is d y by d x, yes you are true d y by d x is f. So, it is f right. So, we need y double of x n this will be tau f by tau x plus f tau f by tau y at the point x n y n. Similarly, all other higher order terms can be computed. Now, which these let us see what happens to our Taylor series.

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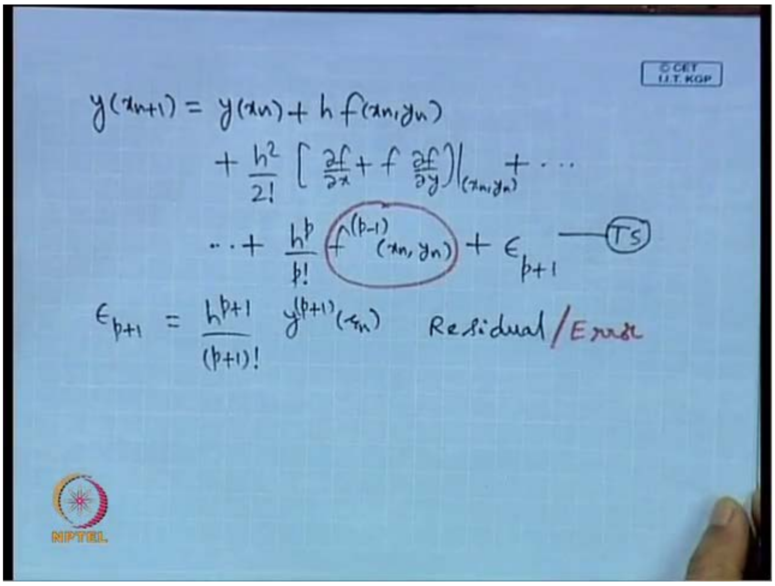

It is y of x n plus 1 x n plus h plus square the factorial 2 tau f by tau x plus f tau f by tau y at x n y n plus the higher order comes h p, p factorial f p minus 1 plus. So, the reminder I would like to introduce term like this. So, this expression I am calling T S. That is Taylor series where epsilon p plus 1 equals h p plus 1. So, this is residual, residual means the error. So, quick remarks observe this term carefully the derivative is p minus 1 x h power p, p minus 1 y is that because actual it should have been h for p y p, but y p is nothing but f p minus 1.

So, this is residual. So, what do you mean by residual, that is error of course, I have not defined formulary what is error. So, we will definitely defined. So, in what way we are going to define this error, yes as I mention somebody is doing up to 3 term. That means considering Taylor series up to 3 terms and somebody else considering up to 10 terms somebody 50 terms. So, we expect there will be a difference with the solution. So, y is that definitely see you consider only 3 terms other person consider 50 terms another person consider 100 terms.

So, the added terms should definitely refine your solutions. So, that it more accurate compare to the solution which is obtained with only 3 term right. So, when you are considering let us say 25 terms of your Taylor series, then what are we doing, we are force to neglect the terms from 26 onwards right. So, you expect that your solutions is accurate up to that level what is that level up to 25 terms 26 and onwards we are neglected.

So, if at all there is a deviation in your solution with that true solution you expect that the deviations contributions is from 26 terms onwards. Suppose, somebody considers 50 terms than the deviation from the exact solution will be from 50 first onwards right. So, we are going to define this residual and discuss about it in a more regress way. So, that means if somebody says I do not mind my solutions is if it is a accurate only up to 1 decimal I do not bother.

So, then how many terms we can consider and keep quite right we will be interested in because somebody is asking you to have accuracy of only 1 decimal. So, why do you bother to compute 100 terms 200 terms it is not required right. So, similarly, somebody would say with this step size, if you consider this many terms, then what will be the accuracy. So, these kind of a trade off various trade off, what does the tradeoffs between if you know the error, means if you know up to this accuracy I need the solution. So, that is 1 quantity the number of terms another quantity, then the step size is another quantity. So, there is a kind of a trade of among this quantity.

