Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 18
Boundary Value Problems- Derivative Boundary Conditions

Good morning, we have been discussing boundary value problems, so far we have discussed linear and non linear two point boundary value problems. So, in case of linear we have seen that the discretization leads to the linear system of equations in particular tri diagonal system which could be solved using Thomas algorithm. But in case of non linear, we have observed that the corresponding non linear system could be solved using this Newton Raphson method, but so far we have concerned only ((Refer Slide Time: 00:54)) type boundary conditions, in the sense only the function values are given at the boundary points.

So, now let us consider boundary value problems where the corresponding boundary conditions involve derivatives, so definitely these types of conditions play a vital role in the overall structure of the problem. So, let us see how far we can proceed with the derivative boundary conditions.

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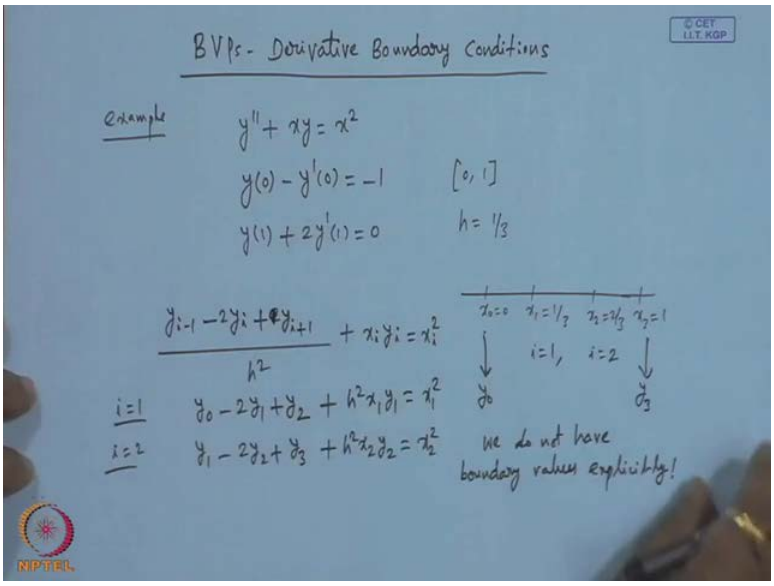

So, BVP's derivative going to conditions, so let us start with an example say suppose the interval is 0, 1 h 1 by 3. So, this means now if you discretize this equation, so we get usual central we have been using. We get this, and then we have to write this equation strictly speaking, so these are the boundary points, so i correspond to one and i correspond to 2 there.

So, if you run for i equals to 1, then we get y 0 y 1 y 2 plus h square x 1 y 1, so then suppose we run for i equals to 2, we get y 1. So, in the earlier case, we have seen where the corresponding boundary conditions does not involve derivative terms in this case we use to have y 0 and y 3. There would have been known y 0 and y 3, we have two equations in two unknowns y 1 y 2, but now the situation is entirely different, so what is the difference we do not have boundary values. Explicitly, we do not have boundary values, so then we have a corresponding boundary condition which involves derivatives, so one simplest solution could be why do not we discretize the corresponding boundary conditions, for sure we can discretize.

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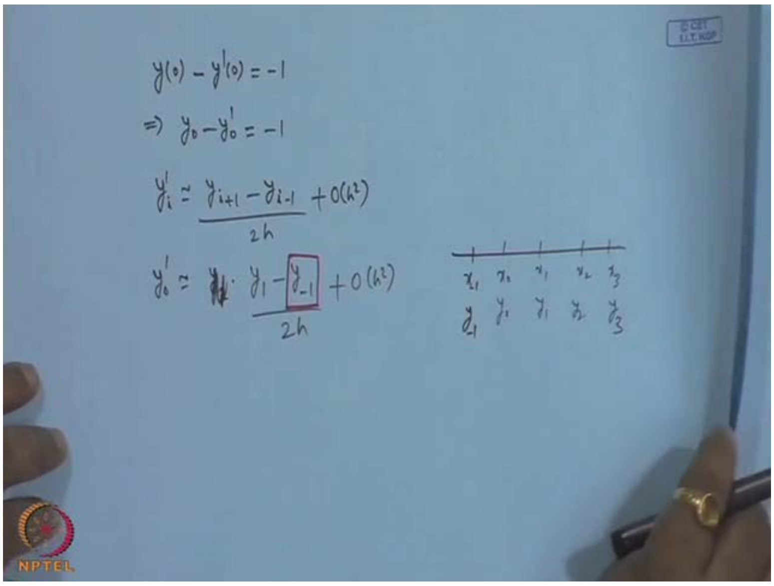

So, for example, consider this, so this involve y 0, so this means y 0 minus y 0 prime equals to minus 1, so let us consider what could be the first order derivative approximation. So, the first order approximation let us use central, this is second order approximation, so it makes sense if you see this is second order approximation. So, it is better to use for the boundary conditions as well so if this is the case then y 0 prime this will be y 1 minus y minus 1 y 2 h.

