Numerical Solutions of Ordinary and Partial Differential Equations

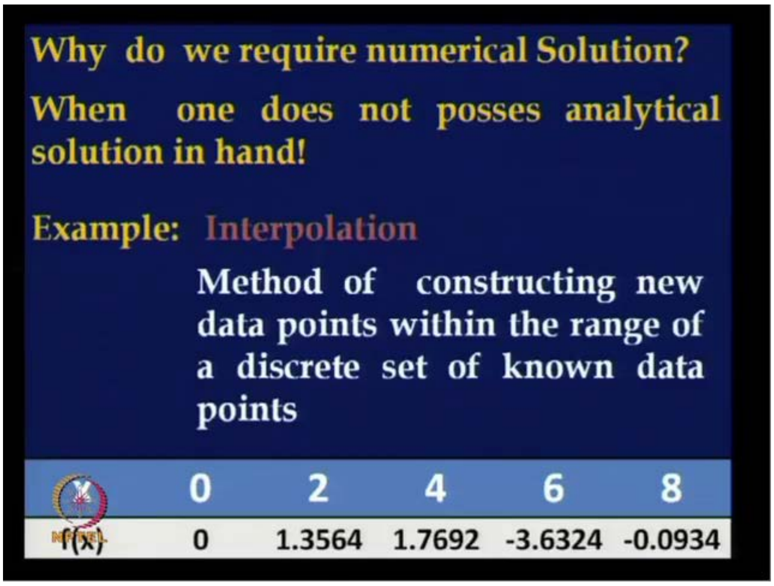
Prof. G. P. Raja Shekahar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 1
Motivation with Few Examples

Very good morning, you know the course title is numerical solutions of ordinary and partial differential equations. So, for the first lecture is I would like to give little bit of motivation with respect to some examples, so as you know numerical when we say numerical solutions of ordinary and partial deferral equations. Yeah, many times we try to take some examples and then try to solve analytically, but sometimes we may not get analytical solutions.

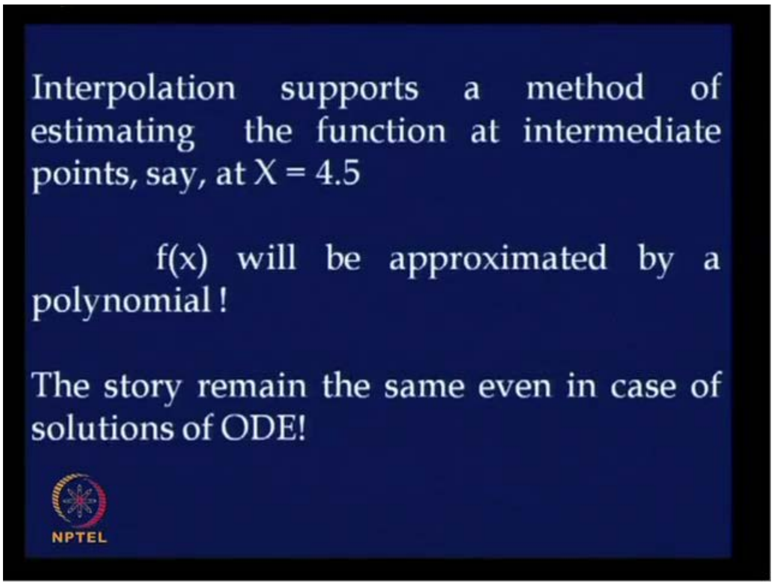
So, then we look for numerical solutions, but then unless we have regress analysis of what kind of methods we work for, what kind of problems we cannot really go for a trial and error. So, the main aim of this course is to learn different methods numerically how to solve ordinary and partial equations.

(Refer Slide Time: 01:24)



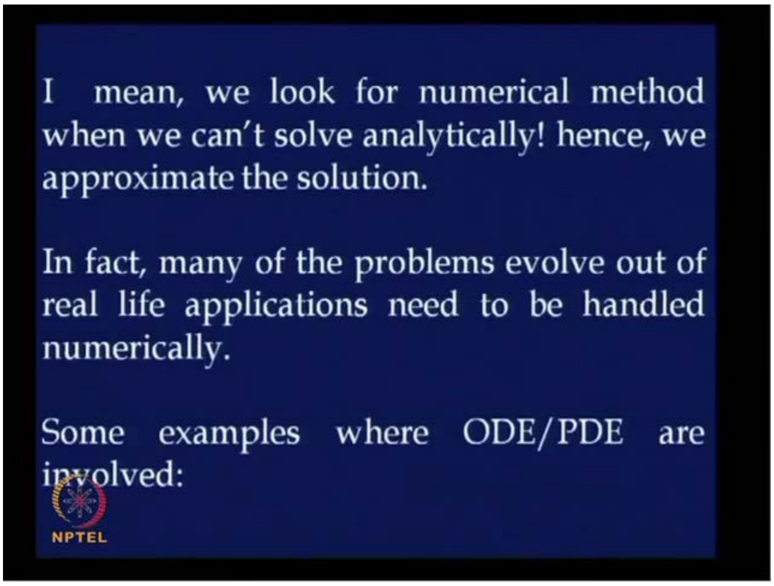
So, the first lecture is motivation. So, let us see as I mentioned why do we require numerical solution? Definitely one does not possess analytical solution in hand, so the simplest example is interpolation. So, what is interpolation? Method of constructing new data points within the range of a discrete set of known data points? So, for example, you are given x and the corresponding function and we would like to get let us say x in between say 2.5 or 4.7 or 6.8, so whatever.