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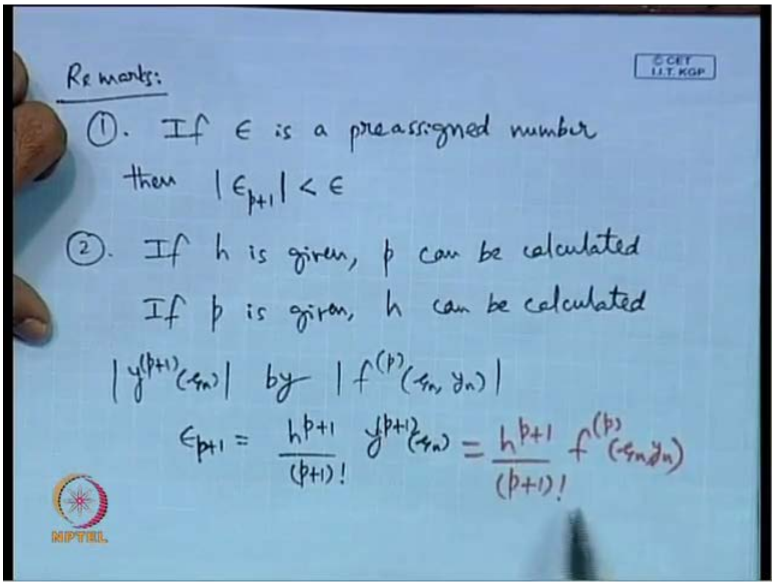

So, lets us have careful look. So, remarks 1 if epsilon is a pre-assigned number, then nod epsilon p plus 1 is less than, see this is the condition on the arrow means epsilon is pre-assigned we expect that you must continue up to this condition. That is your residual must be less than epsilon. Then if h is given step size p can be calculated, if p is given h can be calculated theoretically right, provided we replace by is this if you look at your epsilon p plus 1 it is h for h for p plus 1 by p plus 1 factorial and y p plus 1 of zeta n. So, this is hp plus 1 by plus 1 factorial f p of zeta n y n. So, this our residual, now this residual is less than pre assigned number.

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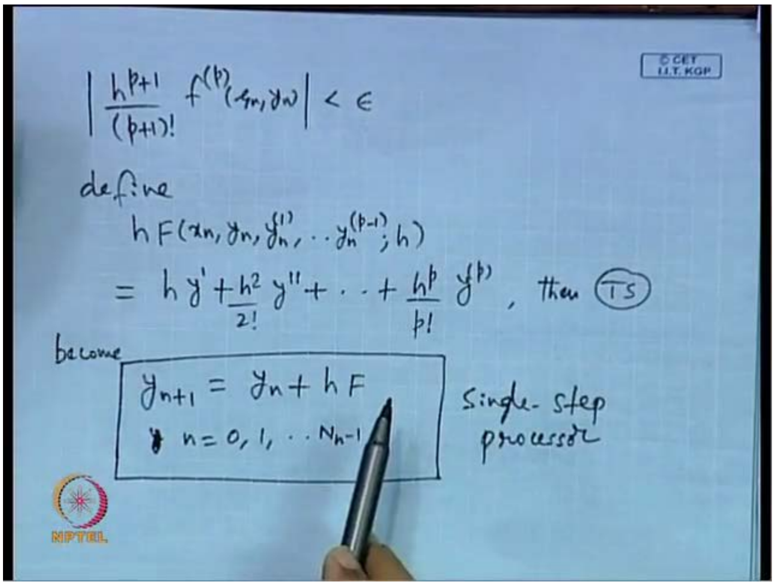

So, mod of less than epsilon pre assigned. So, if h is known f is our hand. So, we have to take the maximum of this and epsilon is known. So, we should compute p. Suppose p known and epsilon is off course known you should be able to compute it theoretically. So, this is trade of essentially that means, if the error is in your hand means if you know up to this accuracy I need the solution and what are the other left out parameters is step size h and number of term. So, you have to play with this two. The more number of terms you expect more accuracy, same time if you jump. So, you have taken at 1 this point then with large step size your jumping.

Then one would expect less accurate solution. So, this trade of is very much important. So, when we come across examples specific examples we will discuss how to do this. Now, what is the process involved in Taylor series. So, let us define h times f of x n y n then y n 1 y n p minus 1 h as y dash y double y p. So, this nothing but the right hand side of Taylor series, but for one term. So, if we define like this, then what was a number we have given T s, then T s become y n plus 1 equals y n plus h f. So, where we need y we need all the quantities. So, it is n is 0 1 say something.