So, what this is x 0, x 1, x 2, x 3 and y 0, y 1, y 2, y 3, but we have term. So, we have this if you think of this x minus 1, so that means we have two equations i 1 2, but then we have we do not have explicitly y 0, y 3. So, then we have gone for the boundary conditions that involve derivative, then if we discretize we are getting an additional term. Similarly, if you discretize at the other boundary we may get another additional term, so let us go back to general framework and try to discuss what would happen in general.

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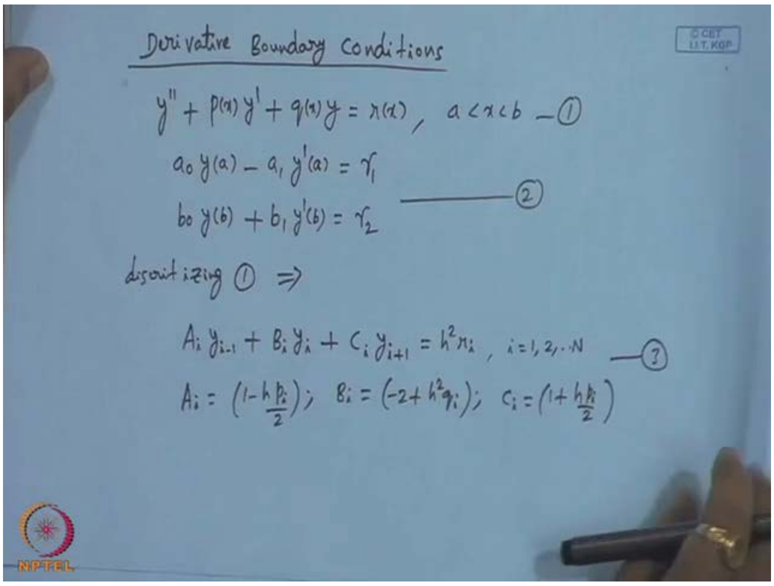

So, consider more general derivative boundary conditions, now discretizing one we have we have done it before where we get this, so let us call this three, so we get now one, i 2 N.

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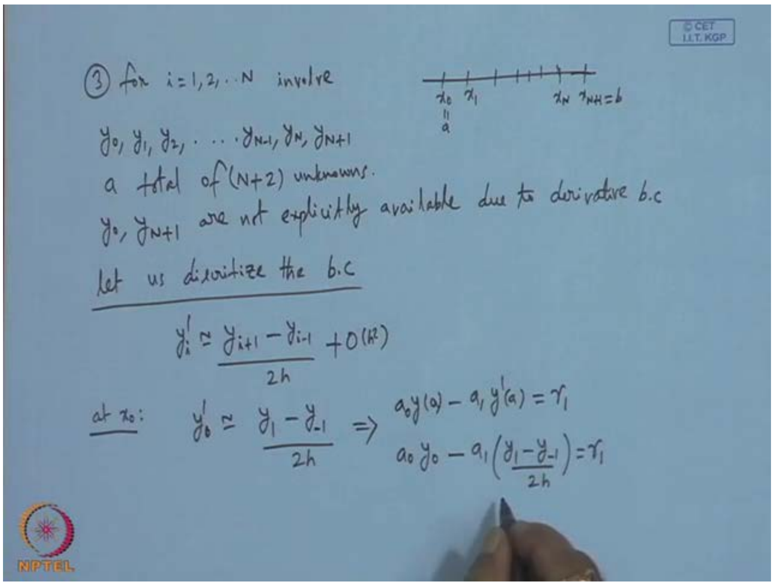

So, if this is the case three for i 1 to N involve y 0 y 1 y 2, its very straight forward, we run i 1, it starts from 0, then we end up with N plus 1. So, this is a total of N plus 2 unknowns, further y plus 1 are not explicitly available, this is due to derivative boundary condition, so however we expect we expect i 1 to n, so N equations, but involve N plus 2 unknowns.

So, still we have not used boundary conditions, so let us discretize the boundary conditions, so with this at x 0 we get this approximation. So, this implies y 0 y, y of a and y dashed of a is to gamma 1. So, this become this is a 0 a 0 y 0 gamma 1, so from this one could get y minus 1, so let us write down a number say this is four.

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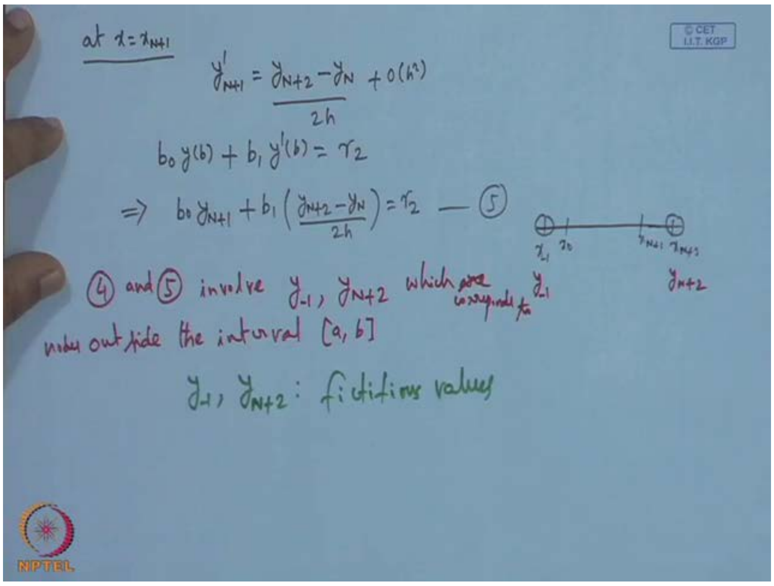