(Refer Slide Time: 02:06)



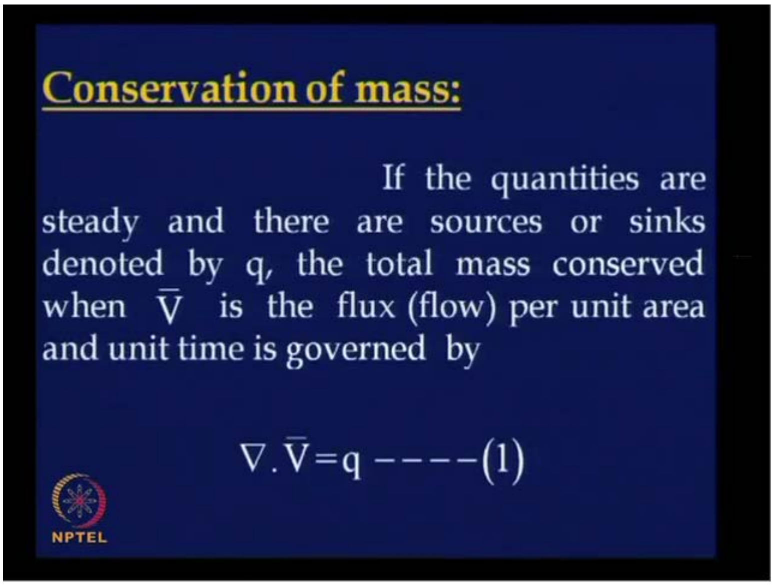
The aim of interpolation is approximate, what is that interpolation? It supports a method of estimating the function at interned points, so f of x will be approximated by a polynomial. Once you get an approximate polynomial, you can get the function value at any point, so the given data points have been approximated by a suitable polynomial. So, then where ever you want we substitute the corresponding value and get the approximate value functional value at that point. Now, more or less the story remains the same even in case of solutions of ordinary differential equations. So, when I say the story remains the same, I mean definitely we also try to approximate the given ordinary differential equations and get the approximate solution.

(Refer Slide Time: 02:59)



So, many problems come from real life applications, they have to be handled numerically because as you know real life situations are really complex. So, we have to handle them numerically, maybe there are few examples which can be handled analytically. So, I would like to explain with some examples how we arrive at an ordinary or partial differential equations, so then how we take it further to solve numerically.

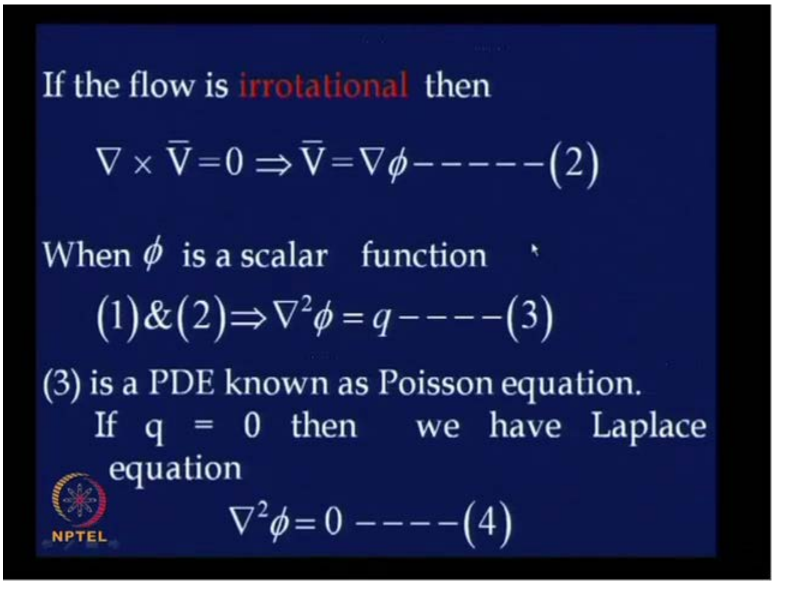
(Refer Slide Time: 03:40)



So, the first example I will consider is very mean in physics, it is very simplest kind of thing, it is conservation of mass. So, what do you mean by conservation of mass? So, you take any flow quantity which we represent by a vector field, so any flow quantity which we represent by vector field v conservation of mass states that the divergence of the vector correspond vector is 0 provided there are no sources and sinks. So, in case when you have source and sinks present, so then the divergence v is equal to the corresponding source or sinks down.

