Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 16
Finite Difference Methods- Linear BVPs

Hello. So, so far we have discussed initial value problems and then some methods both single step and multistep methods to solve the initial value problems. So, then we use them to propose something called predictor corrector methods. So, let us proceed further with something called boundary value problems. So before we proceed for methods, so let us briefly discuss, what is boundary value problem.

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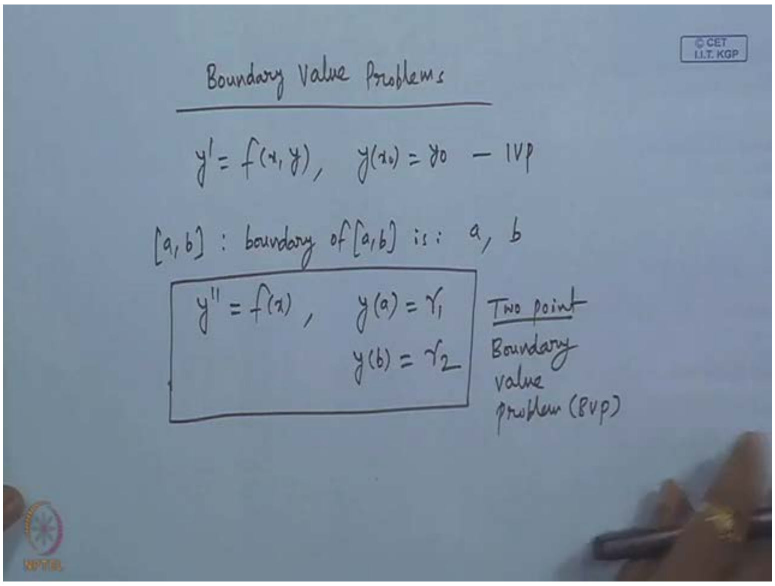

So, we boundary value problems, so we have y dash equal to f of x y and y of x 0 equals to y 0. So, this is initial value problem. So, we know very well that a first order equation requires one condition to eliminate the arbitrary coefficient. Now, since we are proposing a boundary, so for a boundary, we need at least more than one point. So, for example, you consider a boundary. So, if you consider a domain a b, what is the boundary? Boundary of a b is a and b. So, these are the boundary points. Therefore, we expect a kind of second order say this is a simplest I am writing. So, then we need y of a is gamma 1, say y of b is gamma 2.

So, this is called a two point boundary value problem. So, why do we say two point because the value is specified at two points. So, this a two point boundary value problem. Now, what is our aim; to solve such two point boundary value problems numerically. So, how do we solve them? Of course, by somehow we have to introduce the numerical concept. So, how do we do it?

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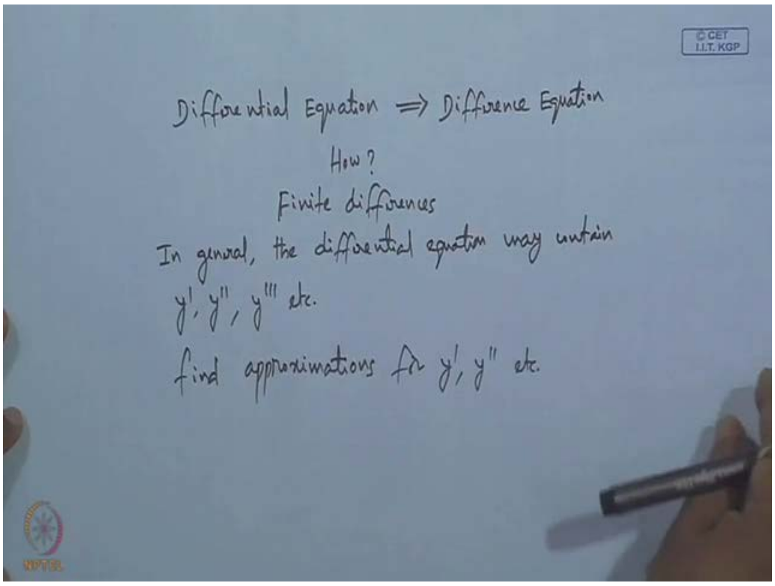

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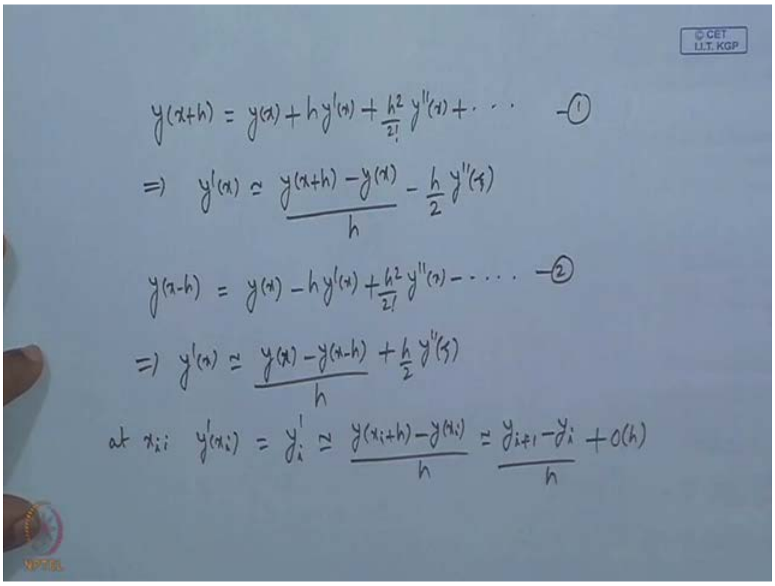

So, how do we do it? We have differential equation. So, we have to convert this into difference equation. Now, the question how, how do we do this? By introducing finite differences, so by introducing finite differences that means see I have considered a simple case y double prime, but in general the differential equation may contain y prime, y double prime etcetera. Therefore, the task is find approximations for y prime, y double prime etcetera. So, how do we find approximations?