So, before we do anything at x equals to y, so b 0 implies that these are different and correspondingly we have these different and correspondingly we have, so four and five involve y minus 1 which are outside the interval a b. I mean which are outside the interval which corresponds to nodes outside the interval because nodes are x minus 1 x N plus 2, so these are called fictitious values.

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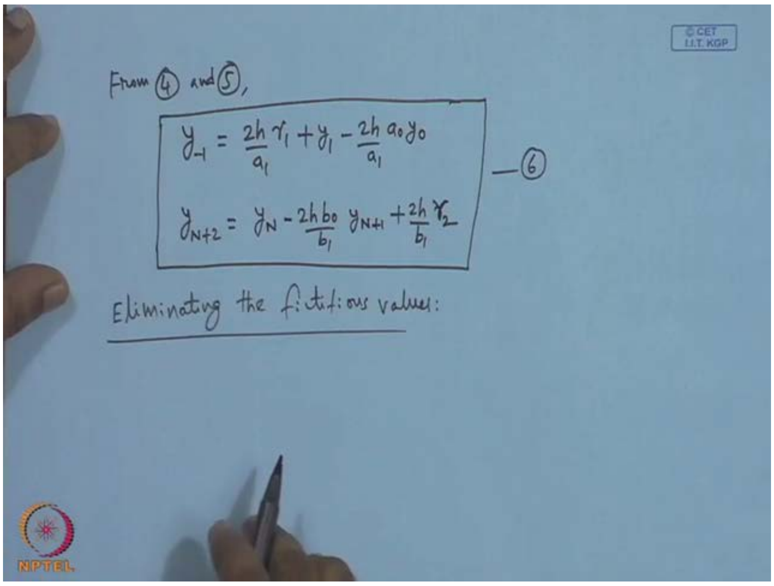

So, let us from four and five, one can express the fictitious values, so the fictitious values have been expressed like this, but unless we get rid of these two, we do not end up with unknown equals to equations. So, what is next step, eliminating the fictitious values.

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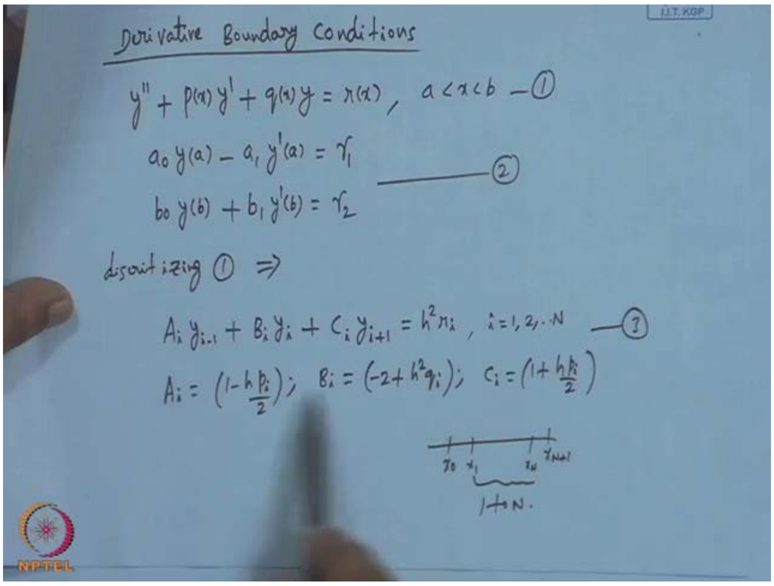

If you recall we have this equation which we were running from 1 to n, so we were running from 1 to N, so that introduced 1 to N we have a N equations.

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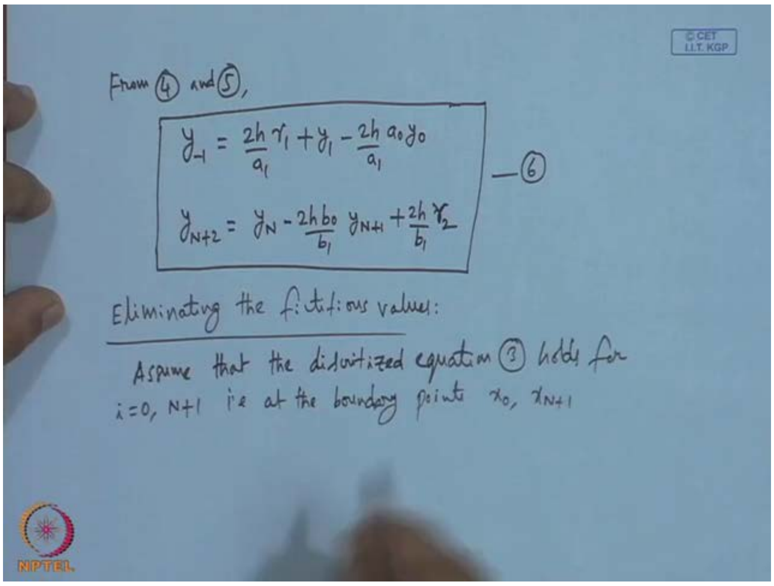