So, I have written here divergence of v equals to q, so where v is the vector field, so which represents let us say a flow velocity is flux of magnetic field etcetera and q is the corresponding source or sinks down. It depends on the special coordinates, so we are considering the study case, the simplest case. Hence, this is the form, so when there is no source and sinks q will be 0.

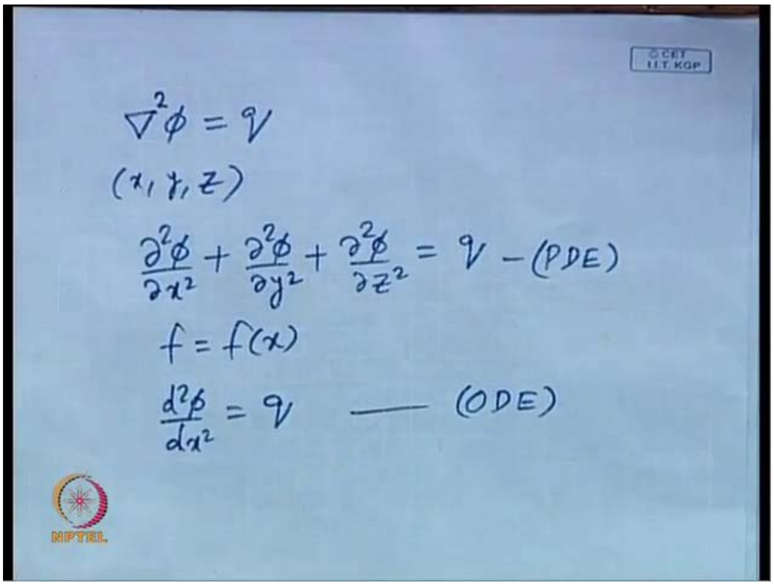
(Refer Slide Time: 05:06)



Now, let us consider another physical concept that is, when a flow is irrotational, we represent generally curl of v is 0, as you know curl of vector represent rotation. So, when curl of v is 0, the flow is called irrotational, of course I am not describing the conditions under which an irrotational field. It can be express as a gradient of scalar; there are some restrictions on the domain, so I have not coded those restrictions. Your velocity field v bar can be expressed as gradient of a scalar where phi is a scalar.

Now, when phi is a scalar function this equation is this, so you substitute v equals to grade phi in divergence v equals to q. So, then the left hand side v is Laplacean, so Laplacean of phi is equals to q, so this is a PDE which is popularly known as position equation. So, this is Poisson equation, so when q is 0, so then the corresponding homogeneous equation is called Laplace equation. So, this is the Laplace equation, now whether it is PDE for example, this particular equation three whether it is an ODE or it is a PDE depends on the coordinates.

(Refer Slide Time: 07:06)

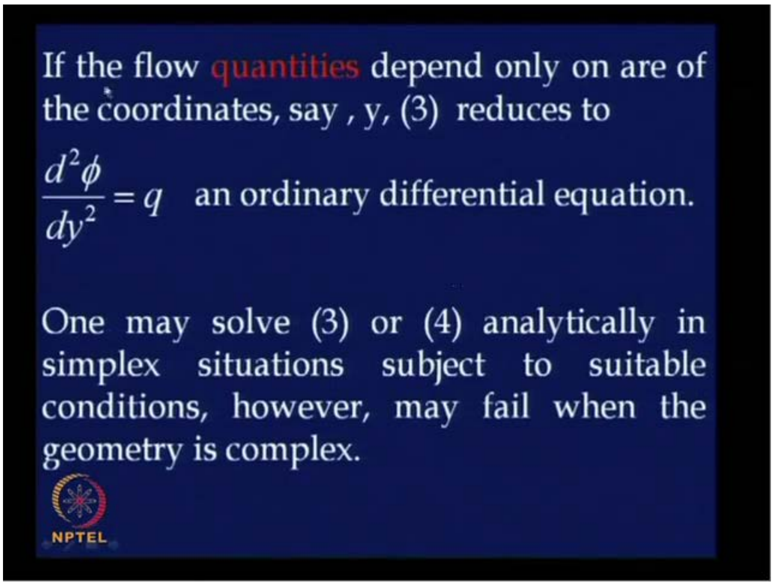


For example, let us see the Poisson equation is the Poisson equation is tau square phi equals q. Now, let us say we are talking about three dimensional, complete three dimensional coordinate system x, y, z. So, then your re proclaim becomes tau square phi by tau x square tau square phi by tau y square tau square phi by tau z square equals to q. So, this is a PDE and what kind of PDE it is a second order PDE right, so suppose your quantities depends only on one coordinates, so that means any quantity f just depends on say function of x.