So, this is the corresponding single step processor. So, we believe that now Taylor series method is a single step method as you can see it will ask only 1 past value however higher order derivatives. So, we are not denying that. So, that as that as do with number of terms, where as all they values whether it is first derivatives second derivatives 50 th derivatives. All the terms are expected only at 1 past point, so that is motivation for single step method. Therefore, will believe that Taylor series method is a single step method, before we discuss more about the error.

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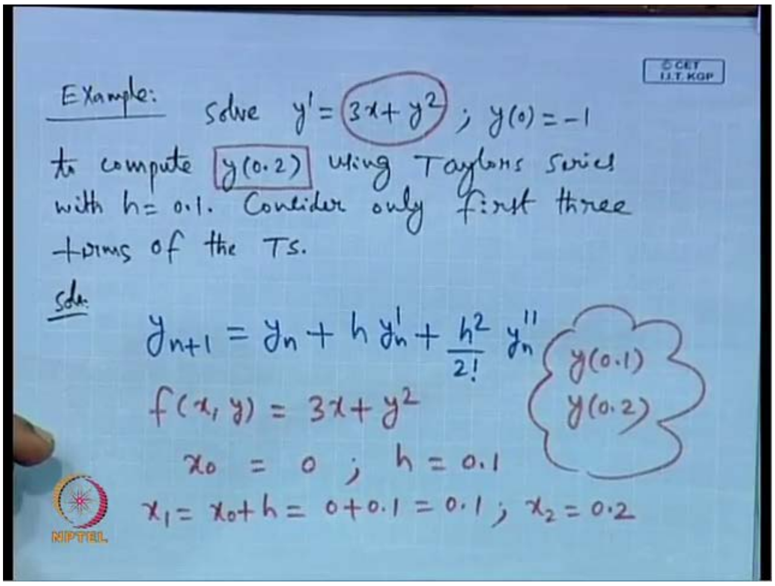

So, let us look at once specific example and see how do we compute the solution using Taylor series. The example say solve y dash d s 3 x y square and y of 0 is minus 1. To compute y of 0.2 using Taylor series with h equals 0.1, then consider only first 3 terms of the Taylor series. So, suppose this is an example. So, what it says for this o d this is the initial condition, we compute y of 0.2 using Taylor series with step size 0.1 now what is remaining the number of terms. So, consider only first 3 terms of the Taylor series. So, let us briefly look at the solutions. So, since it is suggesting to use only first 3 terms. So, let us look at the Taylor series. The Taylor series is given by y n plus 1 is y n plus h factorial 2 this is our Taylor series first 3 terms 1 2 3.

Now, let us identify what is known data from their from the problems the known data is f. This is our f, f of x y equals 3 x plus y square and what else is known x 0, x 0 is 0 then what is known step size 0.1. Now, we need to compute y at 0.2. So, what is required see with this h is 0.1 with x 0 is 0 what is x 1 x 0 plus h that is. So, this suggest that our x 2 is 0.2 that means we need to compute the solution at y o f 0.1 and y of 0.2. So, that means we have to solve for y of 0.1 y of 0.2. So, we solve let us in order to solve what we required, we require to compute this term and this term rest in hand. So, y dash and y double dash you have to compute. So, let us do that.

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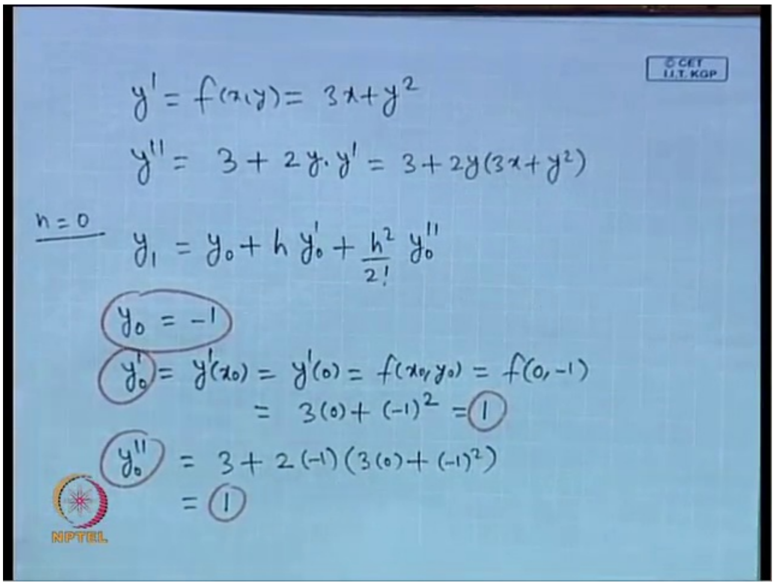