Discretizing Introduced two fictitious values, so we have to eliminate, how we eliminate, we assume that assume that the discretized equation three holds for i equals to 0 and N plus 1 that is at the boundary points x 0, so this is alternative.

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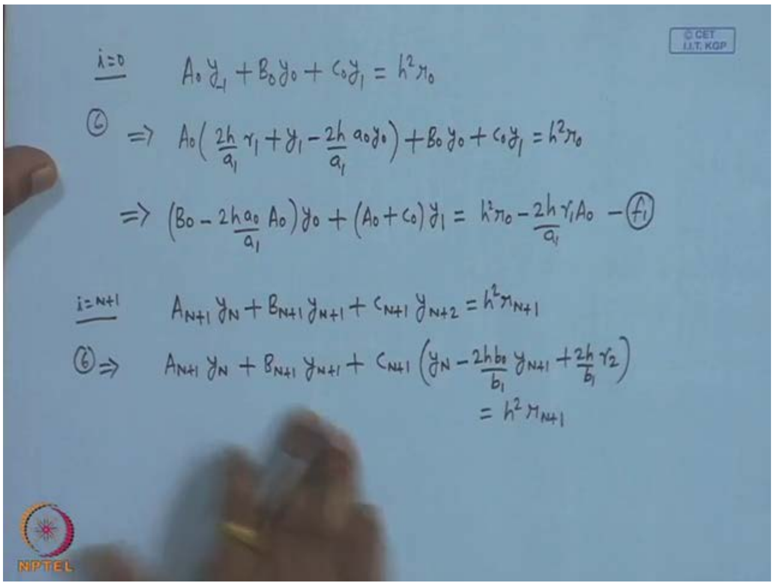

So, accordingly i equals to 0 and we have y minus 1, from six we have to put it y 0, y 1 coefficient of y 0 is b 0 and there is a coefficient here coefficient of y 1 a 0. The rest are known quantities and i N plus 1, again we have six, from that y N plus 2 we had and from this we can collect the coefficients.

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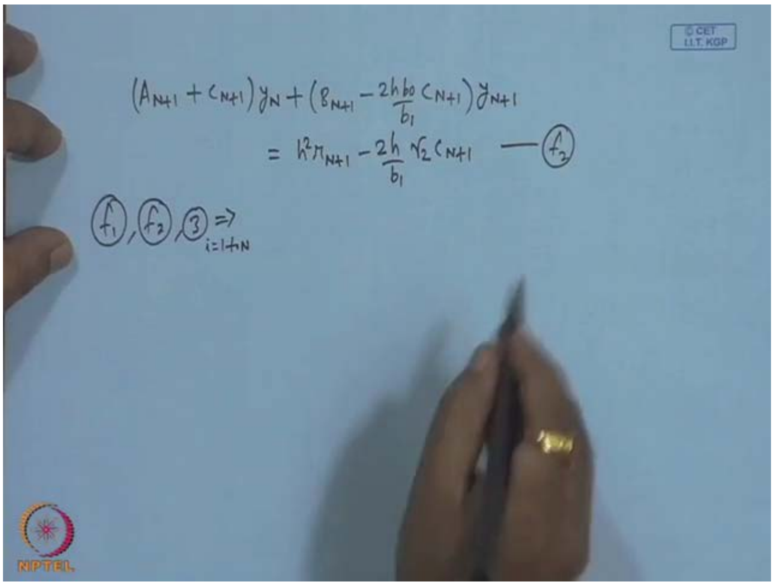

So, one may ask what is the necessity of doing this general, one can solve any problem, well the answer is straight forward, any numerical competition we have to really setup thinking that we would implement on a machine. So, this kind of a general framework would help us so that we could write the algorithm systematically, so this is f 1 and say f 2, now we have f 3 which is the system and we have this f 1 and we have f 2. So, this will set the scenario with a number of unknowns equal to number of equations, of course three i 1 to N.