Then, your PDE reduces to simply d square phi by d x square phi, so this is a PDE whereas, this is a ODE, so these are few examples that means the same physical principles that is conservation of mass which is a represented by Poisson equation here. Depending on the coordinate system for example, when it depends on three coordinates, we arrive at a PDE when it depends just one of the coordinates only then we arrive at ODE.

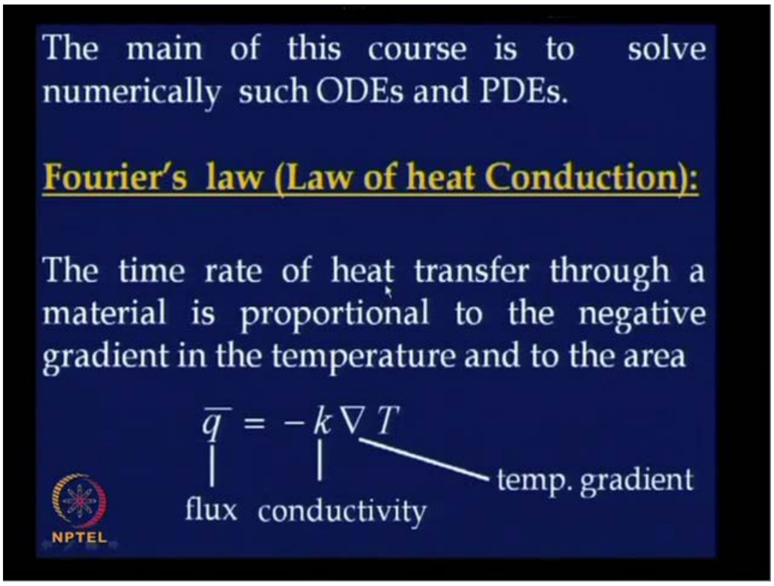
Now, if q is 0, then we have correspondingly the simplest first or second order ODE where d square phi by d x square equals to 0. So, this can be obviously solved analytically, but depending on the complexity of the physical situation, we arrive at some time corresponding equations which cannot be solved analytically.

(Refer Slide Time: 09:33)



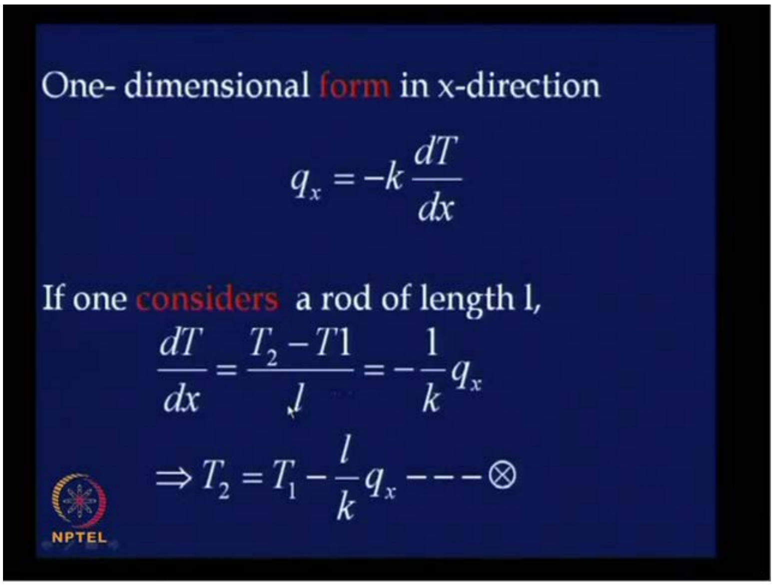
So, let us come back to the Laplace equation, so this is a Laplace equation, now as I mentioned that the flow quantities depend only on one of the coordinates. Then this reduces to an ordinary differential equation as I mentioned, when I solve this analytically in simple situations, but sometimes it is very difficult depending on the coordinates. Sometimes, depending on the geometry, the analytical solution cannot be obtained, so then we have to look for numerical solutions.

(Refer Slide Time: 10:20)



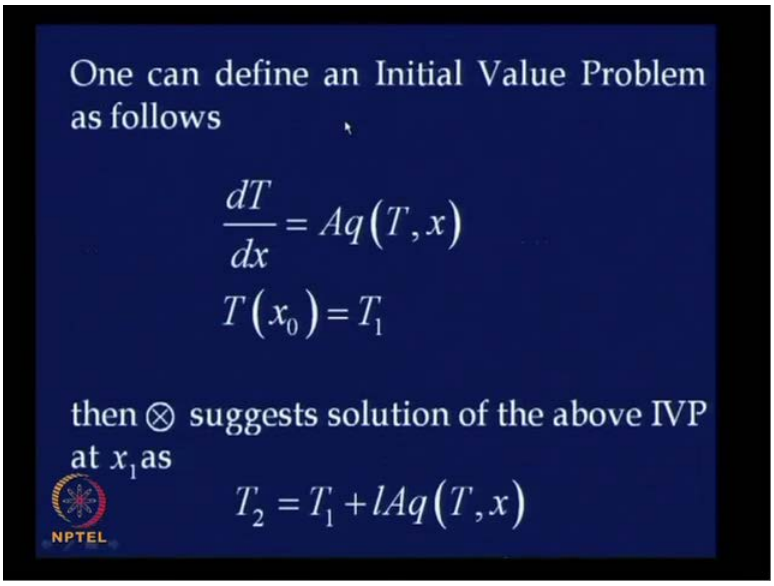
So, the main aim of the course is to learn numerical solution, now let us consider a second example, so the second example is have considered which is very popularly known as Fourier's law. So, this comes on heat conduction problems, so what is that, let us see, so Fourier's law is law of heat conduction the time rate of heat transfer through a material is proportional to a negative gradient in that temperature and to the area. So, q bar vector field which is heat flux and this is related to the heat temperature in this form. So, this is k, the thermal conductivity of the medium and that is the temperature gradient, so this is very popularly known as Fourier's law now one dimension form.