So, consider, so use Taylor series implies y dash of x, so that means we have obtained an approximation for the first derivative. So, similarly, so call this 1, consider, so then this implies is this will be we have to write y of, so we transfer plus, so we have got an approximation, another approximation. So, typically this is known as forward operator, this is backward. So, if one would like to write at any grid point, so if it like to write at any grid point x i y dash of x i, which is this in terms of h, so this is y i plus 1 minus y i of course plus order of h. So, this is a forward approximation.

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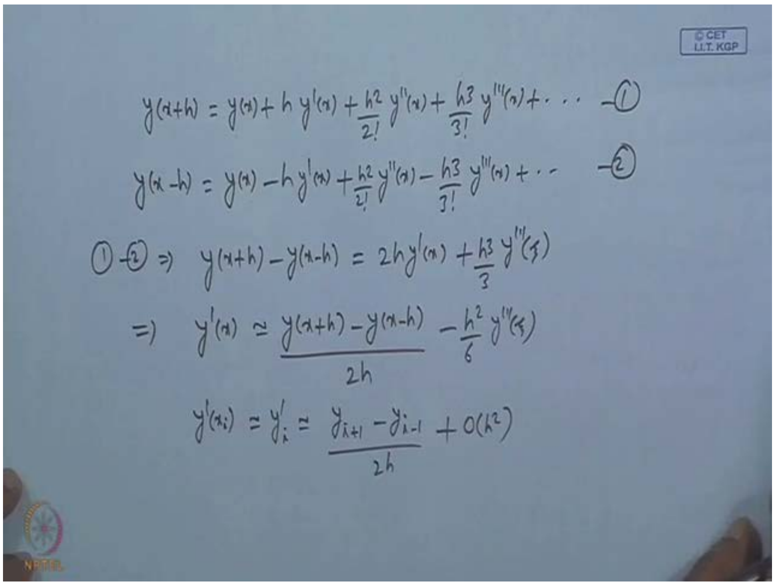

So, now let us proceed further. So, let us write down one more. So, this was our 1. So, this was our 2. Now, our aim is to find approximations for first derivative. So, if we add them, first derivative gets killed. So, let us subtract so then we have on the left hand side, this gets cancelled and this gets cancelled. So, this becomes h cube by, so this is 6. So, this is say this. So, this implies y dash of x is plus or minus, so this is minus h square. So, this is at a grid point. So, that means the earlier approximations, for example forward and this is backward, so they are first order, whereas this approximation, we have second order.

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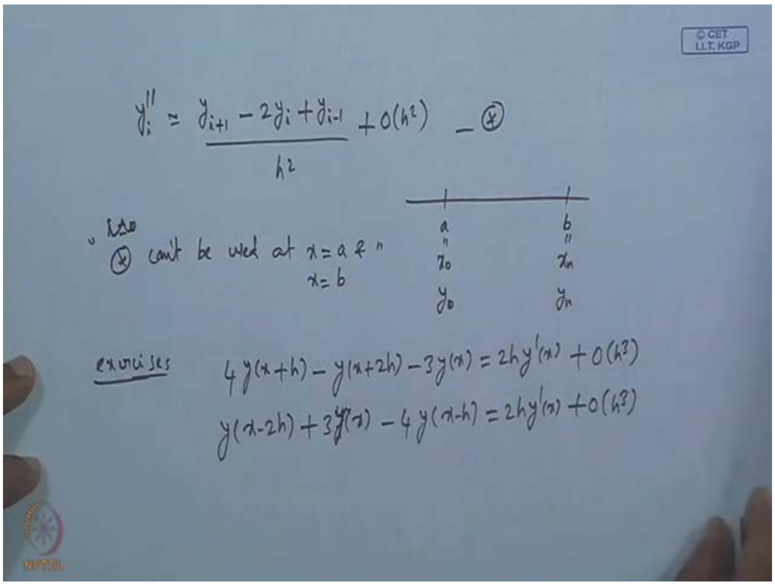

So, this is similarly, in exercise, so for example, so we have 1 and 2. By subtracting, we got this. Suppose we add, what happened? So, these two get cancelled, so these two get cancelled, so this plus this and here we have twice and this becomes twice. So, one may get by h square. So, this can be verified easily. So, if our domain is this, so this is x 0, so this is x n. So, then see for example ii, if one would like to use these, we can use. So, this correspondingly the data starts from y 0 and y n.

So, if we use i equals to 0, so then we have y minus 1, which is beyond this. So, that means in some sense, star cannot be used at x equals to a and b. When I say x cannot be used, we will get back to this. I did not mean literally we cannot use it, we can use it. We get y minus 1 here when we use but x equals to a and when we use at x equals to b, we get y and plus 1, which is the next point use. So, we will get back to this. That is not a big issue. So, now other some exercises, you can try 4 y of x plus h, this is x plus 2 h minus, so this can be derived, 3 y of y of x minus 4 y of x minus h. So, this can be derived. So, these are some exercises.

