Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 13
Stability Analysis of Multi Step Methods

Hello, in the last class we have discussed some general methods for stability aspects of multi step method. So, for example schur criterion we have discussed. So, given a multi step method, how to find out the stability regions using the schur criterion?

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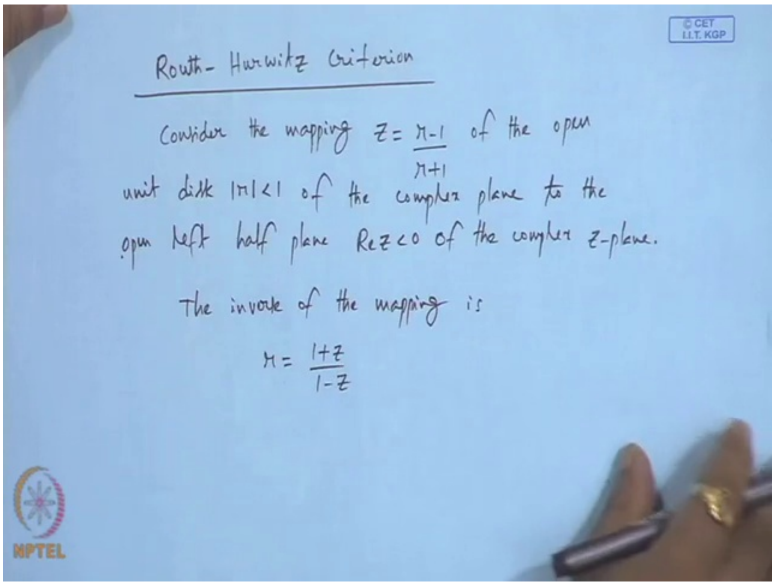

So, let us continue ahead with stability aspects of multi step method. So, let us discuss some further methods. So, one of the popular method is Routh-Harwitz criterion. So, consider the mapping of the open unit disk of the complex plane to the open left half plane, that is real z less than 0 of the complex z plane. So, what this mapping does? This maps unit disc mod r less than 1 to the left of plane in z plane, then the inverse of the mapping is given by inverse of this mapping. So, this is the inverse of the mapping.

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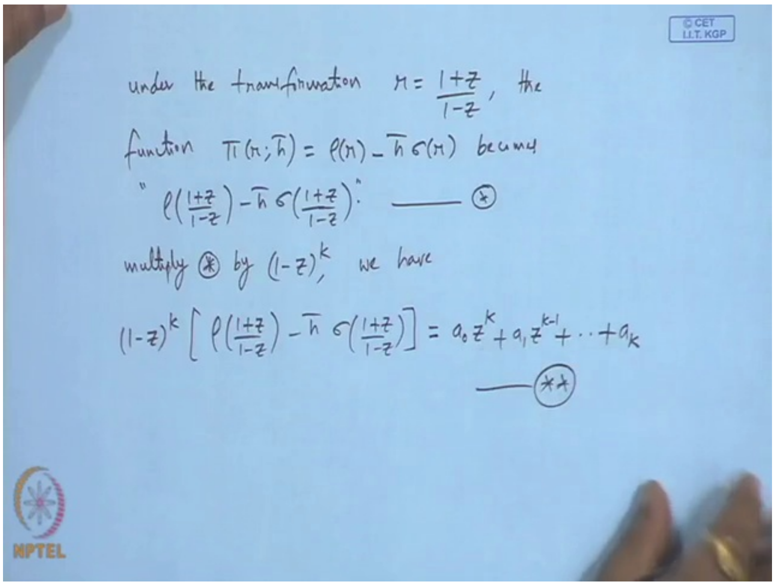

So, under the transformation, under this transformation the function. So, what is this? This is our stability polynomial. So, this is this becomes, so this polynomial becomes this. Now, say this is a star, multiply star by 1 minus z power k, we have. So, we are multiplying this by 1 minus z power k, where k is the order of multi step method. So, we get this and our aim is to identify this as a polynomial in z. So, what was our aim? So, this is order of the method.

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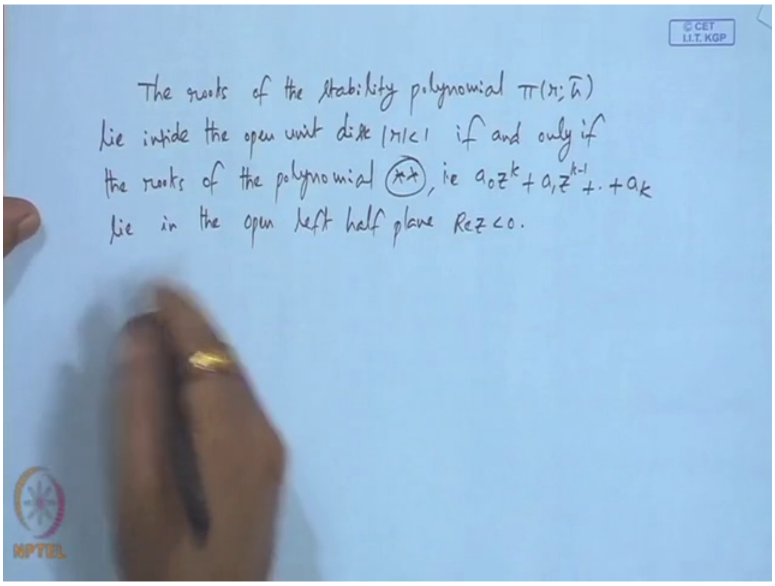

