Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 19
Higher Order BVPs

Hello, in the last class we have learnt boundary value problems with derivative boundary conditions.

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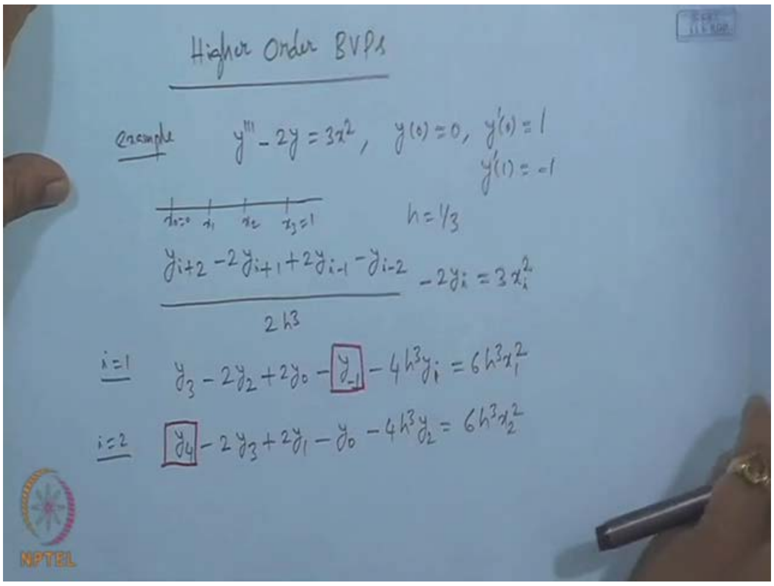

Now, let us learn higher order boundary value problems, higher order BVPs, so let us start with an example first consider this equation, so y dashed of 0 is 1 and say y of 1 is minus 1, say h is 1 by 3. Now, with usual discretization one must obtain for y 3 what should be the corresponding approximation, so we have done it in the first lecture.

So, accordingly y 3 is approximated as minus 2, y i is 3 x i square, now if we run this equation for i 1, y 3 is 2 y 2 plus 2 y 0 minus y 1 minus 4 h cube y i 6 h cube x i square and at i 2, 2 y 1 and this will be y 0 minus 4 is y 1 y 2 six x 2 square. So, what was our line, our line was x 0 is 0 and x 1, x 2, x 3 is 1. So, we have the following fictitious values, which are corresponding to the nodes outside the actual grid. Now, let us look at the boundary conditions, so the boundary conditions this by prime.

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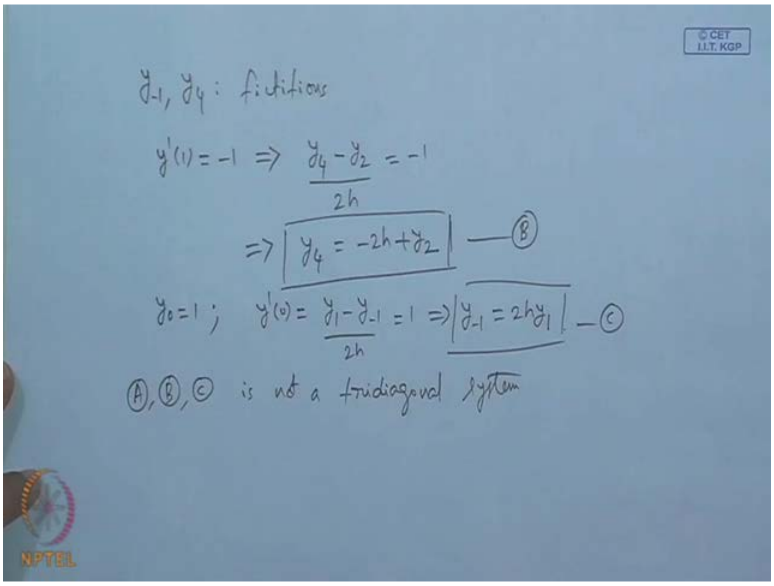

Let us discretize the boundary conditions, so we start as follows so these are fictitious then y dashed of 1 is minus 1 this is y 4 y 2 by 2 h minus 1 this implies then we have y 0 equals to 1 and then y dashed of 0 equals this implies. So, we have these two fictitious values. Now, if you look at suppose we call this a this y 0 is explicitly given, so this gets determined then we have y 1, y 2, y 3 there, so y 0 gets determined so y 1, y 2, y 3. So, three unknowns then this fictitious value is determined and this is determined, however A B C is not a tri diagonal that means there is some issue with respect to the number of unknowns and number of equations, well following earlier strategy.

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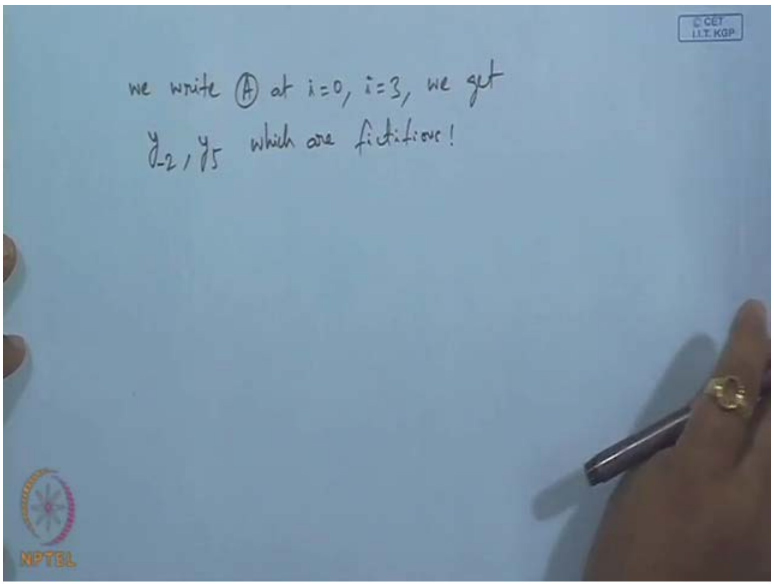

Suppose, we write we write at a i equals to 0, i equals to 3 we get y minus 2 and y 5 which are fictitious. So, it means that one cannot adopt the earlier strategy like you run the equation earlier in the sense which we have done with the derivative boundary conditions. So, one cannot adopt the similar strategy because in a sense of recurs manner you get number of fictitious values if you run further i minus 1, i minus 4 you get more fictitious values, so this strategy does not help.