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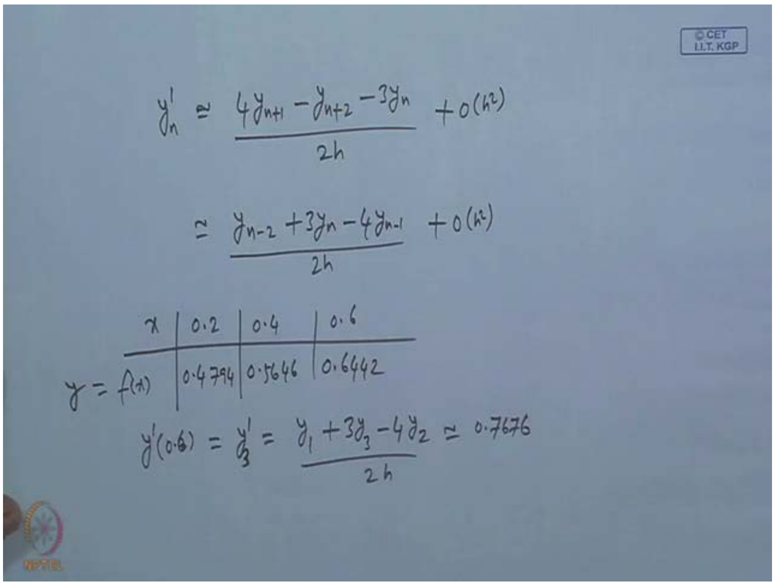

Accordingly, y dash n say this is 4. Also, for example, say suppose this is f of x is given like this. So, one can compute for example y dash at 0.2, so this can be obtained from consider x 0 is 0. So, then this can be y 1 prime, so y 1 prime. So, for example from here, if you would like to compute y 1 prime, I am sorry not, so for example 0.6, so that is x 1, x 2, x 3, y 3 prime, so from here we get y 1 plus 3 y 3 minus 4, 4 y 2 by 2 h.

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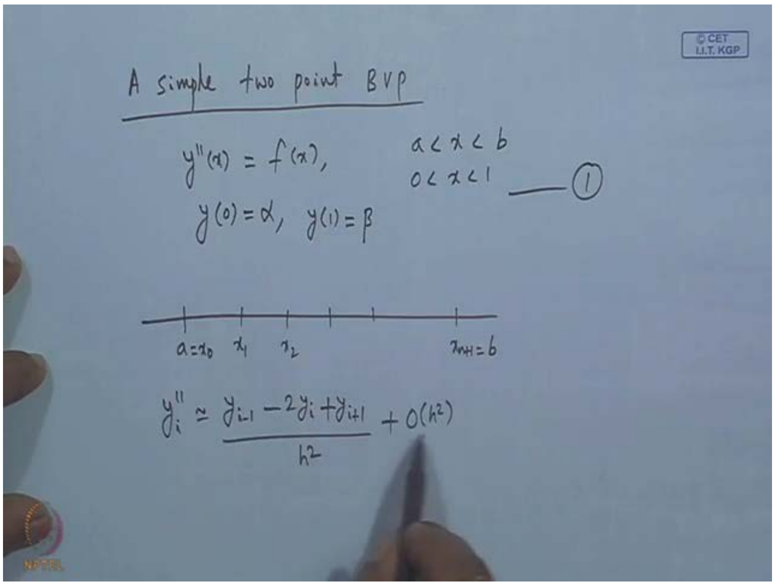

So, this was our y. So, one can compute, we get some value. So, that means using the values, one can compute the derivatives. So, this is an important thing. So, having now introduced the finite differences, let us proceed to solve at least one simple boundary value problems. So, let us try to attempt by replacing the derivatives by corresponding finite differences.

So, a simple two point BVP, so as I mentioned this is very simple and say one can take, but let us take the simple case and y of 0 is alpha and y of 1 is beta. Suppose this is the one. So, this is very simple and one can get the solution very easily, but we have considered such a simple problem just to ensure that the corresponding numerical method really can be implemented and then for such a thing, one can get the analytical solution and one can really compute. So, I will switch over to the notation slightly. So, this is a x 0, x 1, x 2, so on x n plus 1 is b. Earlier, I said x n, so this is not a big issue. So, the number of points, now what was that let us use this approximation. So, this is approximation we are trying to use.

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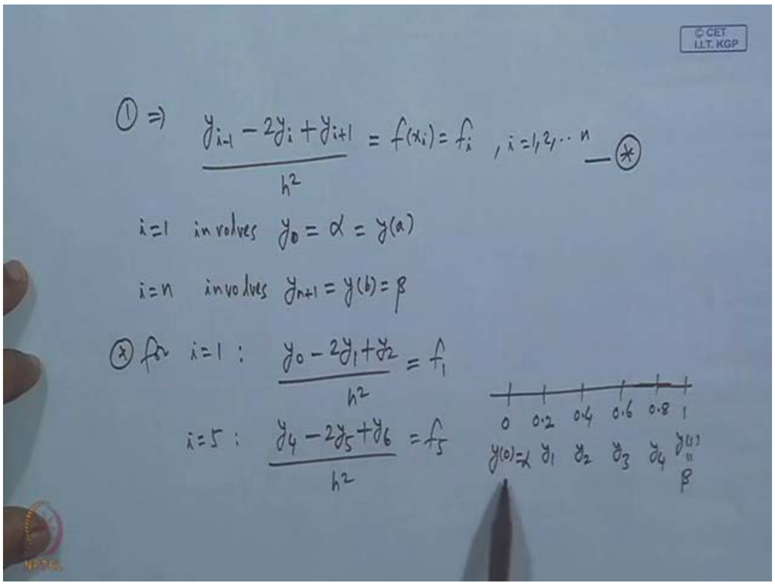

Accordingly 1 becomes, so 1 becomes n. So, this is our corresponding finite difference, replacement for the given differential equation. The case i equals to 1 involves y 0 equals to alpha and the case i equals to n involves y n plus 1 that is y of b of course, this is y of a. So, I make a remark again. I have taken n plus 1 there. So, if you run star for say i equals to 1, we get y 0 minus 2 y 1 y 2 by h square equals to f 0; i equals 1, sorry, f 1. So, this is i equals to 1.