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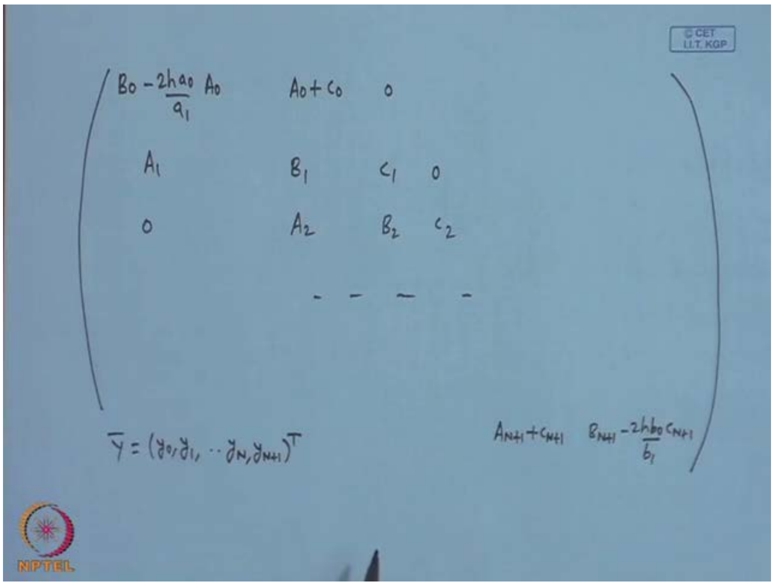

This implies we get a with matrix so on so forth and here we get this matrix, this big matrix where N plus to unknowns, so this is also tri diagonal system and one can solve. So, when you have derivative boundary conditions, what is the rule, the rule is when you discretize a equation just leaving the end points. So, then we have only N equations, but then if you run the equations at end points, then we get fictitious values, so when we have fictitious values what do we do?

We discretize the boundary conditions as well and the boundary conditions, also introduce the fictitious values, now how do we eliminate the fictitious values? We eliminate the fictitious values between the boundary conditions and between the set of equations obtained by running the discretized equation at the end points. So, the matrix is second with respect to the numbers of unknowns equals to the number of equations and we obtain tri diagonal system which we can solve using Thomas algorithm, so let us solve some problems so that we understand much better.

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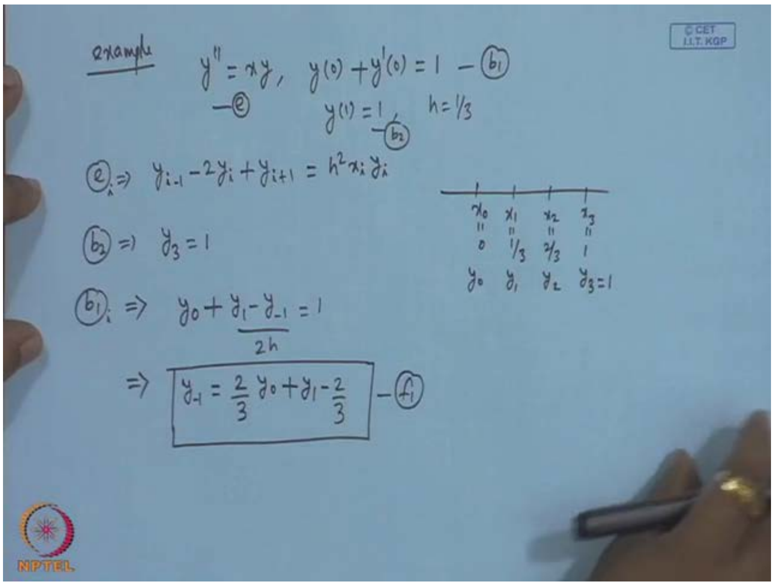

So, consider very simple case and the boundary conditions derivative type only at one end point just for simplicity. So, let us discretize the equations, so this we need to find out y 0, y 1, y 2 in this case, y 3 is given, so we have y 0, y 1, y 2 are the unknowns. So, let this be the equation, say this is e, b 1, b 2, so this is discretization e i, then b 2, this then b 1 i, means discretized that would give y 0 this is y 0 plus y dash, this equals to 1. So, this implies 2 by 3 y 0 plus y 1 minus 2 by 3.

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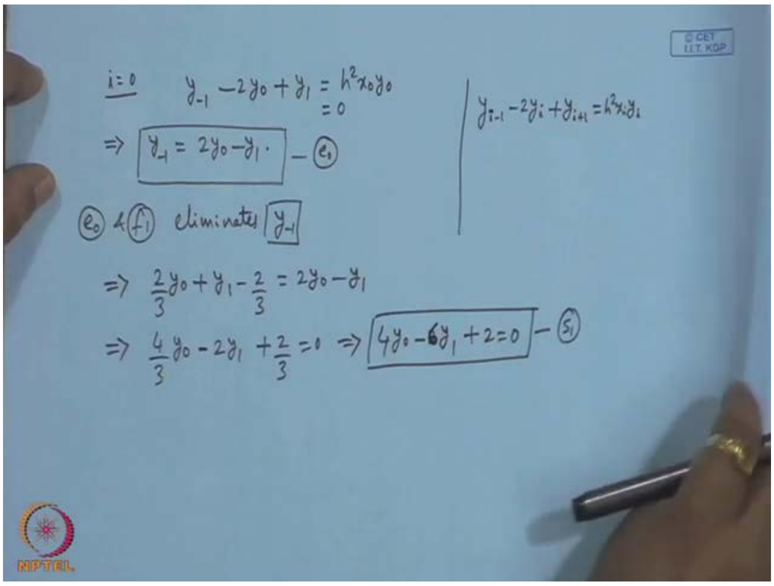