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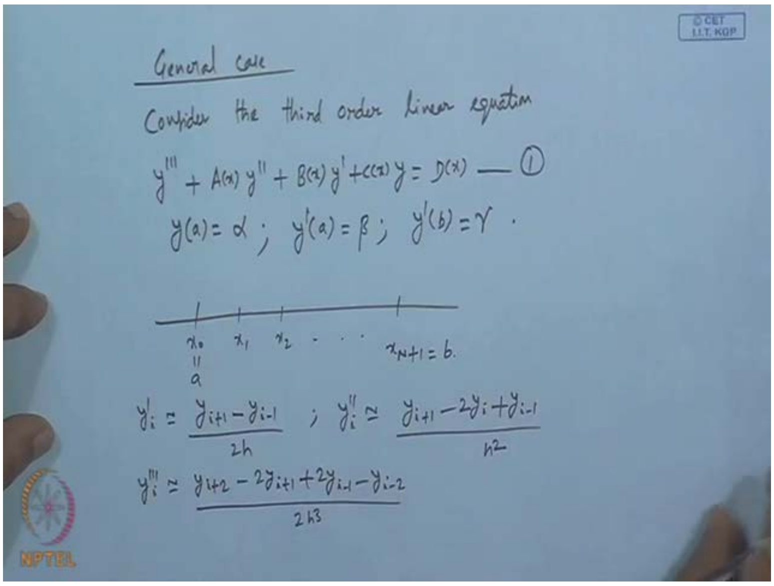

Let us look in to more general case, so this is more general case. So, we consider third order linear equation given by y 3 plus, so when I said general we have all the terms, so that makes life little complex, so 3 2 1 and y 0. So, all the terms are there and since it is third order equation we need a 3 boundary conditions, so y of a is alpha gamma. So, the grid points x n, x n plus 1 equals to b, now what are the approximations, so for first approximation second and for the third.

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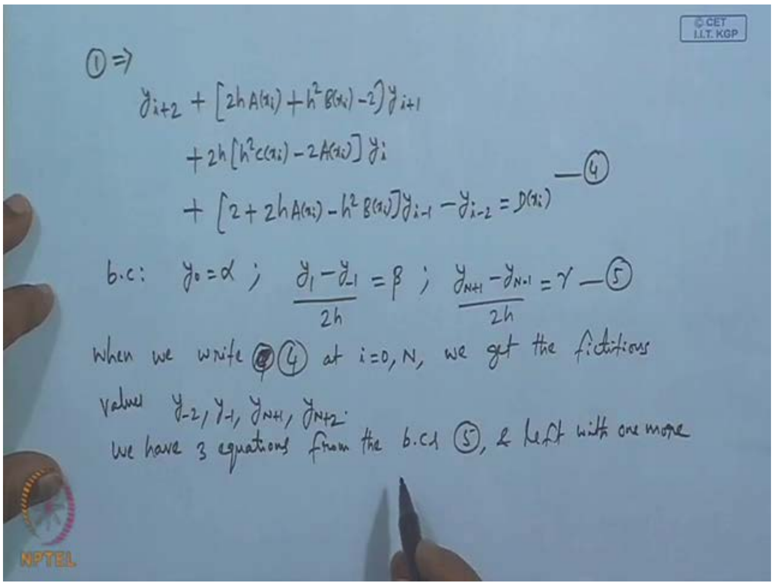

Now, with these approximations, so one gives say a x i, we get this and boundary conditions y 0 is alpha is beta. As I mentioned four and five is not tri diagonal system, when we write four at i is 0, N we get the fictitious values, so these are the fictitious values. However, one can obtain three equations, we already have three equations from the boundary conditions five still, and we are left with and left with one more that is one more equation. So, how do we supply, so here we bank on the order of the equation see the order of the equation is higher order and we know how to reduce it to a system.

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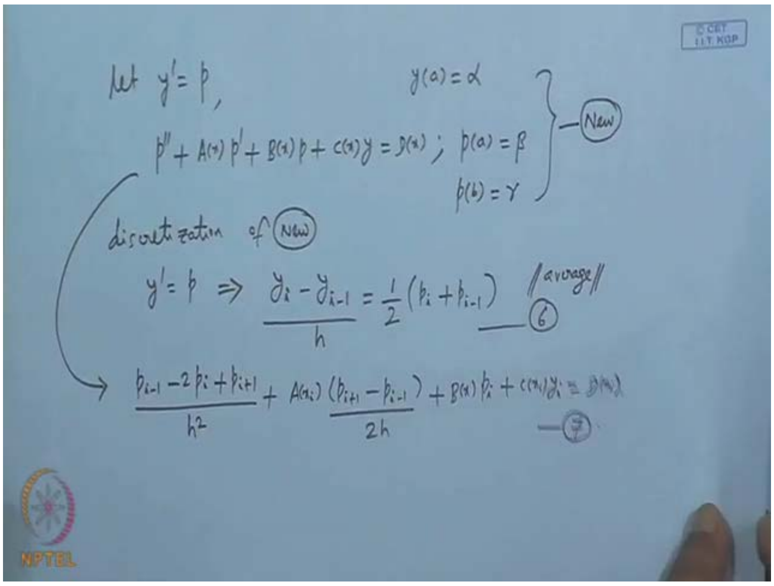

So, we use these let y dash is p then we get the equation and correspondingly y of a is alpha and p of a is beta and p of b is gamma so that is how our system gets reduced. Now, what we do is we discretize the new, so what kind of discretization we propose, so discretization of new as follows this we discretize, so backward there, so this is average or some trapezoidal integration. So, this is for p and for other equation usual central, so what did we achieve, we will see what we have achieved.