(Refer Slide Time: 11:23)



So, one dimensional form, for example in x direction we consider, so then q x is minus k dt by x q x is minus k d t by x. So, for example, if we consider a rod of length l, then the gradient can be represented like this T 2 minus T 1, this should be suffix 1. So, T 2 minus t 1 by l which is minus 1 over k q x, now if you consider only this combination, T 2 is expressed like this. So, you just make a note of it, T t 2 T 1 minus l by k q x, so we will come back to this very soon, so one can define an initial value problem as follows, so same Fourier's law d t by d x is some constant time of T x.

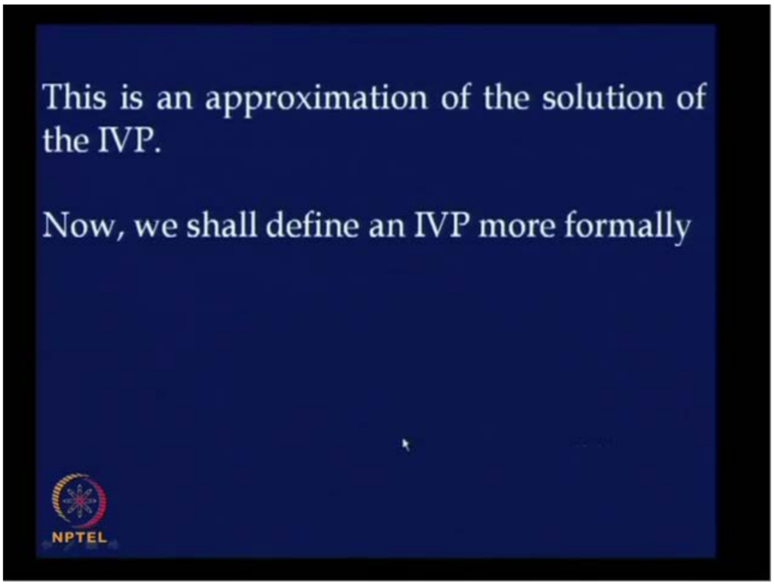
(Refer Slide Time: 12:29)



So, I have considered the simple case one dimension, so if it is more than one dimensions accordingly, then this would be radiant and this also depends on more than one dimension. Then we need at any coordinate what would be the initial temperature, so why do we require only at one point, see this is as long as q. We consider q as only function of T, then d t by d x equals to something. Assuming, this is a linear this is a first order ODE, now how many arbitrary constant exist in a solution, it is 1 because it is a first order ODE.

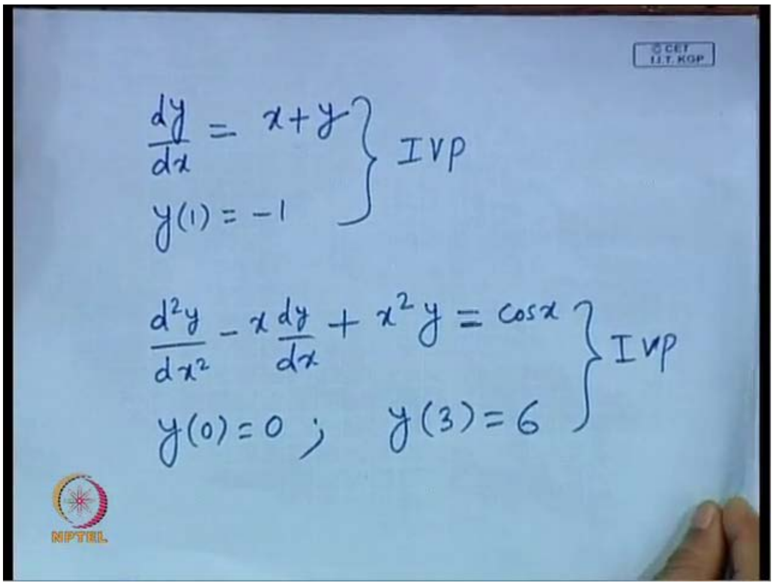
Therefore, one requires initial condition to solve this, so if we define an initial value problem like this, then the star exactly suggest solution of a kind of more or less T 2 is T 1 plus, so what is this T 1 at a known x naught. So, correspondingly T 2 at some other solution is given by this, so this is a kind of solution of the initial value problem. So, that means, if you generalize one would express T 3 is T 2. So, similarly, a general T n plus 1 is t n plus some incremental. So, this is general motivation how an initial value problem may occur and then correspond what will be the solution.