So, y dash is also in hand f of x y which is given by 3 x plus y square y double is 3. So, we can do using the formula because general expression I have computed, but since we have particular case in hand. I would like to do the chain rule directly here. So, y double derivatives of this respect x the 3 plus 2, y 2 y dash. So, this is p plus 2 y, y dash is 3 x plus. Now, consider the case n equals to 0. So, then n is equals to 0. In this Taylor series y 1 is y 0 plus h y 0 prime plus h square by factorial 2 y 0 double prime right.

So, let us write down y 1 is y 0 h y 0 prime h square by factorial 2 y 0 double prime. So, we have y, y dash and y double, now we required y. So, y 0 is minus 1 y 0 prime is y dash of x 0 which is y dash of 0 which is f of x 0 y 0 which is f of 0 y 0 is Minus 1. So, this is 3 time 0 minus 1 square equals 1 right then y 0 double. So, you can do it. So, this is expression in hand. So, directly we can substitute. So, I do not want to do this algebra here this is just for clarity.

Now, directly 3 plus 2 into y 0 is minus 1 3 into x 0 is 0 plus minus 1 square. So, this is this term is 0 this is 1. So, 3 minus 2 this is 1. So, maybe I have taken example where the calculations are simplified. So, we have the known data required in hand is this together with this together with this and this.

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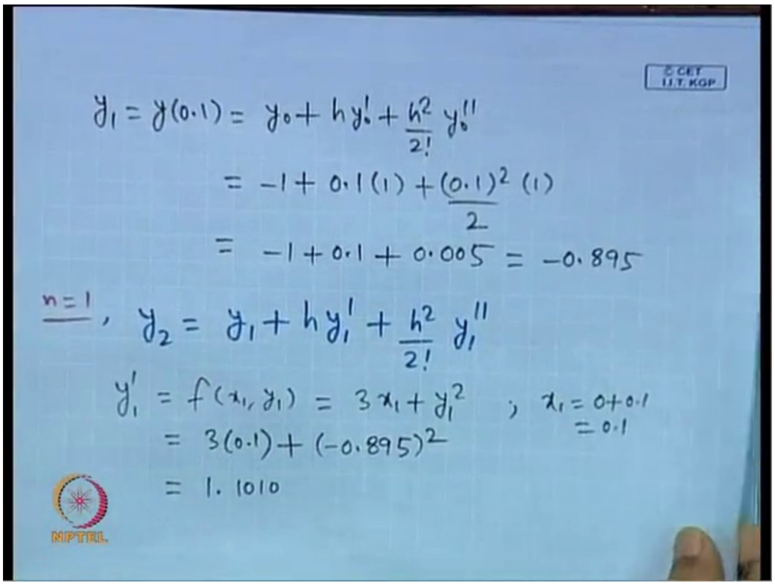

Now, we would like to substitute in star. So, we do it. So, y 1 is y of 0.1 that is y 0 plus h. So, this is y 0 is minus 1 plus h is 0.1 the y 0 prime is 1 plus h factor to y 0 1. So, in this minus 1 plus 0.5. So, this is point so this our y 1. So, this corresponding to n equal to 0. Now, let us take n equals to 1. So, if n equals to 1 then what do we need y 2 y 1 plus h y 1 prime h square by 2 factorial y 1.

So, we have to compute, now y 1 prime y 1 double prime at x 1. So, y 1 prime is f of y 1. So, this is 3 x 1. So, x 0 x 0 h is 0.1. So, x 1 is 0 plus 0.1 therefore, and y 1 just now we have computed therefore. So, this one can simplify we get this right now, next quantity you have to compute this by 1 double.