So, we are multiplying and this we are expressing as polynomial in z. So, then the roots of the stability polynomial lies inside the open unit disk, if and only if the roots of the polynomial double star, so that is of this lie in the open left half plane. So, the roots of the stability polynomial in the original variable r lie inside the open unit disc if and only the roots of the polynomial double start.

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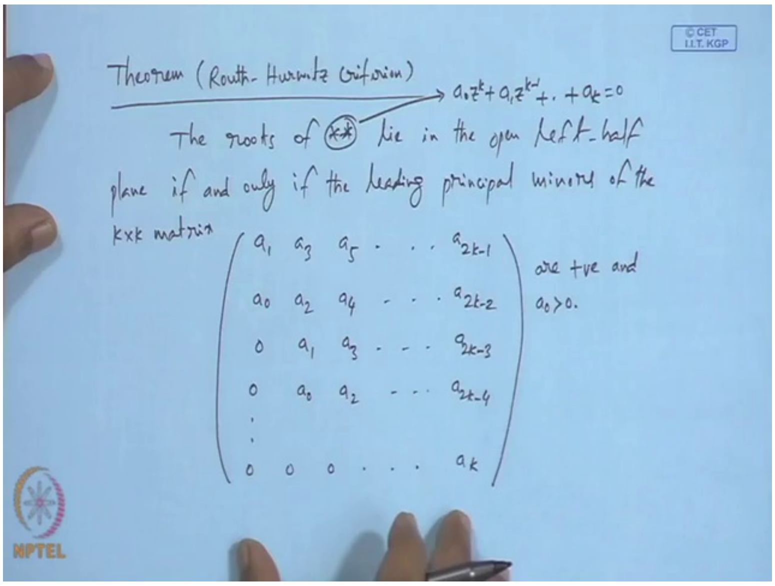

This is this lie in the open left half plane. So, then the criterion the roots of double star lie in the open left half plane if and only if the leading principal minors of the k cross k matrix. So, this is a, the roots of double star so that double star is of the form. So, lie in the open left half plane if and only if the leading principal minors of this matrix are positive, and a 0 greater than 0 are positive leading principle minors of this matrix are positive and a naught greater than 0.

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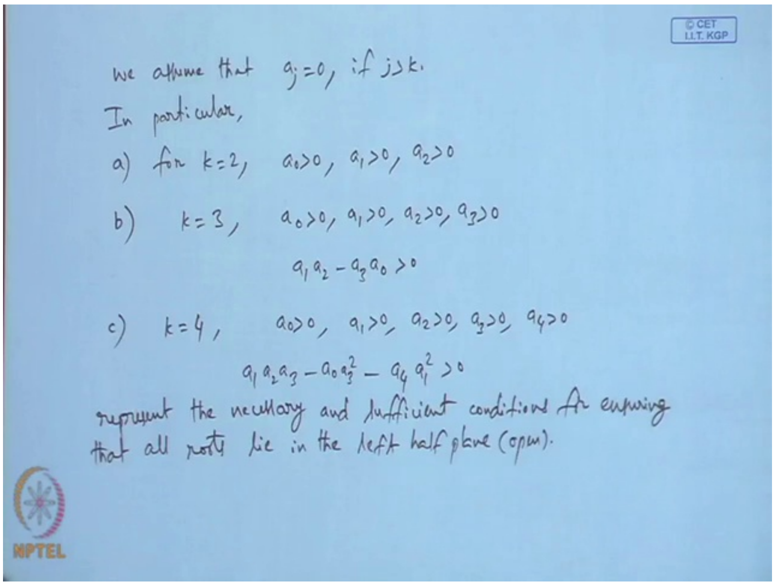

We assume that a j is 0, if j is greater than this, in particular for k equals to 2 the condition leads to this and k equals to 3 and k equals to 4. These represent the necessary and sufficient conditions for ensuring that all roots lie in the left half plane. Of course, open. So, this is these are the necessary and sufficient conditions. So, to start with, so we have used a transformation, what was the transformation? z equals to r minus 1 and the inverse of this is this.