Similarly, suppose i is 5. So, we get y 4 minus 2 y 5 y 6 so on. so forth. So, that means within the suppose, for example, take the case of 0 and 1 and one has to decide what is a suppose h is 0.2, 0.4, 0.6, 0.8, 1 and we know the values here y of 0 is given alpha, y of 1 is given beta. So, what are the unknowns? y 1, y 2, y 3, y 4, so these are the unknowns. So, we have to obtain these unknowns. So, we have to obtain these unknowns. So, that means if we run i from 1 to i 1, 2, 3, 4, for i 1, this point is involved but that is given and that for i 4, this point is involved, but that is given. We will be left with a system for y 1, y 2, y 3, y 4.

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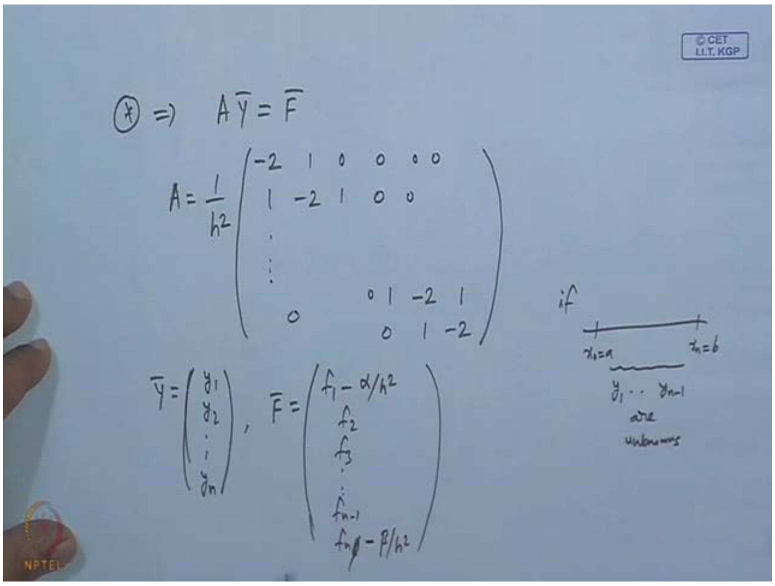

So, let us try to put it in the system form. So, this further the star implies where the structure of A has to be decided. So, for a general case, one can write. So, I have given for example, so if you run y y 0 is known, so in the first equation we have coefficient of y 0, there is nothing because it is known. Then, coefficient of y 1 is minus 2 and coefficient of y 1 is 1. Let us look for a i equals to 5, coefficient of y 4 is 1, y 5 is minus 2, y 6 is 1 that means apart from the end, the equations in the middle would contain three terms. The equation at the end points contain two terms.

So, one can expect for A, so h square I am not cross multiplying. This will be, all are 0s. Similarly, so this will be the corresponding matrix A. Why? I could explain again. Look for the first equation. So, the coefficient is coefficient of y 1 is minus 2 because the unknowns are, these are the unknowns.

So, coefficient of y 1 is minus 2, y 2 is 1. So, because this has to be multiplied by y 1, y 2, y 3, y n minus y n here, so minus 2 times y 1 plus y 2, which we retain. So, similarly, it is better to write down what is Y in this case, y 1, y 2, actually to have a symmetry have given y n plus 1 as b. So, may be I can add a remark here, if x 0 equals to a, x n equals to b, the unknowns will be y 1 are unknowns because accordingly the h will change undoubtedly.

So, this is the thing. Now, what will be F bar? Look at that in the first equation f 1 is known and y 0 is also known, so y 0 by h square, so for example, y 0 is alpha, so alpha or h square is known. So, we can shift it. Similarly, to the last case, we know the last entry y and plus 1, which is beta. So, beta by h square can be shifted. So, what will be our F bar? This will be f 1 minus alpha by h square, f 2 f 3 f n minus 1 f n by so f n minus beta by h square because this for example, 5 this is the last point. So, then y 6 must be beta. So, we know beta. So, that will be shifted, beta by h square will be shifted. So, this is our system. So, I write it again.

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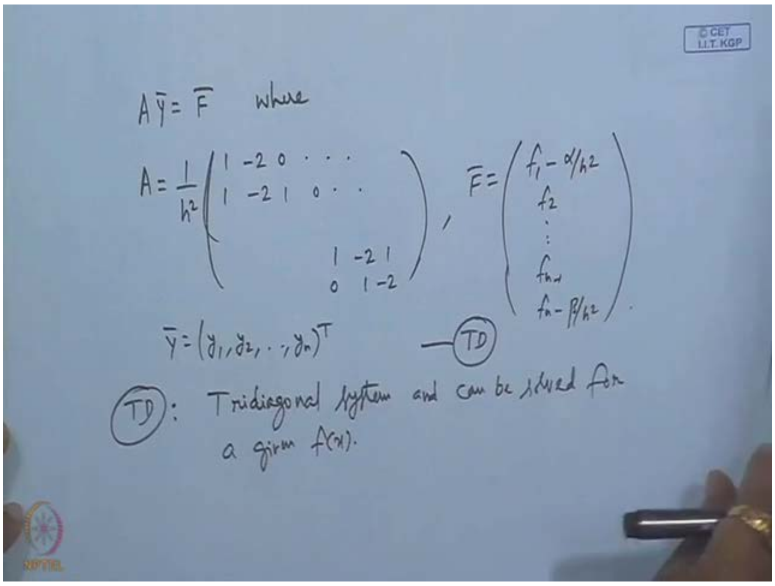

So, we have A Y bar equals to F bar, where A is 1 over h square, 1, minus 2, 0 and here 1, minus 2, 1, 0, here 1, minus 2, 1, 0, 1, minus 2. Then, F bar is f 1 minus alpha by h square, f 2, f n minus 1 and transpose. So, all this, now what is the feature of this TD? It is indeed a triangular system, look at the entries. So, this is a tri diagonal system and can be solved for a given f of x. So, I will get back to little later how this can be solved for a general case.