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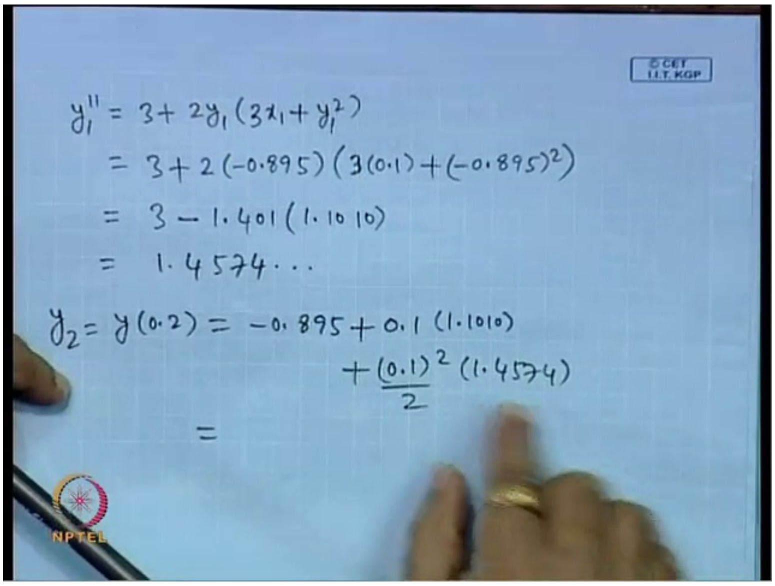

So, y 1 double is 3 plus 2 y 1 3 x 1 plus y 1 square equals 3 plus and 2 into minus 0.8953 into 0.1 0.895 square. So, this is 3 3. So, let us give this minus here to this is 1.401. So, let us you can also try you can also try. So, this is this is of course some value we get it. So, this is exactly y 1 prime right. So, we have it this is exactly 1.1010. So, this is 1.401 and it is final we get something like this 1.4574 y 1 double right now. So, once we get this we have the solution in hand. So, it is a straight substitution y 2 equals y of 0.2 equals y 1 y 1 is minus 0.895 plus h into y 1 prime. Just now we got 1.1010 plus x square by 21.4574.

So, this one can compute. So, we get some value. So, y 1 is a y of 0.1 and y 2 is y of 0.2. So, you can compute at higher grid points. So, you can, but remember we have used only the first 3 terms only right. So, if somebody would like to try further. So, there will be slide modification in the values what we have obtain. Now, quick remark here is, if you see while computing y 1 we would have rounded off.

For example, there is a 0.1 square is there. So, we would have a rounded it some accuracy. Then we have put down that then further y 1 has been used to compute y 2. So, we again cut down to some terms and then put down. So, every time you cut down. So, one kind of firm error is the number of terms the number of a terms. We are throwing after a second level. So, the other is at each computation you are throwing some digits and using that as an information to compute further values.

So, these are named with a specific terminology, but of course they are the errors the first one that is neglecting after certain number of terms is also termed of kind of errors. Then neglecting few digits and then using that information further is also termed as a kind of error. So, this more technical term exact technical terms I would define little later.

Now, let us look at the thrown part. In this example we have calculated using the first 3 terms that means we thrown from fourth term onwards. So, let us look at this fourth term. So, that we understand what is happening. So, let us look the fourth term. So, what is that?

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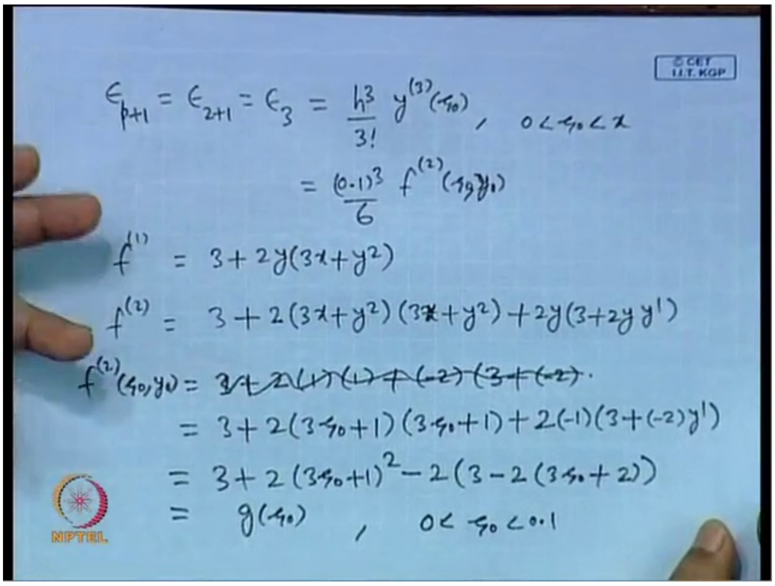