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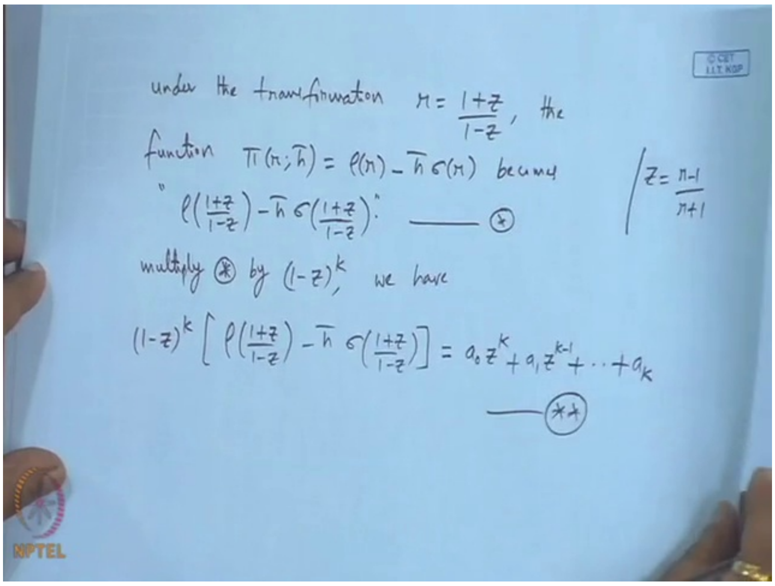

So, under this transformation the stability polynomial reduces to this, then we have to multiply by this factor. As a result of the multi step method, the linear multi step method of order k we are getting this factor. So, then multiplied by that we get a polynomial z.

So, then the Routh-Hurwitz criterion states that the roots of this polynomial lie in the open left half plane if and only if the leading principal minors of this matrix are positive. And a 0 is greater than 0 and in particular we have the these being the necessary and sufficient conditions for ensuring that all the roots of the polynomial in z to lie in the open left half plane, ok?

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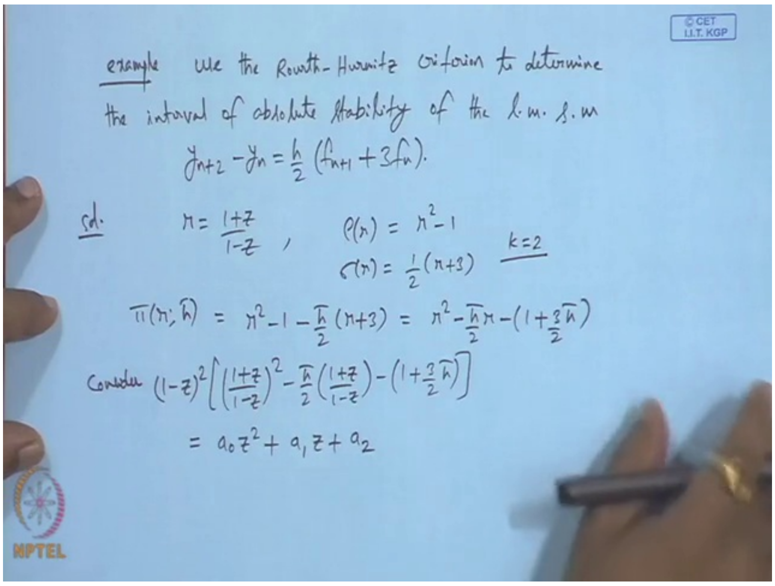

So, let us see how it works out with a example. So, use the Routh-Hurwitz criterion to determine the interval of absolute stability of the linear multi step method. So, the given linear multi step method we need to determine the interval of absolute stability. So, we have r equals to this is a map, then for this method we have. So, this is given by r square minus 1 and this is given by accordingly, we have r square minus 1 is h bar.

So, this will be the stability polynomial which is, so this is our stability polynomial and what is the order of the method 2. So, we need to consider this into variable change to this transformation. So, accordingly this should be written in the form, so we can simplify this.

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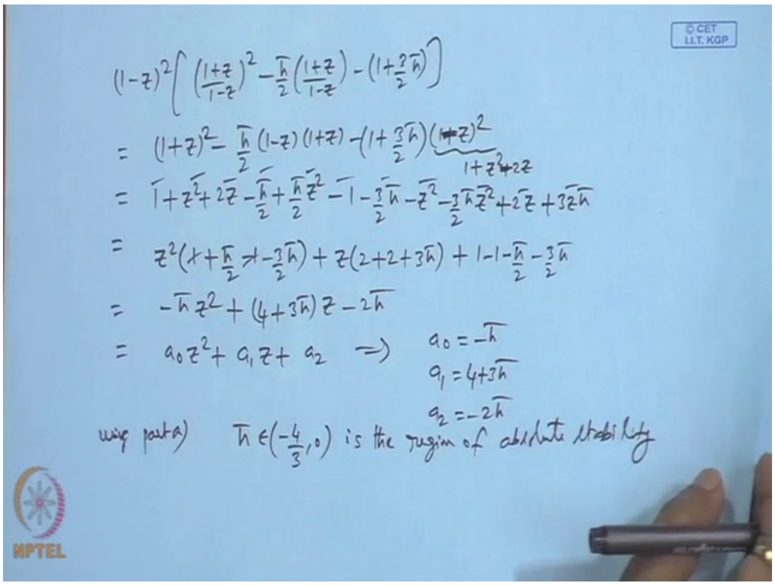

So, here this get cancelled minus 1 term there, so 1 minus z. So, this will be 1 minus z square. So, this will be 1 minus z square. So, minus h by 2 z square and this is usual expansion. So, we multiply, sorry this is minus, this must be minus. So, this will be these terms then so this is minus h bar z square plus 4 plus 3 h bar z. This is this 2 h. So, this is a 0 z square.