So, this is f 1, now let us run the equation i equal to 0, so y, so from here we are running this, we had our equation, we had this, so when i equal to 0, we get this. So, this implies plus h square x 0, so this is 0 because of x 0, so we get this is say this is e 0, so e 0 and f 1 eliminates y minus 1. So, this would lead to, we can simplify this, so this is 2 minus 2 by 3, 4 by 3 y 0 minus 2 by 1, this 6 by 1 plus 2 equals to 0, so this is one equation, so this is let us say s one, now i equals to 1.

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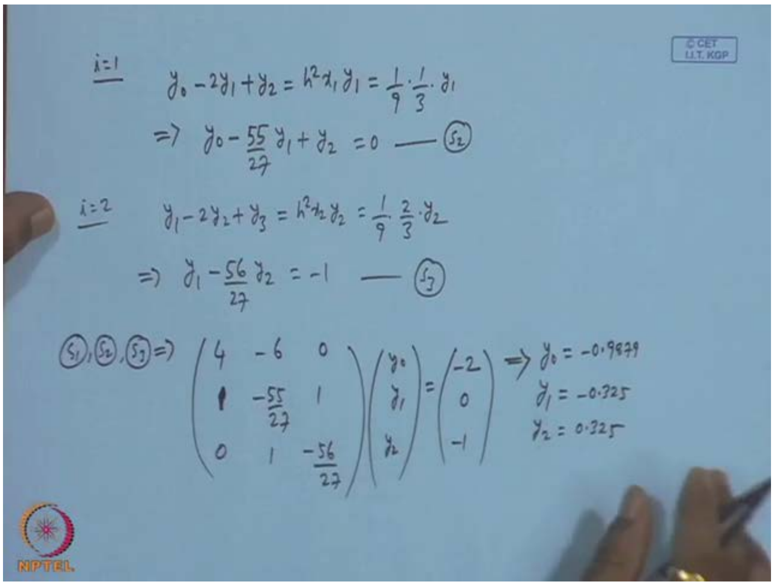

We get again i equals to 1, so this we have h square is 1 by 9, x 1 is this, so this will be 27, 54, so that will be minus say this is s 2. This implies from s one, s two, s three implies so we have used y 3, y 3 is also 1, so we have also used that this is gone, so then we have 4, minus 6, 0, then this is 1, 1, 0, 1. We have minus 2, 0, minus 1, so one can solve this, it is simple system, so we get y 0 is minus point, so the desired solution we get it.

So, when we have the derivative boundary conditions we have seen, of course this example it is just one of the boundary conditions contained derivative, but if we have both as well similar procedure can be adopted. So, before we see that situation let us go for little bit of theory on the linear two point boundary value problems.

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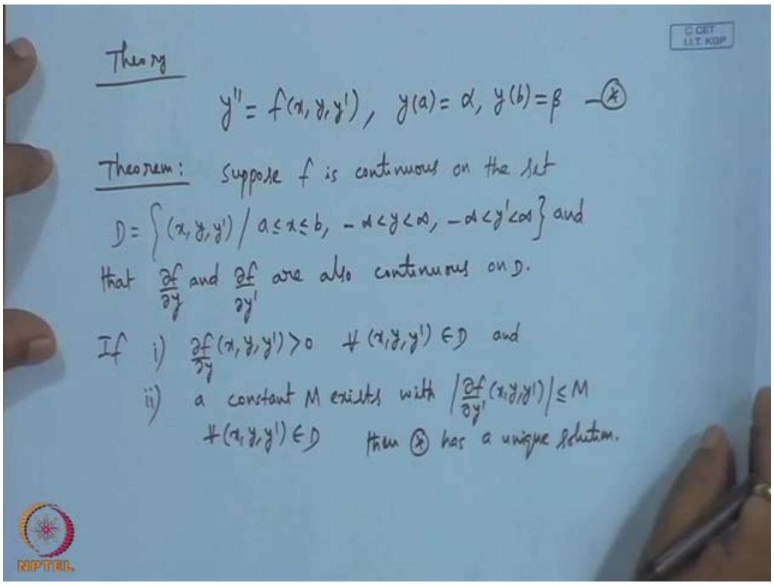

So, some theory we consider, we have seen already, so this is our equation and we have not talked about for IVP, we have talked about existence uniqueness. So, it would be better to talk about the similar situation in case of two point boundary value problems. Suppose, you say this, suppose f is continuous f means, of course f is given in star at suppose f is continuous on this set. That dou f, dou y, dou f y dashed are also continuous on d if d is greater than o for every this and 2 a constant m exists with less than or equals to m for every d, what is this?