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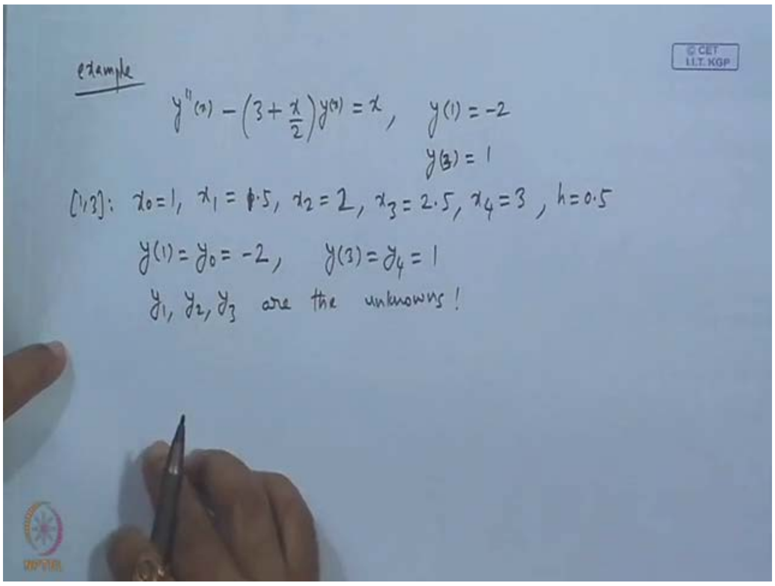

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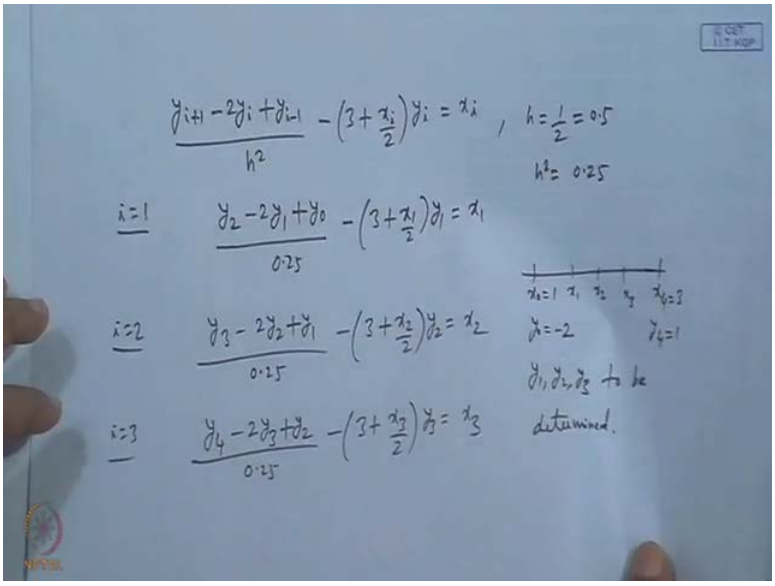

Now, let us look at the scenario with an example before we generalise. So, the domain is 1, 2. Then, say x 0 is 1, x 1 is 0.5, x 1 is say 1.5, x 2 is 2, let us extend it to 3, so x 3 is 2.5, x four is 3. So, this is, so that means the domain is 1, 3. So, x 0 is 1, x 1 is 1.5. So, accordingly h is 0.5. So, accordingly y of 1, which is y 0 equals to minus 2, then y of 3, y 4 is 1. So, how many unknowns then left; y 1, y 2, y 3 are the unknowns. So, these are the unknowns. So, now consider the discretisation. So, we have this equation. So, we have to discretize this.

So, if you discretize, so this is a discretized equation. Now, i equals 1, so this will be y 2 y 1 y 0 h square. So, h is h is this, so h square is, so h square is 2 5, so minus 3 plus into y 1 is x 1. So, then i equals 2, so y 3, y 3, y 1. So, if you look at this our x 0, 1 and x 4 was 3, so in between x 3 or the points, so y 0 is known to us, y 0 minus 2 and y 4, 1. So, if you look at these in the first equation, we have only two unknowns and again in the last equation, y 4 is known only two and the middle of course. So, here to be determined, so this has to be determined. So, we can simplify and put it in a system.