Now, for k equals to this is the necessary and sufficient condition accordingly. So, using part a, we get h bar to be is the region of absolute stability. So, this is Routh-Hurwitz criteria, right? So far these stability methods, these two general methods, one is schur criterion, other is Routh-Hurwitz criterion. So, they give you using some kind of approach, the Schur criterion via Schur polynomial and then Routh-Hurwtiz criterion via in kind of a transformation.

So, but there are another general methods, in the sense you try to after all it is a difference equation, so you try to find the characteristic polynomial and find the roots so that like a general recurrence relation you get the solution. See once we have a recurrence relation which is also a finite difference method, right? So, we can get the solution using the roots and hence analyze what happens to this solution, ok? So, let us see one such method.

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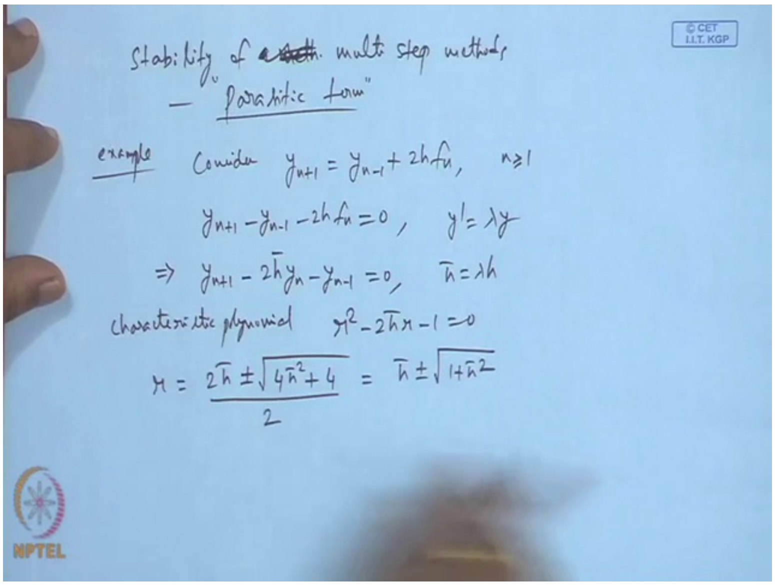

Stability of multi step methods parasitic term. So, we will see what is this parasitic term? So, let us see with an example, so consider the multi step method. So, obviously this is for n greater than equals to 1 so then so with reference to absolute stability the reference equation is lambda y. So, accordingly these become 2 h bar y n, where h bar is, so the characteristic polynomial r square minus this, right? Now, the roots are given by 2 h, so this is given by, so these are the roots.

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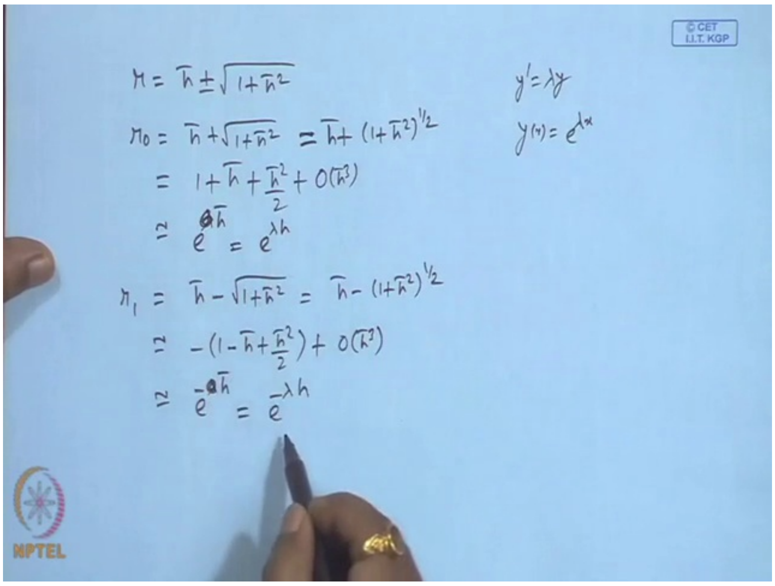

So, let us analyze, so what were the roots r equals to. Now, consider first root, call it r 0 so this one plus so this is h bar plus. So, having power series expansion we get 1 3. So, I write that first, then h here, then h square by 2 plus the higher order terms. So, if you try to approximate see what was our equation and naturally the exact solution must be of this form.

So, if we approximate and try to identify with the exact solution y a of course, this is indeed e power lambda h bar. Now, consider r 1 h bar minus, so this is h bar minus, so this can be approximated as minus 1 minus h bar plus h bar by 2 plus third order terms. Now, this can be approximated as minus e power lambda h bar, I am sorry this is lambda which is e power h and this is e power minus lambda h, ok?