Suppose, f is continuous on this set and both the derivatives with respect to y and y prime are also continuous on this domain if this is positive and there exists a constant that means if the magnitude of this derivative is bounded. So, then star has a unique solution, so if you recall we have for IVP's, we have assumed records existence satisfying Lipschitz condition and similar thought of existence theorem for the two point value problems.

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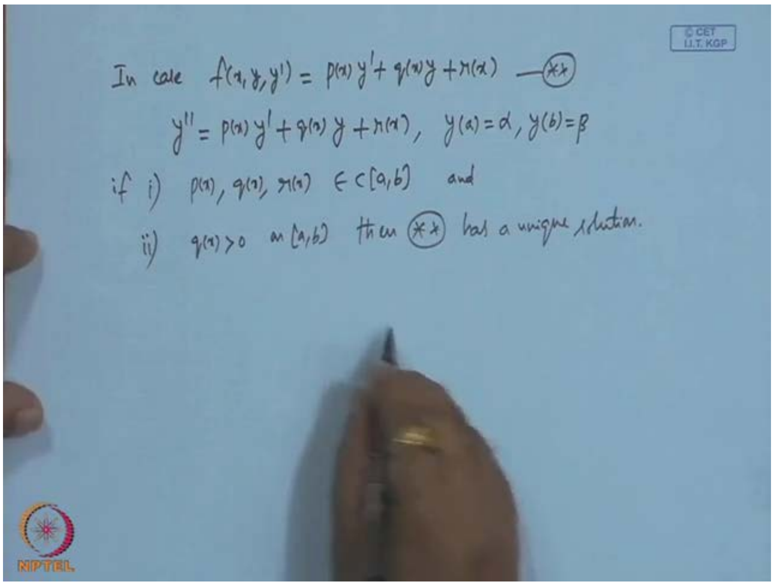

Now, let us consider in case of this form, then we will have similar to what we have considered in the last lectures, so then in case of this if p x, q x, r x, they are all continuous on this and 2 q x is positive on a b. Then, double star has unique solution, now with respect to this uniqueness, we get a little bit of idea, so that means if you have initial value problems we need Lipschitz type conditions.

In case of two point boundary value problems, we have continuity as well as positiveness and some kind of boundaries of the derivative terms with y dash etcetera. Now, if you observe for the linear two point boundary value problems without derivative boundary conditions we end up with tri diagonal system. So, then under what conditions tri diagonal system can be solved and things like that let us see little bit on that.

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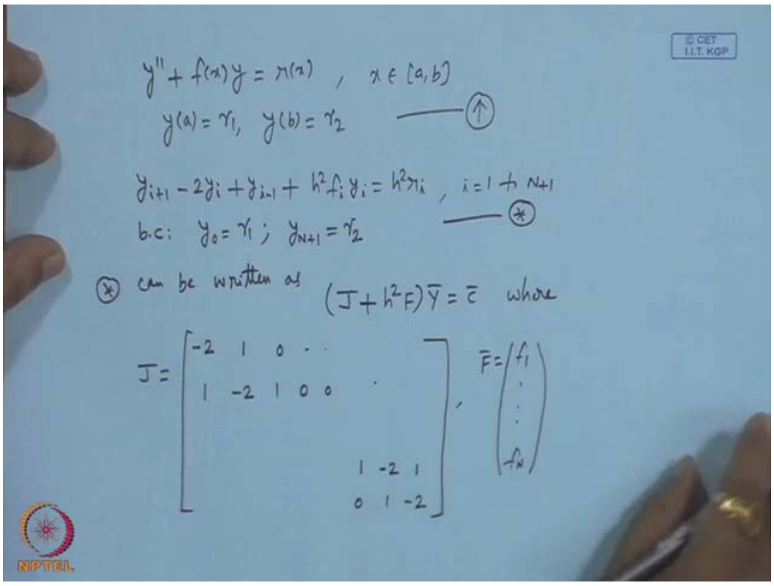

So, we have say special type just we consider x belongs to this with y of a is gamma of one y of b is gamma 2, so then when we discretize and the corresponding boundary conditions. So, star can be written as star can be written as j plus h square f 1 y equals to c, so we have already noticed that this produces a tri diagonal system the entire thing.

So, here j is of the form minus 2, look at this the coefficient of y i, then coefficient of y i minus 1 coefficient of y i plus 1, so accordingly this we are giving it to f 0, then 1 minus 2, 1, 0, 0 then 1. So, we get this and f bar contains f 1 f n, then then c could contain gamma 1 of course, so this is we split it actually y, I could have plugged in and we could have written this matrix in a different way, but I just put it like this.