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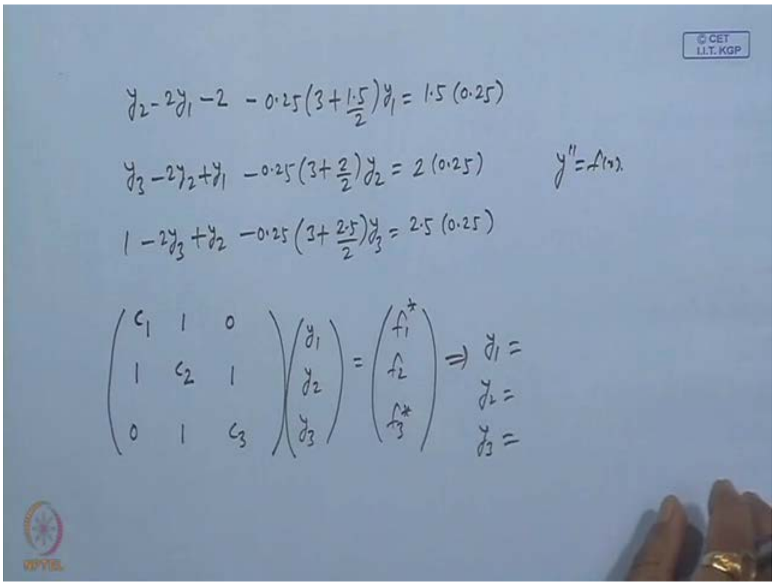

So, if you put it, we get the system. So, let us do it y 2 minus 2 y 1 plus y 0 y 0 minus 2, then minus 0.23, 3 plus x 1 was 1.5 by 2 into y 1, x 1 was 1.5 into h square goes there. So, this is then the second equation y 3 minus 2 y 2 plus y 1 minus x 2, x 2 was and here x 2 2 h square, then y 4 y 4 we know that is 1.

So, accordingly one can simplify, we get. So, here the coefficients of y 1 got changed because of our earlier equation y double f of x, but in this case, we had a part which is depending on y. Therefore, naturally there will be a contribution. So, one cannot expect minus 2 and 1. So, here we get some coefficient, some coefficient c 1 and y 2 coefficient is 1, y 3 coefficient is 0 and similarly, here y 1 coefficient 1, then here c 2 because y 2 has minus 2 and this part. c three is 1 and in the third equation, y 1, y 2 coefficient 1, and then this has c 3 and here we get f 1 f 2 and f 3.

So, this is a structure which is tri diagonal and one can solve for. So, this is the scenario in general, but the question to be asked is very important. How well these, see when we solved, we are getting y 1, y 2, y 3 that means the solution at the grid points, how well this is approximating the true y of x, so this is an important question. So, it depends on the discretisation scheme; true it depends on discretisation scheme, but if we look at it, we get a system and then we are trying to solve. So, let us look little more carefully into this matter.

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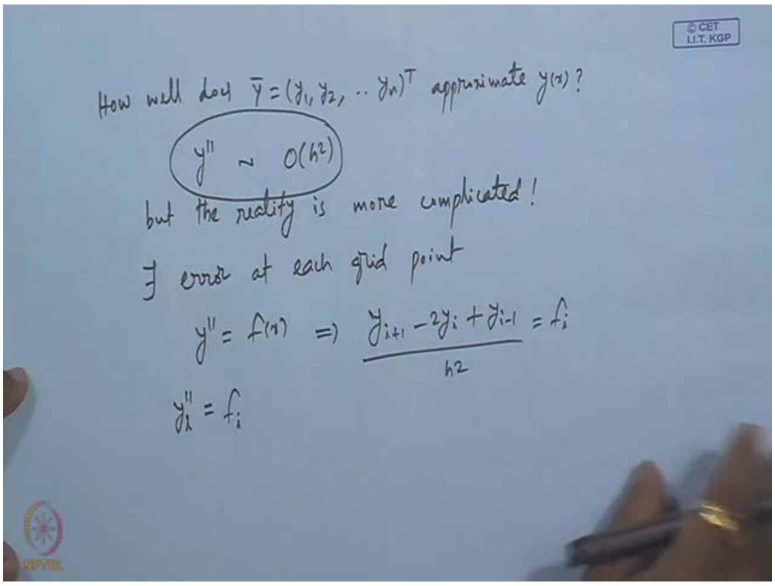

So, the question to be asked how well does say the approximate solution Y bar m y 1 y 2 transpose approximate y of x? So, then somebody would say well we have discretized, we have discretized y double prime by a scheme, which behaves like this. So, therefore, the corresponding approximation must be of this order. Are you following? See, these are at the grid points. Remember, these are at the grid points; y of x is the true solution. So, this is the vector notation I am using.

Now, the question to be asked is how well this solution approximates the true solution. Then, immediate answer is well, we have approximated y double by o h square approximation. Therefore, the corresponding approximation is this. But, the reality is more complicated. The answer may be true, but the reality is complicated because there exists error at each grid point. Why? What we are saying, so take the case of y double equals f of x.

The discretization we proposed y i plus 1 f i, so this is of second order, but then when we get the system, we are saying y i double prime must agree with f i. So, this is what we are saying at each grid point. So, that means you take a y i, which is computed from this and then you apply the corresponding second order approximation, then we should get f i. So, this process is expected to give similar, but it is not so obvious.

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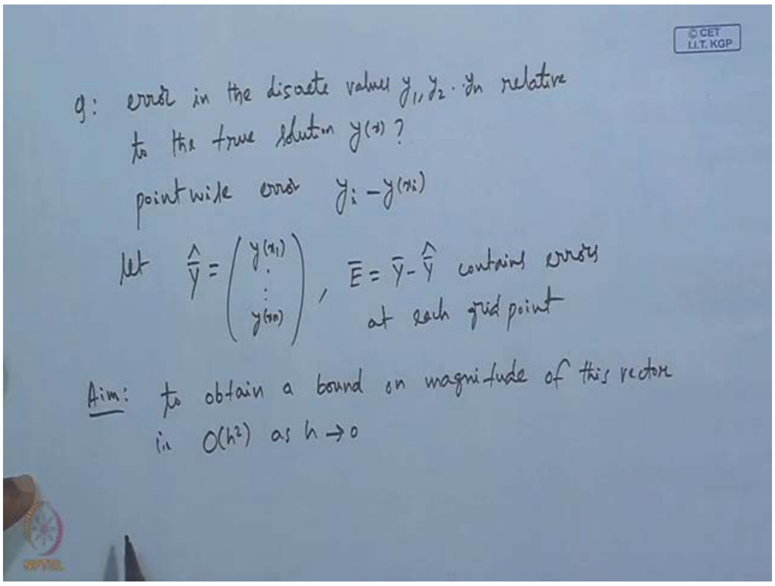

So, let us look at this further. So, what was the question to be asked, error in the discrete values y 1, y n relative to the true solution y of x, so what is the error? Immediate concern was immediate answer point wise error y i minus because at a grid point, y i we know and it should have been y of x i, so the difference will give you.