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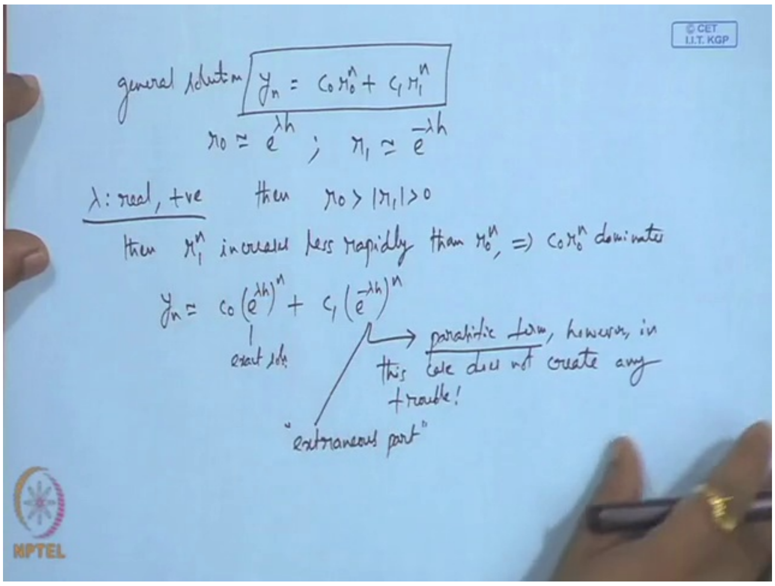

So, one you have this type and other is this type. However, once we have the roots the general solution y n is given by c 0 r 0 power n c 1 r power n, right? So, there we have r 0 is behaving like e power lambda h and r 1 is behaving like this. Now, let us take real positive then what happens, lambda real positive. So, we need to compare because the general solution is this.

Now, we need to identify which one is dominating and really the solution of the difference equation gives stable solutions, right? So, then what happens if lambda is real and positive r 0 is greater than this. Then what happens? This grows faster than this. So, what happens r 1 power n increases less rapidly then r 0 power n, implies c 0 r 0 power n dominates, this dominates.

So, that means the other term, so out of your solution c 0 e power lambda h n e power lambda h n. This is dominating this for this is a case which is the exact solution and this is the parasitic term. However, in this case does not create any trouble. So, why do we call this parasitic term and then why we conclude all this? Of course, this is exact solution for the given solution, but this is an extra. So, this is also called extraneous part and this is the parasitic term, but however in this case does not create any problem. The overall solution grows as per the exact solution. So, that means for these values of lambda the solution is stable.

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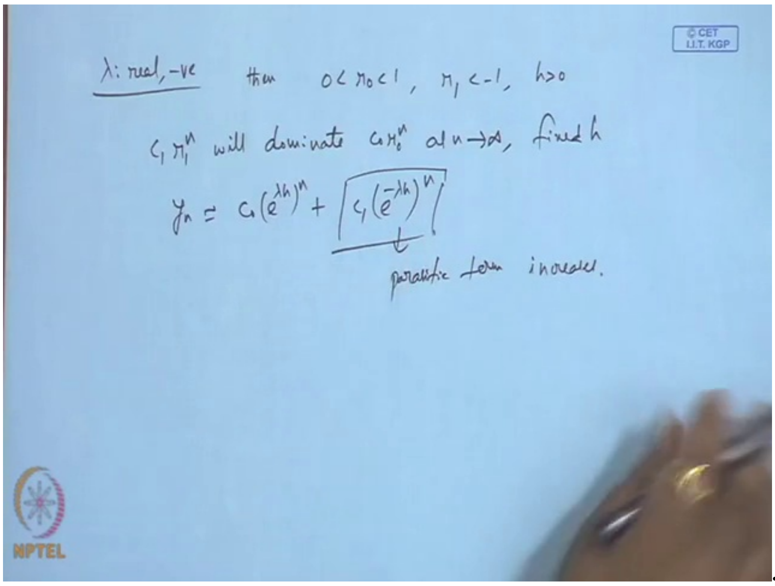

So, let us consider now the other case, lambda is negative. Then for of course, h is positive. So, then c 1 r power n will dominate c 0 r power n, as n goes to infinity for a fixed h. Therefore, so the parasitic term which is h, so this parasitic term increases. So, this creates a problem, so that is how so for lambda real and positive it agrees with the true solution, whereas for lambda negative this parasitic term creates a problem. So, this is another kind of a stability analysis, ok?