We have a taking on the higher order equation we have converted into a system, so this serves as one more equation this serves as the one more equation that we require. So, let us see how to proceed, so from six we have and seven, sorry a i, y i, this is a i y i plus b i p i minus 1 plus c i p i plus d i look at that this contain p i minus 1 p i plus 1 and y i.

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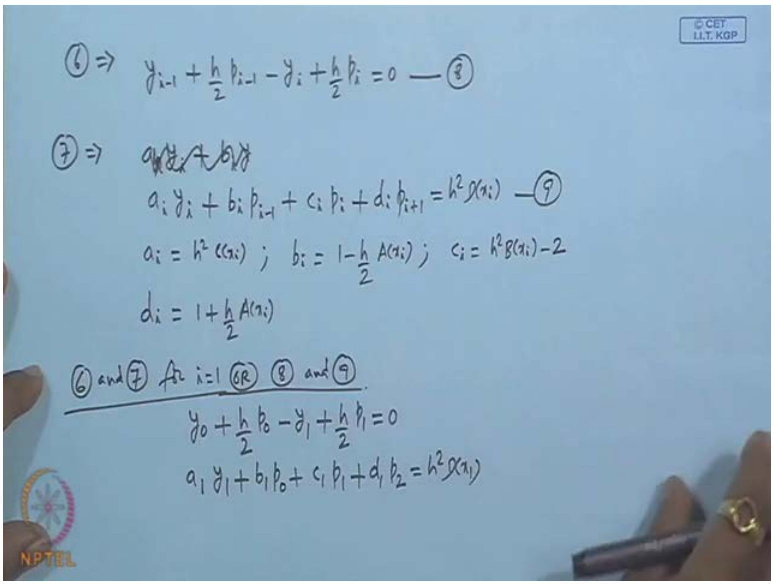

So, this is written in this from, where a i is c of x i b i is and this can be verified very easily. Now, we have to somehow adjust the number of equations and the number of unknowns, so how do we do it. Let us proceed step by step, so let us write down six and seven for i equal to 1, sorry eight this six and seven means or eight and nine. So, from here when we write for i equals to 1 y 0 plus h by 2, p 0 y 1, p 1 equals to 0 and from here a 1 y 1, b 1 p 0, c 1 p 1, d 1 p 2, so these can be written as…

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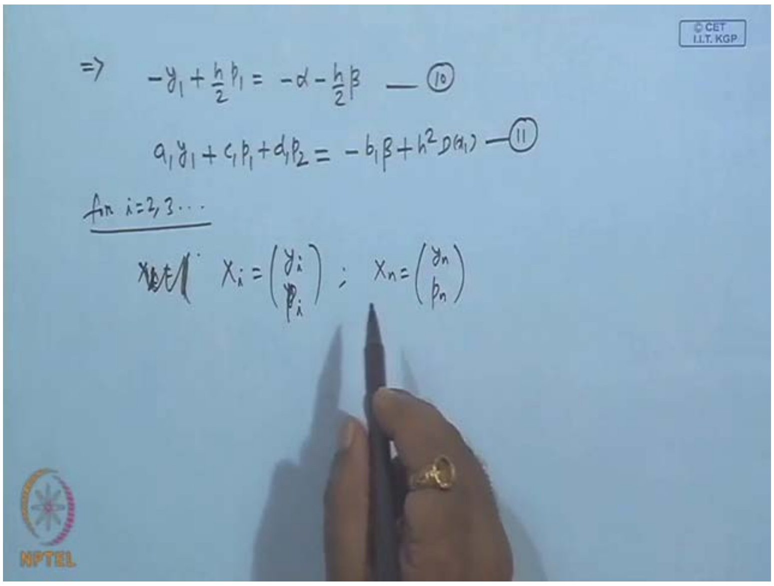

How did we do that, look we have y 0 alpha and p 0 is beta, so these are known we have pulled it to the right hand side. Then from the other one then we have to be careful, so what I have done these two, eight and nine we have written for i equal to 1. Now, these two that means these two we need to run from two onwards. So, let us look at it if we run for two y 1 and p 1, y 2, p 2, so you have form this we have a vector what is a vector y i p i kind of vector it is see, look at it y i minus 1 p i minus 1 y i p i.

So, at least the index is matching, but whereas here there is slight deviation, thus y i p i minus i p i plus 1, so let us look at it, how we can place this? So, this can be written as you introduce some notation, let us introduce notation x n is x i, x i y i, p i. So, if we introduce this, so in general any x n is y n, so there is a coefficient.