(Refer Slide Time: 14:42)



This is a kind of an approximation when we say T 2 plus T 1excreta, because the gradient has been approximated by this. Now, we are in a position to define IVP more formally, so let us look at few examples before we define more formally.

(Refer Slide Time: 15:12)

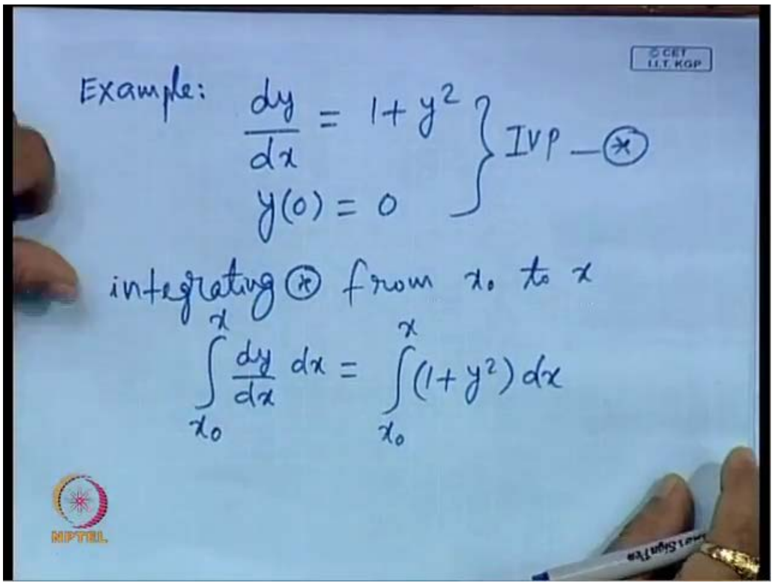


So, for example, d y by d x equals x plus y, so this is a first order ODE as I mentioned we need how many initial condition we need one, why we need one because this is a first order ODE. So, let us define your condition as y of may be 1 equals some quantity, so this can be considered as initial value problem. So, similarly, one can consider examples say minus x d y by d x plus x square y equals say cos x. So, what kind of defines like ordinary definition is this second order ordinary differential equation. Now, to solve it we need two conditions, so let us say y of 0 is 0 then y of 3 is 6.

So, this is also initial value problem, so we would like to consider such initial value problems and device numerical methods to handle them. So, before we define more formally when we say numerical method how we arrive at a numerical method what is a motivation I would like to explain with reference to simple method, which is a kind of semi analytical. So, why do I say semi analytical is not really numerical, but it is numerical, so it is definitely an interesting because any numerical method we expect a kind of iteration.

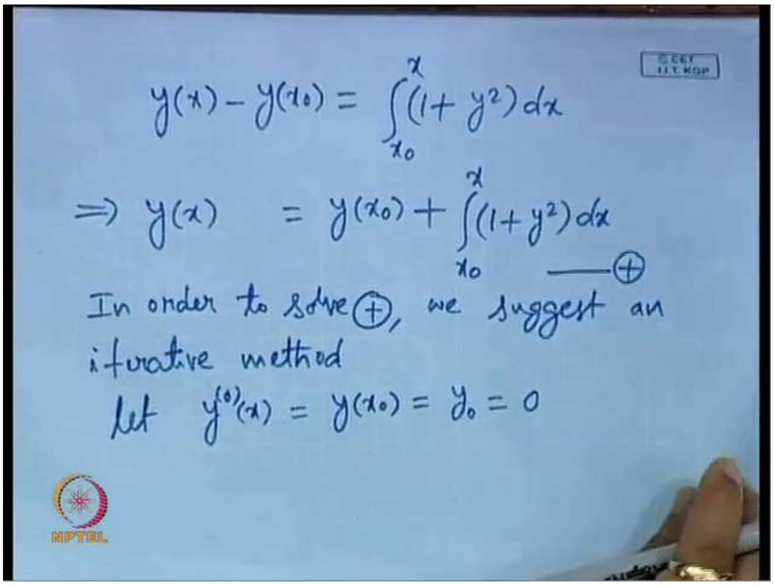
So, what do you mean by iteration, you plug in something and then get an improved version, then again you plug in that process and get improved version so on. So, for it let us consider one such semi analytical method which suggests what a numerical solution is, let us see that.