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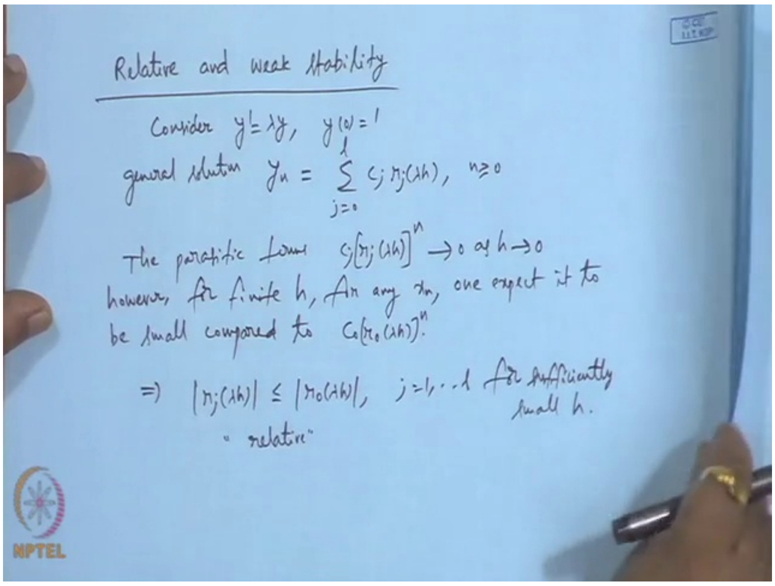

Let us see this little more from little theoretical point of view, relative and weak stability. So, relative and weak stability. So, with respect to what it is relative we will see. So, consider this, so general solution is given by. So, for a general method this is 0 to n c j r j lambda h, right? The parasitic terms power n goes to 0 y a, this is true because as h goes to 0 the parasitic terms goes to 0. However, for finite h for any x n 1 would expect it to be small compared to c.

So, that means we are bringing out the concept of relative stability. So, relative with respect to, what? So, relative with respect to the a h, the principle exact r 0. So, this implies for sufficiently small h. So, this is relative is a comparison the parasitic terms with the exact, so extraneous terms with the exact, right?

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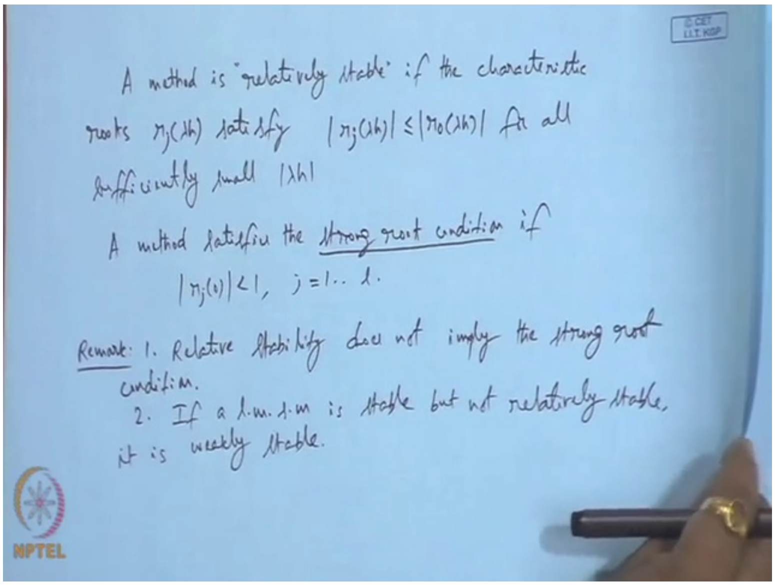

So, a method is relatively stable if the characteristic roots satisfying for all sufficiently small. So, that means let there be some parasitic terms, but compared to the exact, these extraneous must be small for sufficiently small lambda h. The impact satisfies the strong root condition if a h, these roots which are, which would contribute to the extraneous terms if the magnitude it.

Of course, for a h 0 if the magnitude is less than 1 then we say the strong root condition, right? So, remark relative stability does not imply strong root condition and if a linear multi step method is stable, but not relatively stable. It is weakly stable, so relative stability does not imply strong root condition and if a method is stable, but not relatively stable, then we say it is weakly stable, ok?

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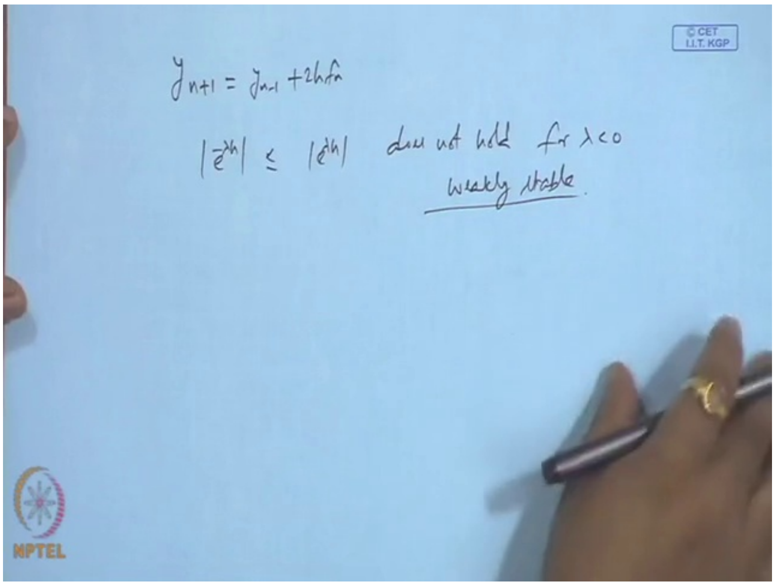

So for example, the case y n plus 1 plus 2 h f n, so far in this case we have seen. So, this does not hold. So, hence this method is weakly stable, so that is how we can categorize the relative stability and weakly stability. So, we have some criterions to systematically capture the interval of stability region of stability, but there is another concept to analyze the parasitic terms and then say how the method behaves, ok?