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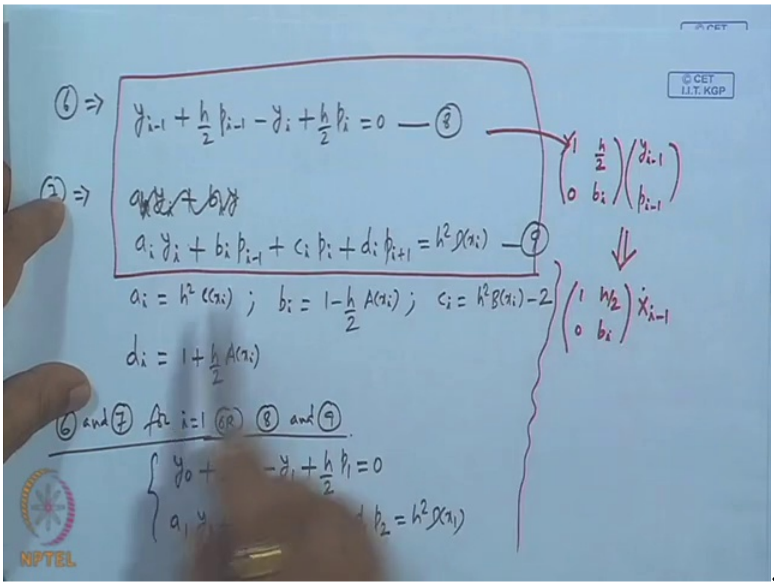

If you look at it suppose look at this equation y i minus 1, p i minus 1 if you like to reproduce this coefficient of this must be 1 there and coefficient of this must be h by 2 there. Similarly, for y i and p i so that means from here, if we plot for y i minus 1 there is no coefficient. Therefore, we expect 0 there where as for pi minus 1 we have b i, so what did we achieve these two equations.

So, two terms we have produced and these two terms, yes we have produced so that means these two equations can be put in a matrix form. So, this further can be 1 h by 2 b i and as per our notation this can be x i where x i is x i minus 1 y i minus 1 p i minus 1. So, what we are going to do, we have to follow very carefully these two equations for 2 to n, we are trying to put it in the matrix from and with a vector notation, so if we do that we get.

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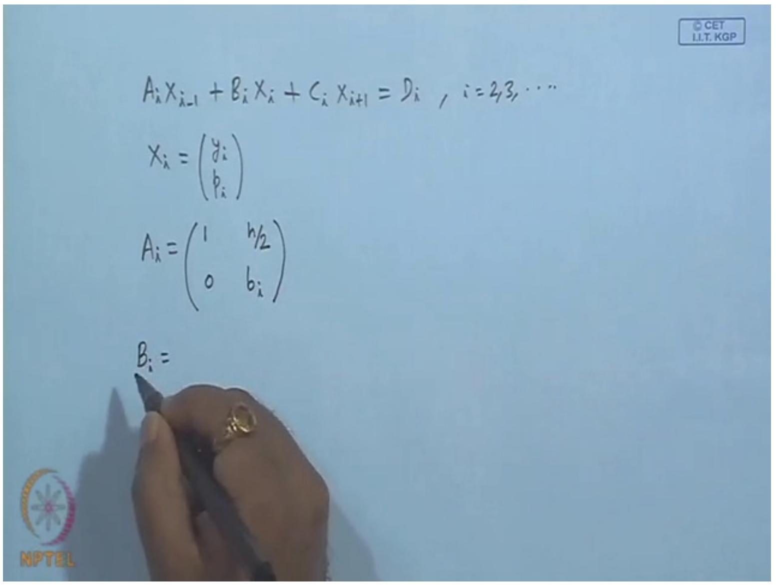

This can be written as A i plus B i plus C i equals D i and what is our notation X i then we should be able to write down the matrix A, so again I go back x i minus 1 which means y i minus 1 p i minus 1. So, we need the coefficient matrix a i for this, so what should be A i the coefficients coming from first equation the corresponding coefficients from second equation, so which we already have written down look at it.

So, here one comes and from here h by 2 from there here no y i minus 1 before 0 and we have p i minus 1 we have b i, so this must be our a i, so what was our a i 1 h by 2 0 b i. Let us look at next we need b i x i and what is x i y i p i look at that so here we have minus 1 minus 1 for y i h by 2 for p i.

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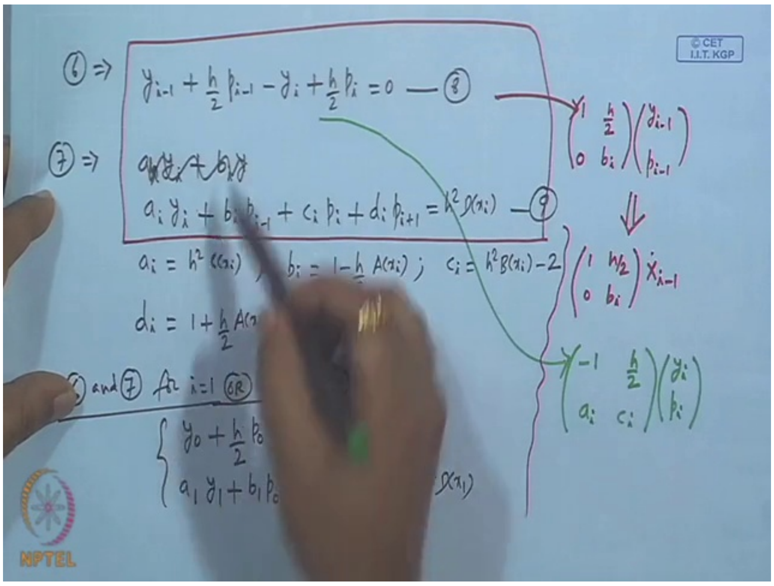

So, which means this minus 1 there h by 2 there this is multiplied y i p i, look at this and what is multiplying here a i is multiplying y i and c i is multiplying. So, we are getting the corresponding coefficients so the two equations in the matrix form.