So, our p was 2 because h square, this is p plus 1. So, this is epsilon 2 plus 1 this is epsilon 3 this is. So, let us say we are considering this at first. So, that means while computing 0.1 in this. So, what is this and y 3 is f 2, y 3 is f 2, y 3 is f 2 right f 2 of this. So, for this example let us look at this quantity we have only f 1 in hand. So, what is that f 1 is 3 plus 2 y 3 x plus y square. Now, f 2 3 plus 2 y prime that is 2 y 3 x plus y square and keeping y prime is f.

So, that is 3 x plus y square plus 2 y into 3 plus 2 y by prime right. So, let us compute this f 2 at zeta 0 by 0. So, y 0 is for this problem y 0 we have consider is minus 1. So, if we 3 plus 2 into x 0 is 0 y 0 is minus 1. So, we get 1. So, this we get 1 plus y 0 is minus 1. So, minus 2 and here 3 plus this is minus 2. So, this not correct because x 0 3 plus 2 x zeta 0 plus 1 into 3 zeta 0 plus 1 minus 1 3 plus minus 2 y this is wide prime.

So, this for the time being I am putting next line. I can simplify 3 plus 2 into 3 zeta plus 1 square minus 2 into 3 minus 2 y prime is 3 zeta 0 plus 2. So, essentially this will be some function of zeta 0 this will be some function of zeta 0. Now, this zeta 0 is varying in this interval. So, let us say you are computing zeta 0. So, 0 less than zeta 0 let us 0.1 here computing next solution it is within this interval. Now, we have to analyze this. So, how do we analyze naturally we have to look for the maximum.

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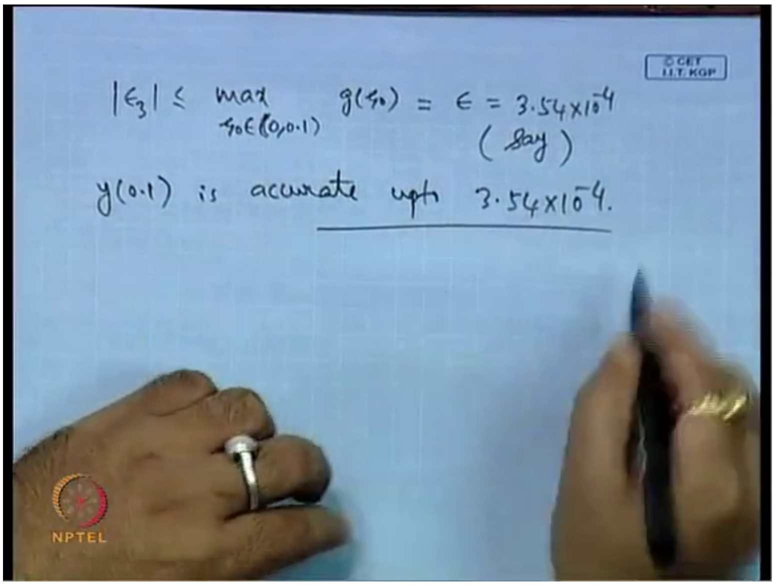

So, mod epsilon equals is less than are equals to maximum of zeta 0 in the interval 0 to.1 of g of this. So, this value if you compute then 1 can say y of 0.1 is accurate up to. So, this we get some say epsilon say we are getting this as some 3.54 into 10 minus 4. Say not for this problem because say bit lengthy calculations. So, I am just a showing you for example, that means y of 0.1 is accurate up to 3.54 into 10 minus 4.

So, this is the residual. So, I have not yet mentioned what is the error different technical names of various errors, what kind of errors I told you the number of terms there is 1 name and then we cut the digits there is another name. So, we formally learn coming lectures. So, this is how the Taylor series method would give you approximate solution. So, the assumptions are first of all you would able to compute higher order derivatives as time as possible then one can get approximate solution up to some accuracy which depends on the number of terms one would consider.

Thank you.