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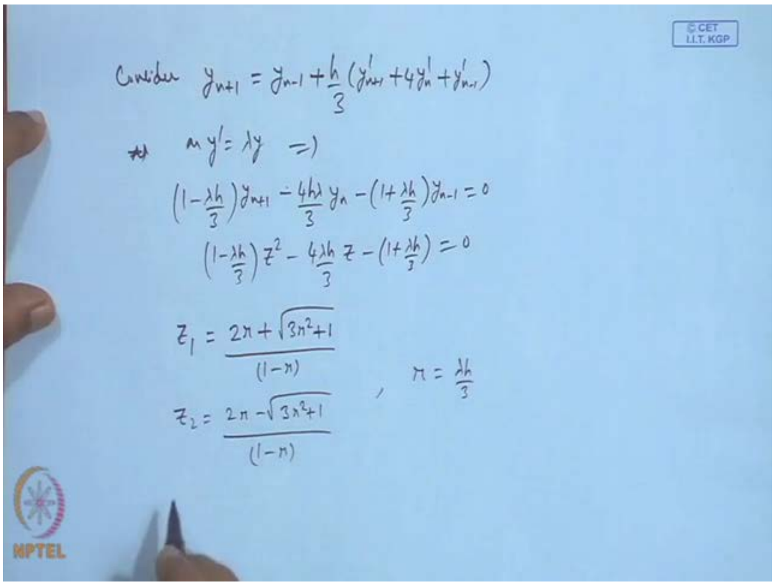

So, let us analyze with some more examples. So, for example consider right. So, this implies on y dash we get. So, we have, so this will be the characteristic polynomial, then roots. So, if r is this we get these roots, ok?

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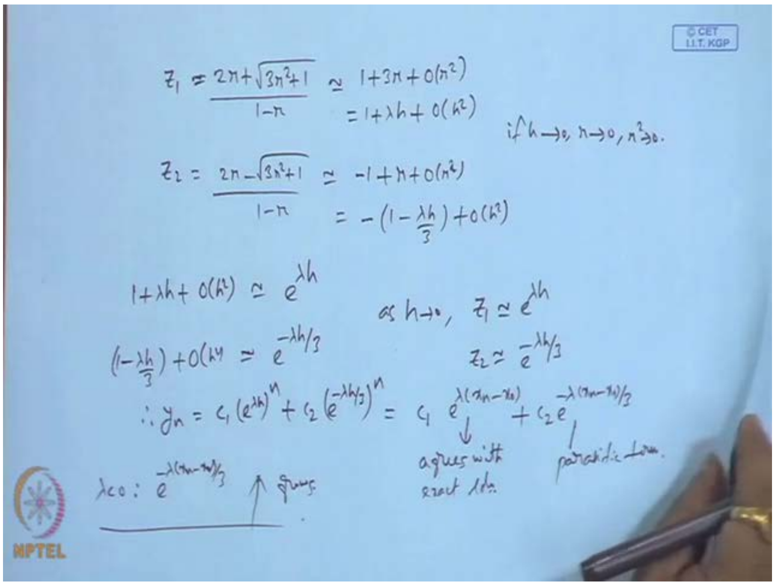

Accordingly z 1, z 1 which is so this would be written as this can be written as. So, this is for what if h goes to 0, r goes to 0, r square goes to 0. So, with that we can linearize and try to get this. So, you can identify, so we have done it in the last example. So, 1 plus lambda h equal e power lambda h. So, as h goes to 0, z 1 behaves like, it has to behave like this. Therefore, y n c 1 z 1 power n plus c 2, then so this is c 1.

So, this agrees with exact solution and this is parasitic term. Now, if lambda is negative, this grows and creates trouble. So, this is another example where we could analyze. Now, for these multi step methods there is a small technique, given your characteristic first and second characteristic polynomials are r h o of zeta and then sigma of zeta. So, for a particular method if r h o is given, can we determine the corresponding sigma?