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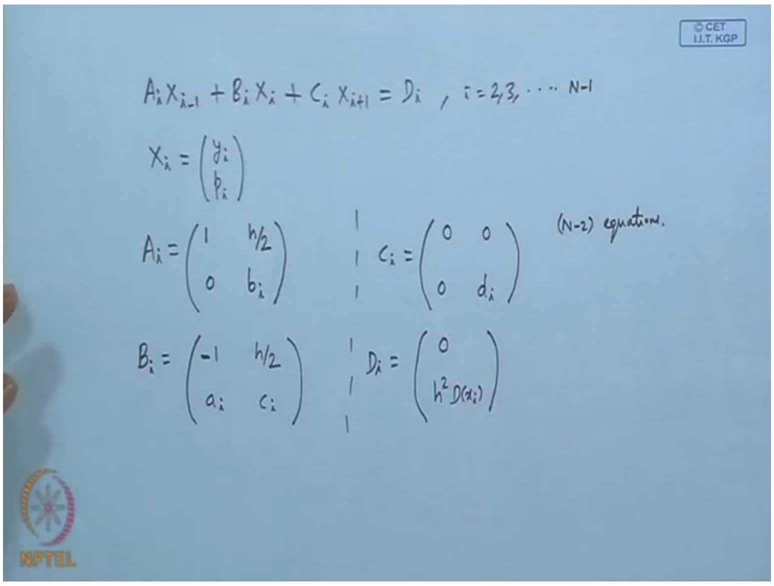

So, B i must be minus 1 then C i X i plus 1, so X i plus 1 look at that, so here we do not have any y i plus 1 p i plus 1. So, the first entry must be 0 whereas, here we have p i plus 1 therefore, so I am not explaining like this because it is it is a straightforward. Now, there is no y i plus 1 p i plus 1, so their first entries must be 0 the matrix, but whereas here there is no y i plus 1. So, the first entry of the second must be 0 and we have D i there, so correspondingly C i must be 0 0 0 d i because it is multiplying x i plus 1.

Then, d i must be column matrix we have exhausted all the terms here and here and then the third one there is nothing on the right hand side where as here we have third one. There is nothing on the right hand side where as we have this term while this be this notation wise this is matrix this is the unknown function. So, there are equations and this from i 2 3 and minus 1 and these are how many n minus 2 equations, but we had written something for one, so for one we have written, so this also can be putted in matrix form probably.

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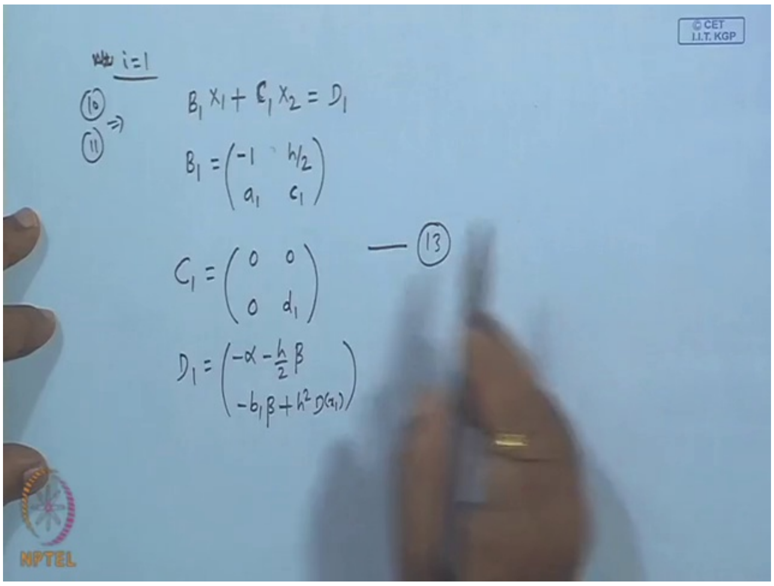

So, let us put it i equal to 1 case can be Written as B 1 X 1, C i X 2, so we have not have to give some number 12, 10 and 11 i equals to 1, 10 and 11. They can be putted like this how, look at that here we have y 1 p 1 which is nothing but exactly or vector x 1, so from here y 1 p 1 is a 1 c 1, so our matrix b is more or less structured. So, b 1 minus 1 h by 2 a 1 c 1 this is small c coefficient then the matrix c 1 matrix c 1.

We do not have the next because c is multiplying x i plus 1, so when i is 1 x 2, but we do not have any two terms there c is multiplying x i plus 1. So, when i is 1 x 2, but we do not have any two terms there, so we get 0 0 and 0 d 1 and d 1 is minus alpha minus h by 2 beta minus b 1 beta plus h square d of x 1. So, this is 13, now we have covered i 1 i 2 to n minus 1.

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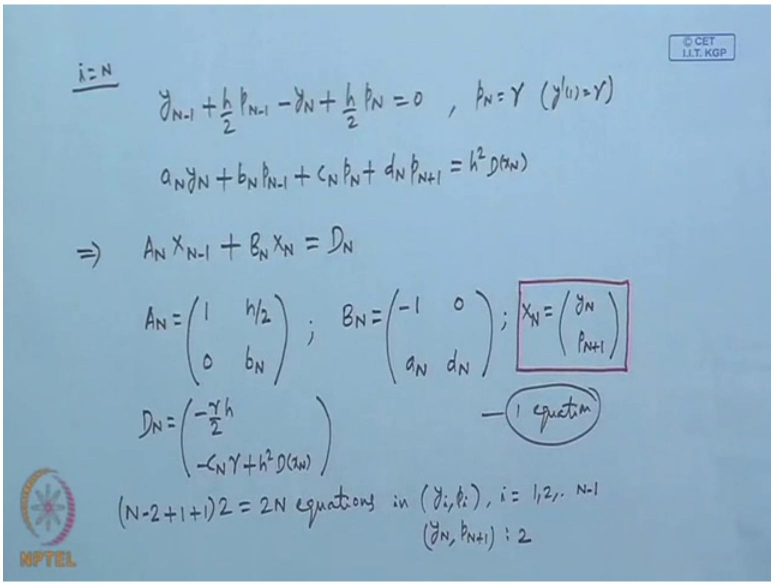