So, let denote this by true, then this vector, so this contain errors at each grid point naturally because this is approximated and this is true. If you populate vector, so this error vector contain errors at each grid point. Then, what is our aim; to obtain a bound on magnitude of this vector and show that it is this. So, to obtain a bound and magnitude of this vector that is it is really like this, so how do we do it?

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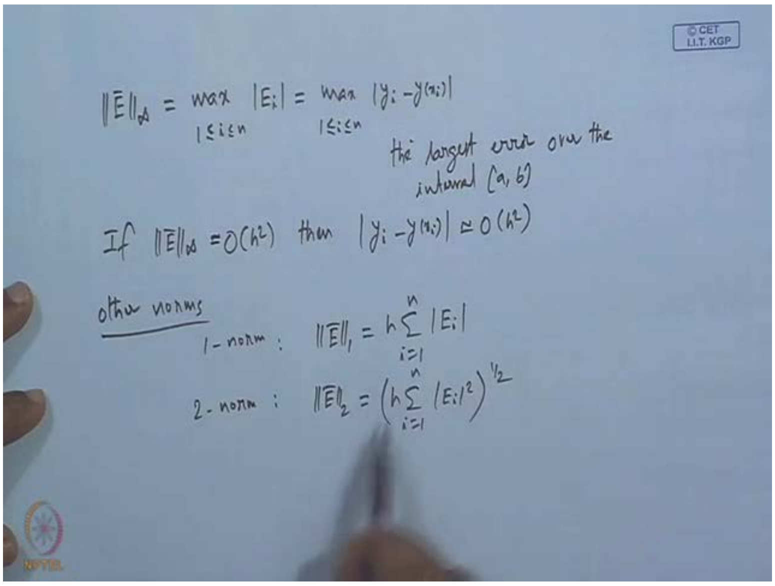

So, consider since this is a vector, so we have to consider some norm, so this let us consider this maximum. What is this? This is nothing but the largest error or the interval because at each fixed, point point wise error and then you are taking the maximum. Therefore, if this, then point wise must be because we are taking the maximum of, so if the sup norm of the error vector e bar is this, then definitely the grids it is of this order because we are taking the resultant is coming of this. So, then what are the other norms, other norms? So, there is 1 norm, so which is like this and there is a 2 norm. So, one can refer some standard books to know more about this 1 norms and 2 norms.

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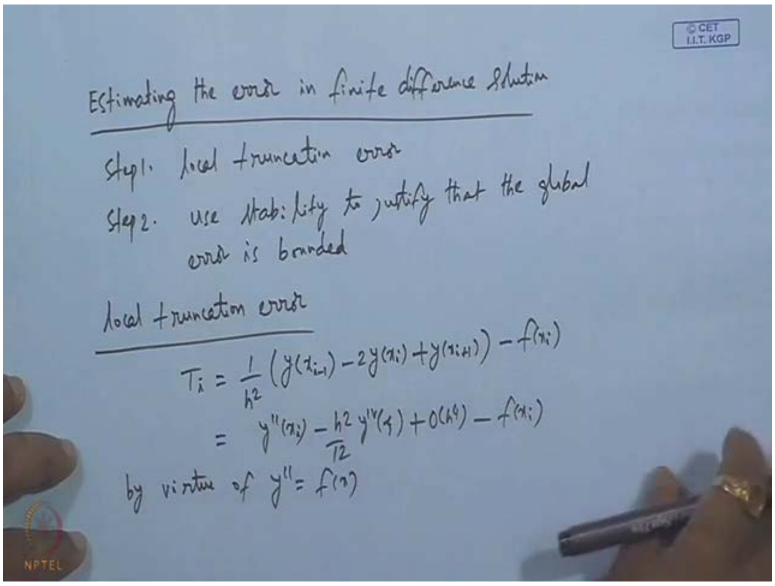

So, now what is our next aim; estimating the error in finite difference solution. How this is done? Step one local truncation error, step two use stability to justify that the global error is bounded, bounded how; bounded in terms of local truncation error. So, that means you have a finite difference approximation. Now, what is the local truncation error that is bringing in then; so that is the disturbance. Initially, how do you ensure that it is not getting magnified?

Therefore, you have to use stability aspects and conclude that global error is not really bounded. It is not getting magnified. So, let us look at the local truncation error. So, so we are replacing the approximations by the exact, then only we get the error minus f of x i. So, now this is y double of x i minus because this approximation is subject to this because for y double, we got this approximation and this is the error. But by virtue of y dashed of equals to f of x, we have this must be agreeing with this.