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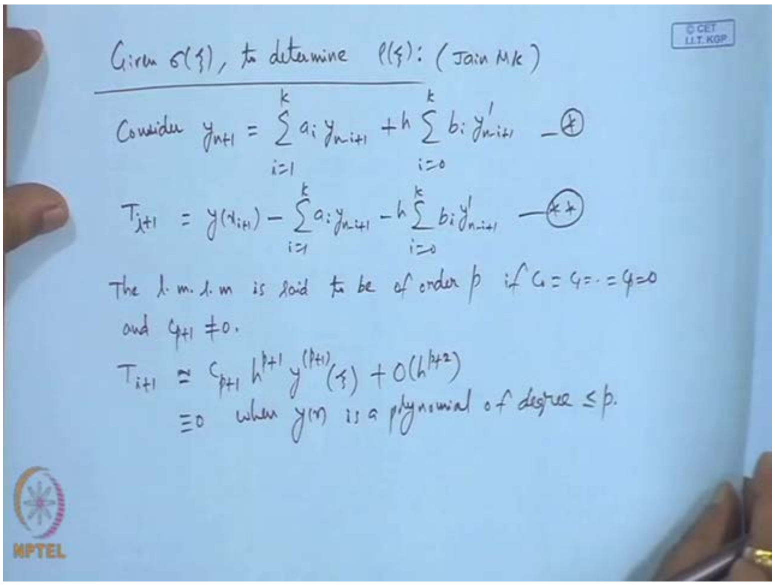

So, this is another small trick involved a h. So, let us try to do that. So, given this how do we determine, so one may also refer the numerical solutions book by Jain. So, to know lot of examples, so consider, so these are general linear multi step method case, step method, then the error. So, this is the error, then recall this linear multi step method said to be of order p if C is 0. So, this gives a p th order method. So, accordingly T i plus 1 reduces to right. So, this is identically 0 when y x is a polynomial of degree less than r equals to p, right?

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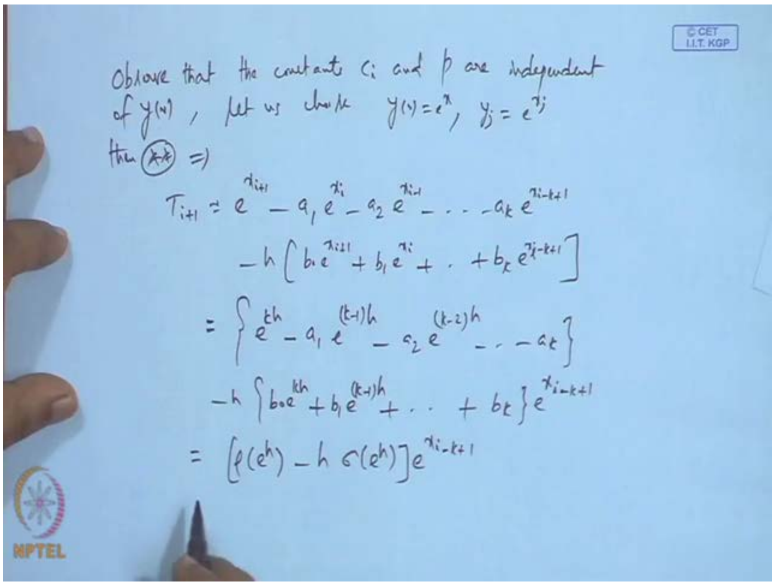

Now, observe that the constants C i and p are independent of y of x. So, let us choose y of x equals to this, then double star, what was our double star this. So, this grows like, so this is e power k h minus a 1. So, this can be written as r h o of e h minus h, so it is right. So, T i plus 1 this must be behaving like this, ok?

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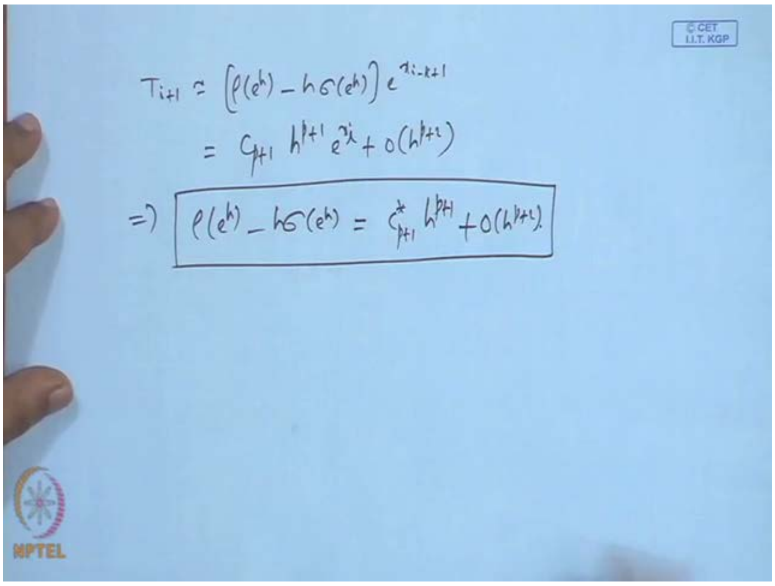

So, this implies we can get some new constant because there are some terms left out after getting cancelled x i, so that I have put it to this constant. So, this is the relation using which when r h o is given. How to determine sigma? So, this will suggest us. So, let us work out some problems, how to determine exactly for a given example using this relation in the coming lecture.

Until then thank you, bye.