Then we are left with i n, now i n case gives us however, so from where we would get this. Of course, from here the matrix, so this can be putted as a n x n minus 1, where see x n minus 1 that is y n minus 1 and p n minus 1. So, we have the terms very much one and here we do not have y n minus 1, so p n minus 1 has the term then b n, look the remaining left out we know p n what was p n gamma it is a boundary condition because y dashed of 1 which is p n. So, this term is gone to the right hand side d n, so we can expect minus 1 there and there is no as such p n left, where as we here we have y n and p n plus 1.

So, we had look at carefully this is n minus 1, so y n minus 1 p n minus 1 is done then x n, so we have to write x n. So, x n should contain y n and p n, so p n is known, so we shift to the other side and even here we know it. So, this will be a n, d n with a definition x n equals y n, p n plus 1 because this is known to us, so we push it to this side. Therefore, unknowns y n and p n plus 1 that is a small adjustment, so you can make a note of this, because the symmetry is lost and then we are in position.

So, d n must be p n is known, so it has gone minus gamma by 2 h and here we know this is gamma, so minus c n gamma we have. So, let us compile n minus 2 equations and then this is a another equation, so n minus 1 and this is another one equation. So, total n equations, but in what sense they are n minus 2 plus 1 plus 1 and each is two equations and equations in how many unknowns 2 n unknowns because these are two and these are two times n minus 1. So, what is the achievement by putting in this matrix form by converting into system the number of unknowns and the numbers of equations have been matched. Now, eliminating the unknowns we get, let us see carefully look at this, so d is for 1 to n minus 1, so then this is for 1.

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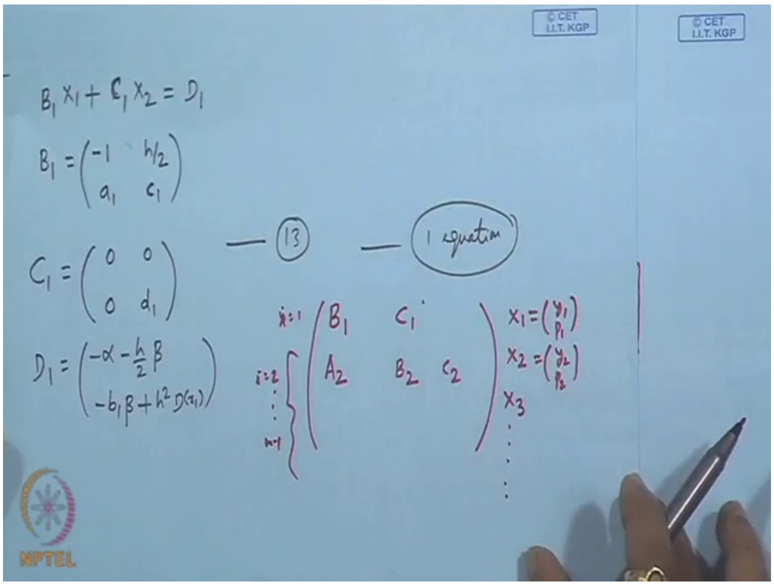

So, if we write a matrix this is all matrices right, so suppose B b 1 and C 1 there this must multiply. Let us say in a bigger matrix these two must multiply x 1, x 2 then 2 so for 2 to n minus 1 it is like this, so let us start the index to 8 must multiply x 1 so that means we must expect 8 there then b 2 must multiply x 2 b 2 there then c 2.

So, this continues each of them is column vector for example, what is x 1 and what are they these are matrices in turn, so this corresponds to and equals to 1 and i equals to 1 and i 2 to n minus 1 and the last what we obtained the last one i equals to n. So, this will also come at the bottom and this entire thing which one I am talking about this is you see it is a block, but then again it is a tri diagonal system.

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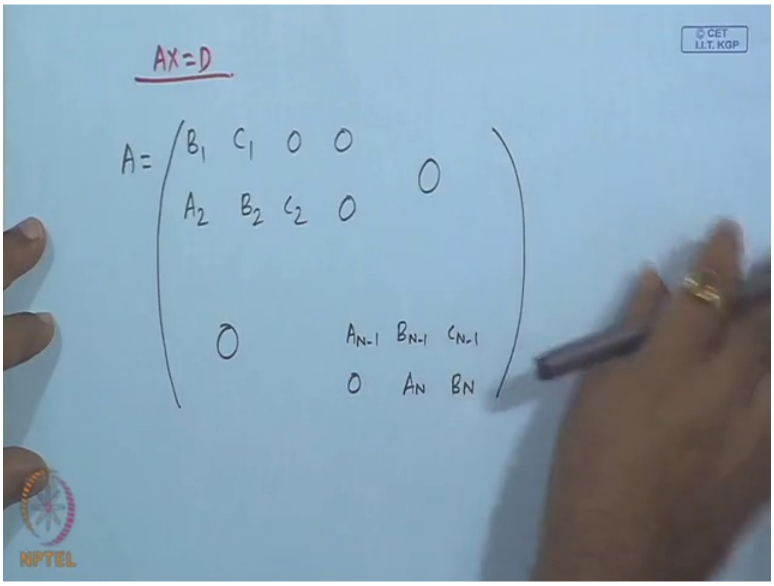

So, it is a block tri diagonal system, so we get as follows A X equals to D some global where the structure of A follows from the earlier matrices, these are null matrices then A 2, B 2 and C 2 like this.