(Refer Slide Time: 18:21)



So, I will explain with respect to an example, so the example I consider is d y by d x is 1 plus y square, so as I mentioned, this is a first order ODE. So, I would need one condition, so I have taken these conditions very simple. So, this is initial value problem and first order, now when I said semi numerical, which means to some extent we proceed analytically and then switch over to numerical. So, that is the motivation you integrate, say this is star, integrating star from x naught to x, so x naught will be x d y by d x, d x equals integral x naught to x 1 plus y square d x. so this is the step which is definitely analytical.

(Refer Slide Time: 20:15)

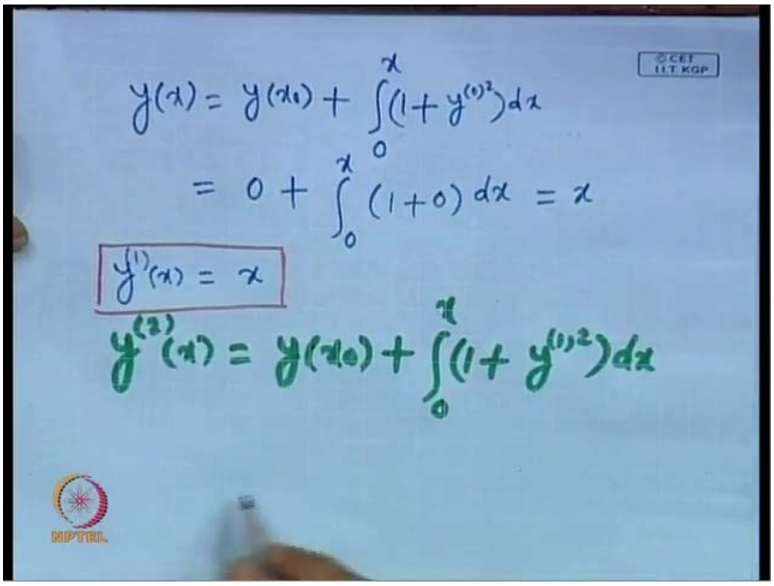


Now, let us expand this, we integrate d y by d x d x so that we get it as y of x minus y of x naught equals integral x naught x 1 plus y square d x. So, if you perform integration on left hand side we get this, so this can be written as y of x equals y of x naught plus, so now at this stage if you look at it the corresponding differently equation has been reduced. An equation which involves an integral, hence you may call this is an integral equation so that the next question is how do we solve it? Remember we have not yet made use of the initial condition, now how do you solve it if you look at it right hand side also contains why we are looking for y where as right side also contains y.

Therefore, how do we proceed further, so this is where I defined this method as a semi numerical, because now we cannot proceed further in some analytically. Therefore, we would like to bring in the numerical concept in some in the sense a kind of iterative process, so in order to solve say this is a an iterative method. So, what is the iterative method, we have our initial condition in hand which we have not yet made use of...

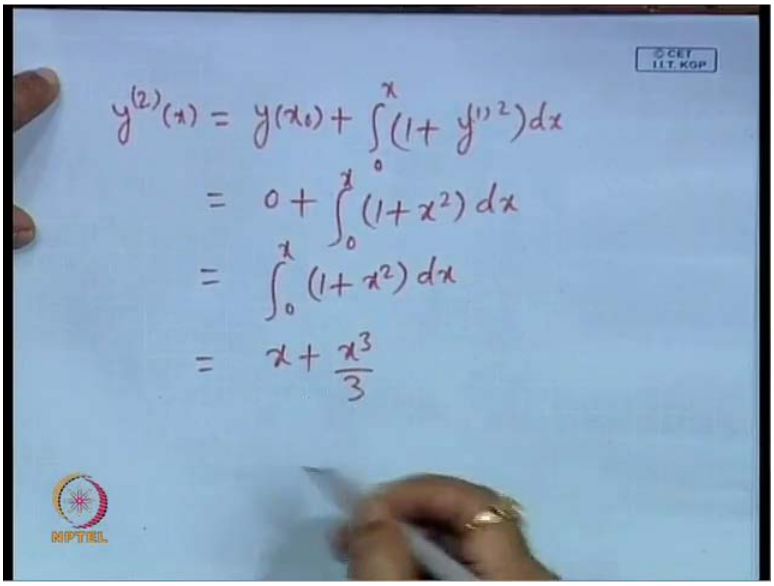
So, we would like to pick up that initial condition and see left hand side, we are trying to obtain so that should be our answer. So, whereas right hand side contains a same quantity y, therefore what is a iteration as I mention whatever approximate in hand you put it in your process and get slightly refined. So, here we have right hand side y, now we plug in our initial value as a first approximation. So, let y 0 of x equals y of x 0, so this is our initial approximation, now for this problem y 0 is defined as 0, so let us see what will happen.