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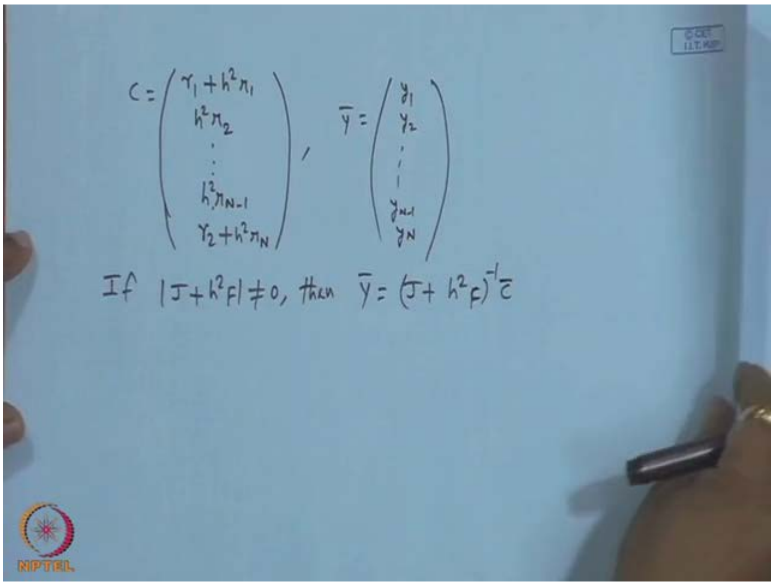

So, this is a tri diagonal system if the determinant is non zero, then y equals to that means is invertible, so c and we get the solution from the tri diagonal system.

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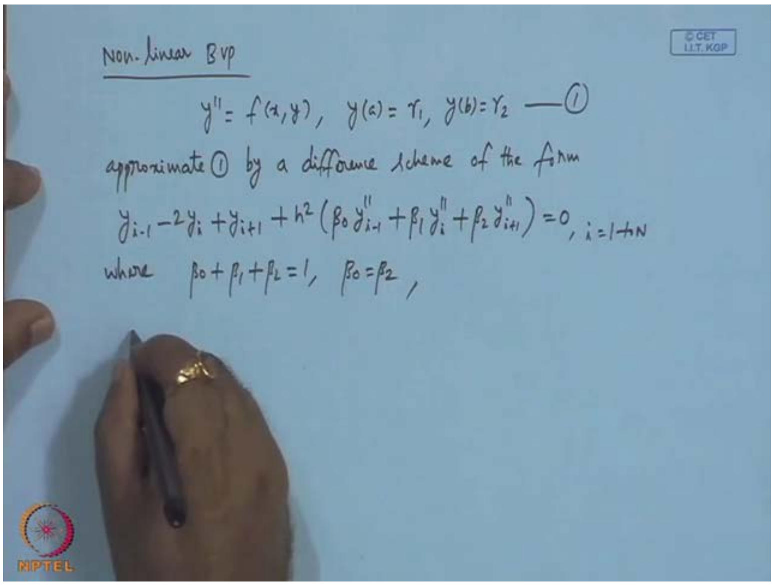

So, this with a linear case, so let us see little non linear case, so we consider there are several ones. So, I am just continuing, sometimes one sometimes star, so then the discretized, so what we use because of this non linearity we make a special remark. Suppose, approximate one by a different scheme of the form subject to that means these are some weights subject to these conditions. So, this entire thing has been approximated by this, but with a specific condition, then the discretized equation can be put in this form.

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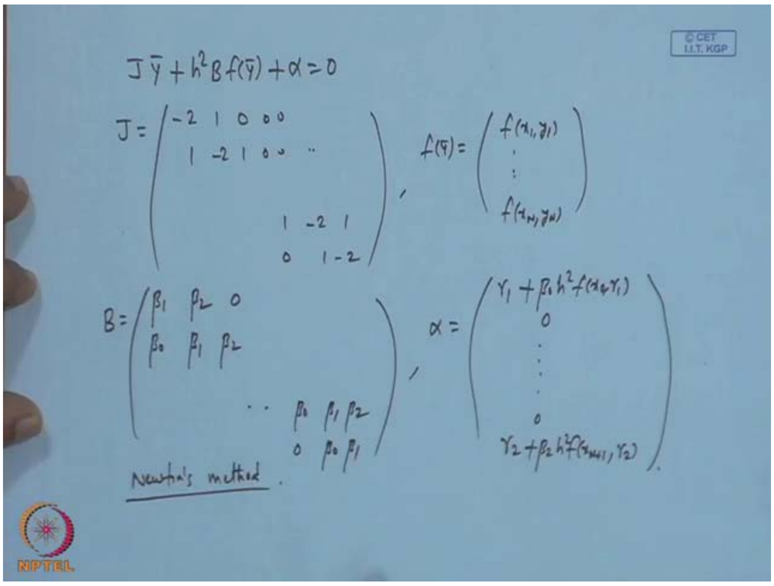

So, the discretized equation can be putt in the form where again j would contain f of a and b and alpha gamma 1 beta 0 h squared 0. So, this system is because we have f contains this which is a non linear type, so again one can solve Newton's method and similar conditions on this matrix holds corresponding to linear case. So, with this we have settled the matter for linear and non linear BVP's. Of course there are other techniques as well, but within the context of finite difference methods.

We discussed when we discretized the cases where we end up with tri diagonal system and even we have derivative boundary conditions, we end up with tri diagonal system. So, we will see we have settled the matter only for second order linear two point boundary value problems. So, when we consider higher order what happens we will see in the coming lectures.

Thank you.