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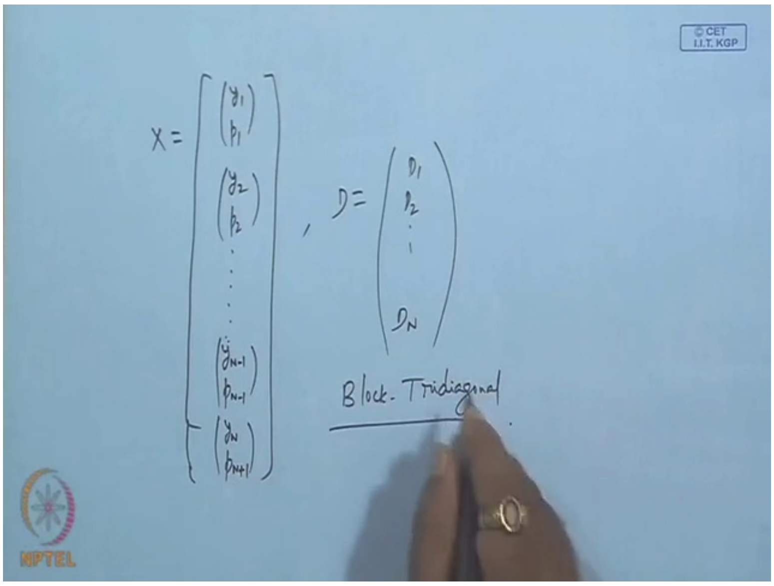

What is our X, it is better I write X looks like this see x 1 is x 2. So, this is our X and so this is in fact I should write here this is a block tri diagonal system, so with a reference to BVP, what did we do.

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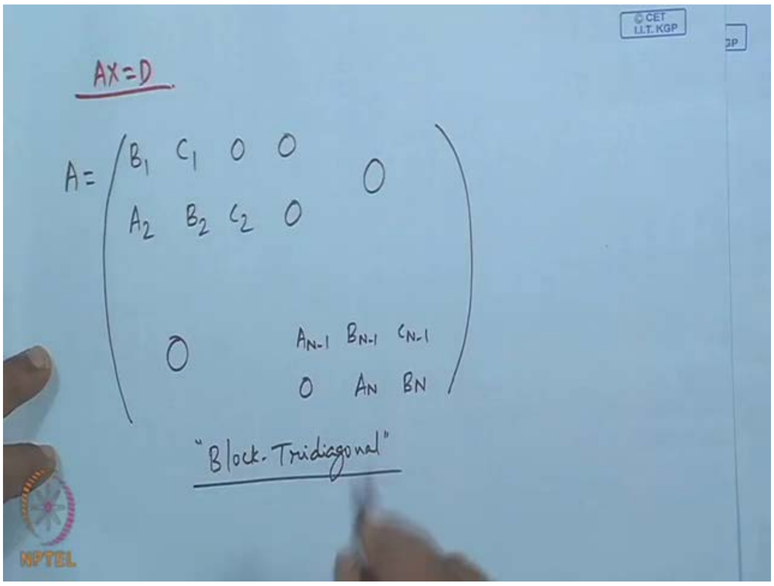

If you discretize the equations definitely we are short of one equation and this supplied by really writing the higher order equation in terms of a system. So, this system will help us to supply the additional one more equation and since this is a system we have to solve simultaneously, so we obtain block tri diagonal matrix, so let us try with an example.

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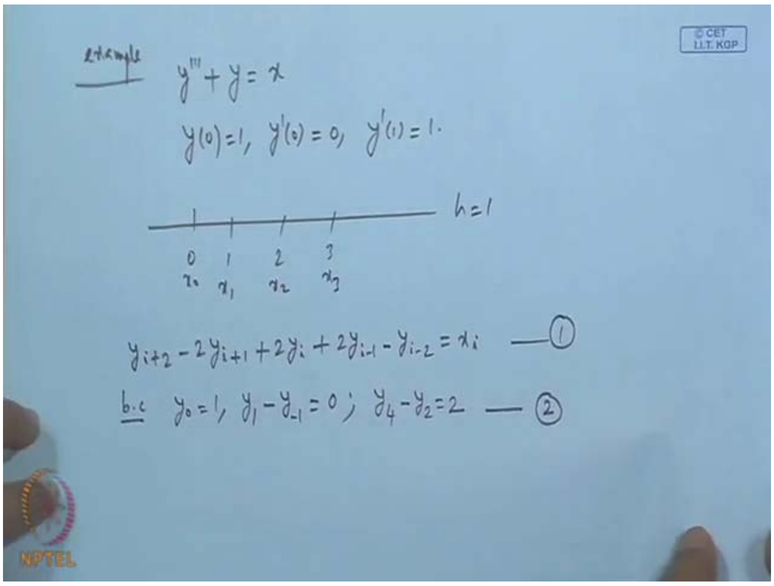

So, this simplest, so for calcification purpose I have taken h equals to 1 say 0, 1, 2, 3, so then with usual discretization, so discretization of this 0 discretization of this because two the denominator goes there.