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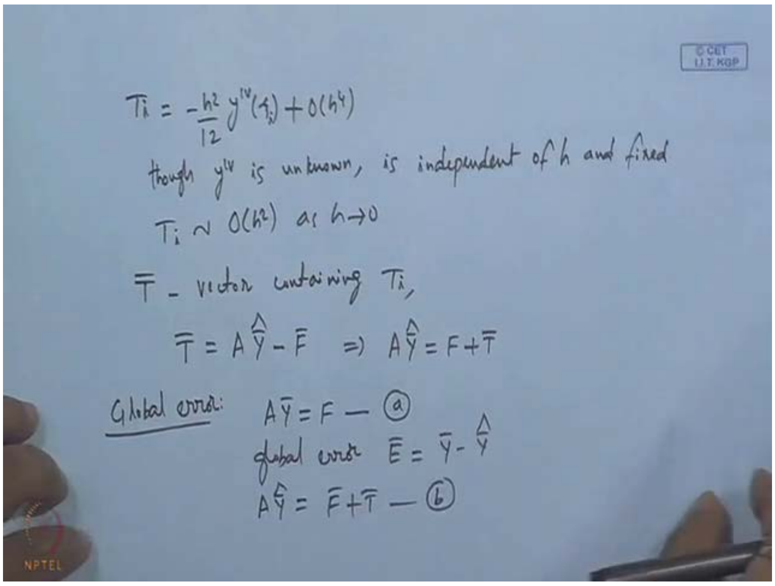

So, we have T i, but of course we do not know, but we believe that this independent of h. So, the remark is though y 4 is unknown, it is independent of h and fixed. So, therefore, T i behaves like this because this is fixed a constant, which is independent of h. Therefore, the overall behaves like this. So, let us define now T bar is vector. See, this is at a grid point, certainly at a grid point. So, T bar is vector containing T i, then what would happen? T bar is a true minus this. This implies is this. Now, global error, so we have A Y equals F, this was our approximation. Then, the global error E bar was defined as this, then above we had A is F bar plus T bar, so from subtracting these two...

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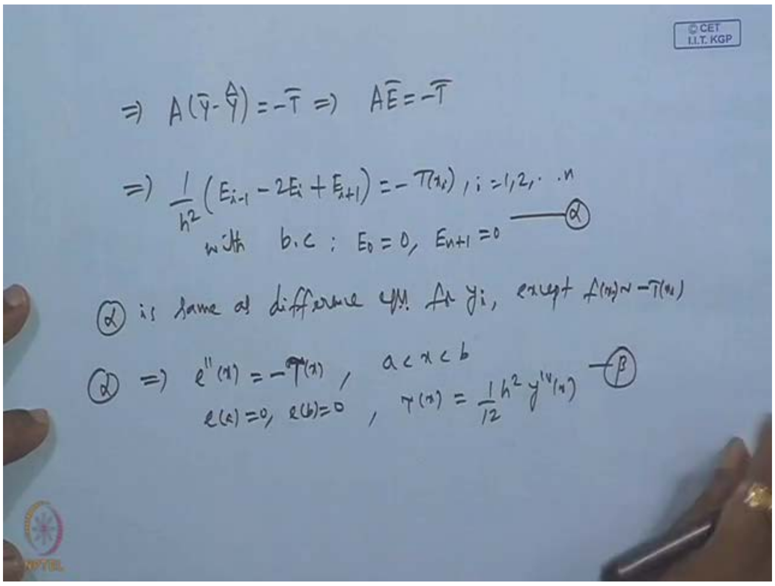

...we get A. This implies, so this implies we know the matrix A, so we can write it like this at each grid point with boundary conditions E 0 is 0, E n plus 1 is 0 because we use the exact values at the boundaries.

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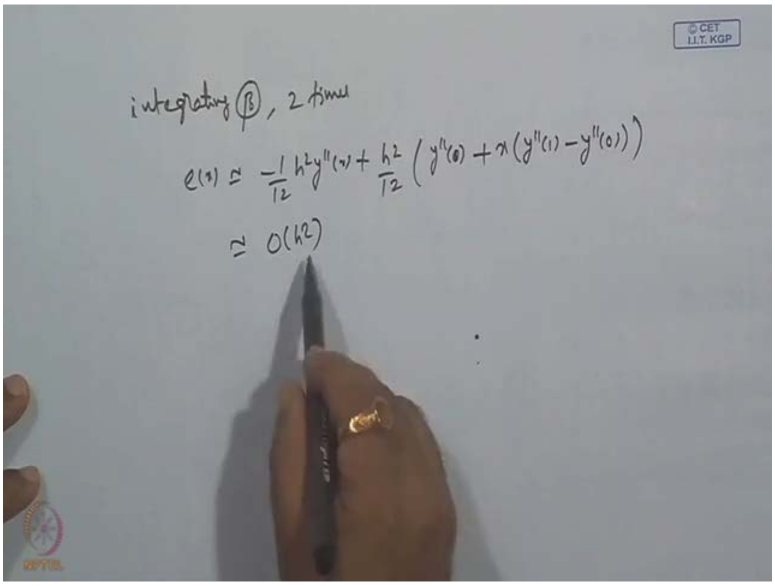

So, say so this is same as difference equation for y i except f of xi is replaced by minus T of x i. So, this can be thought of as, alpha can be thought of as e double prime x equals to minus some tou of x as a as a function. So, this can be thought of as in this interval where e of a equals 0, e of b equals to 0. Tou x is 1 by 12 h square y 4 of x, so say this is beta.

Now, integrating beta two times, we get sometimes ab, sometimes 1 0, but I want show you just it is at the end points and this is of order this. So, this is the global error. So, showing this global error as order of h square is an essential task because it is not that we are getting as a result of discretisation. Also, you have at each grid point, you have an error.

So, then when you operate your double derivative operator, discretisation operator, so you get the corresponding thing agreeing to the function on the right hand side, but this is with respect to max norm, we have this difference as error vector and then using that, really one can show that the global error is also of order h square. So, then of course, one has to show that the method is really stable and converging. So, we can discuss in the next lectures. Thank you.