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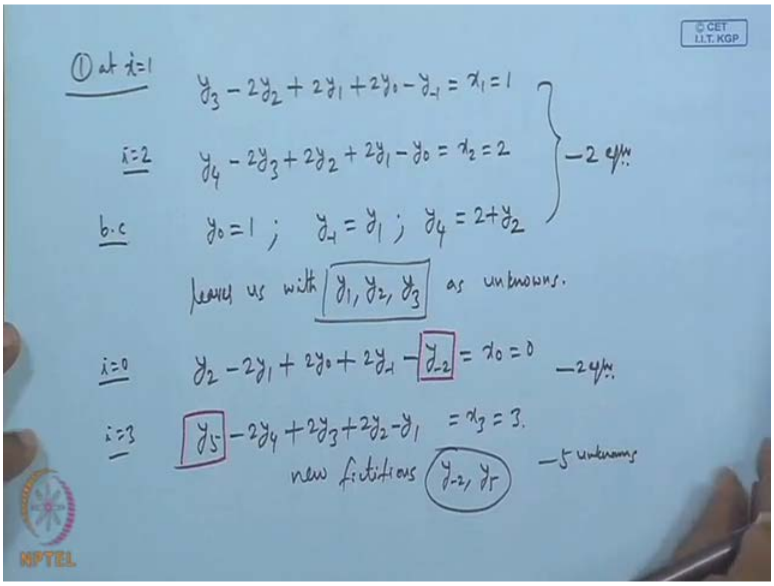

Now, we write one at i equals to 1, so this then i equals to 2, so however here and from boundary conditions we have, so this gives us. So, for example in this y 0 is given and y minus 1 can be substituted y for can be substituted. So, this whole thing leaves us with y 1, y 2, y 3 as unknowns, now at i equals to 0, y 2 y 1 y 0 at i equals to 3, so what happened? So, these are new fictitious, so two equations, so because these are all redundant. Now, this reduces to two equations, but y 1, y 2, y 3 and five unknowns.

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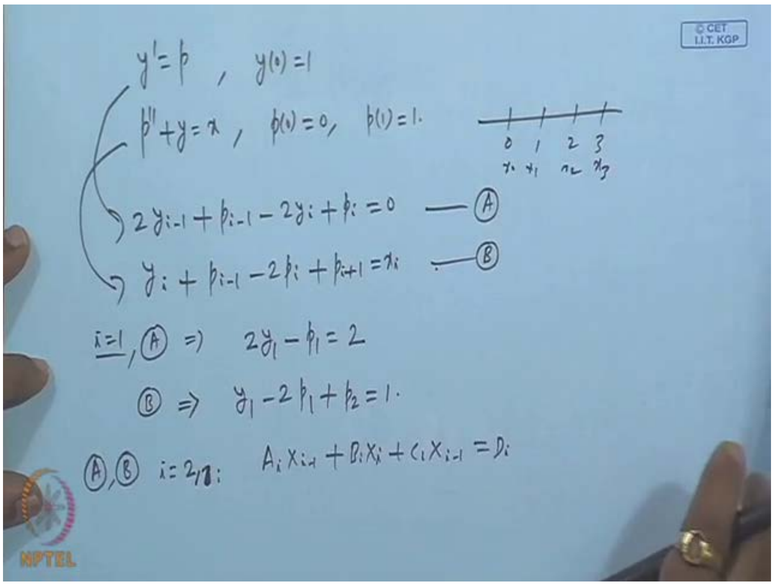

We have to really depend on y dash equals to p, so then p double plus y is x. So, correspondingly this one and p of 0 is 0, so with the discretization of this we get so that means discretization of this discretization of this so and then we have let us have some number. So, i equals to 1 a gives us 2 y 1 minus p 1 is 2 because I have simplified you can do it then B gives us right, so this is we have done for it a general case. Now, for A this A and B for i 2 and 3, so this can be written as, in fact i 2 only one we have because our grid was 0, 1, 2 and 3, so this is x 0, x 1, x 2 and x 3, so this can be written as D i.

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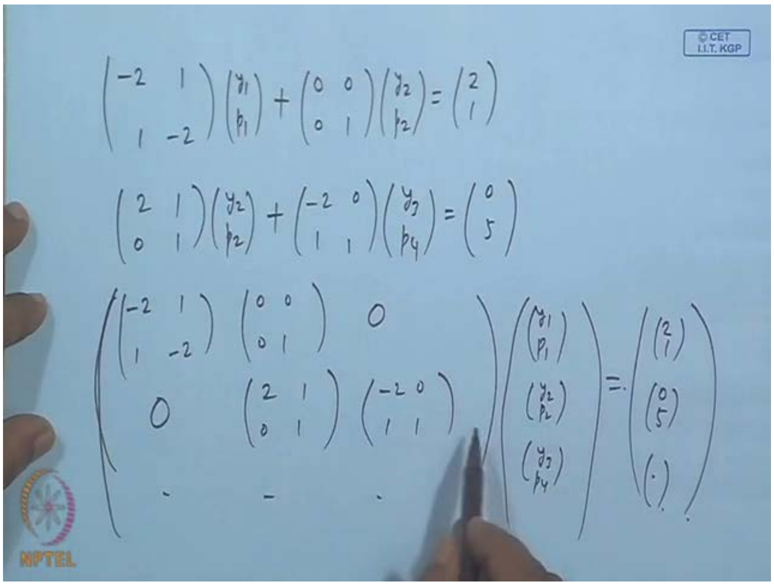

So, we if we try to do that we arrive at so this is 1 set and another set. Ultimately, we get another matrix here and this one, so this will be the tri diagonal system which we have to solve. So, this gives the structure for the higher order boundary value problems, so like initial value problems higher order one reduce it to system and can solve it. Similarly, we have been reducing to system to support one more equation, but then we end up with block tri diagonal system which one can solve it. So, there are other methods for solving non linear boundary value problems, so we look at them in the coming lectures, until then, bye.