(Refer Slide Time: 24:02)



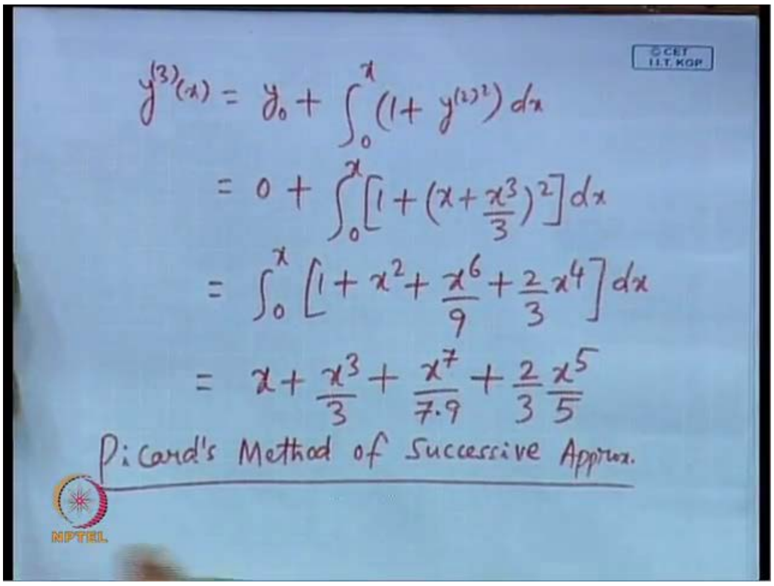
So, y of x which is y of x 0 and x 0 is 0, 0 to x 1 plus y square will be now y 0 approximation d x. Now, y of x 0 is 0 1 plus and this is 0. So, this is simply x, so we get a first approximation, therefore we denoted this, so we denote this y 1 x equals to x. Now, we expect that this is a refinement over the initially get; now we would like to make use of this and get the next approximation.

(Refer Slide Time: 25:54)



So, let us see how do we do it, so we defined y 2 x as y x 0 plus 0 to x 1 plus y 1 square, so this is what we defined. Now, what is our y 1, now we should plug in that the right hand side, so let us see y 2, x is y of x 0, x 1 plus y 1. So, this is again this is 0, 1 plus y 1 is x, so this is 1 plus, so this is x cube by 3, so this is our y 2, this is our y 2, now we would like to improve up on this.

(Refer Slide Time: 24:02)



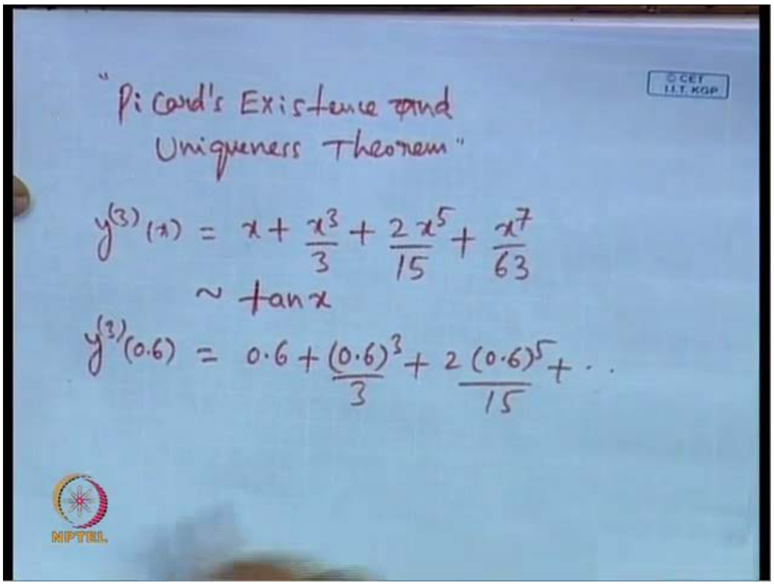
So, how do we do it, defined y 3 process to x 1 plus y 2 square, so this is 0, 1 plus y 2 is x plus, so this is 7 into 9 plus, so this what we get y 3. Now, what is story next step, so we got the expression, so why I said it is a kind of semi numerical. It is looks as if an analytical method because you are seeing only the expressions of r known numeric, but what is the motivation behind it? You get some initial approximate in hand, you plug in that improve and you get a refined version plug in get a refined version.

So, this is a kind of numerical where as the expression in hand is looking like a analytical, so what we would like to do, now suppose we need solution at a particular point. Since, we have the expression you definitely plug in whatever we have obtain and you get the solution, but how do we know that this is really exact solution of the given initial value of problem. So, there is an issue, what is that issue we have done up to three terms that means we find and we have obtained y 3, but one of your friend let us say thus it ten times. So, then what do you expect the number of terms will be more, so naturally when you compute the value at a particular point using whatever just now we have obtained.

Let us say your friend used ten terms minus ten refinements and then the person gets a huge expression and get a value using that expression, then if at all you have analytical solution in hand and try to compare. So, the solution applied by your friend appears to be more closer to the analytical solution, why is that because your friend used more number of terms. So, that means the moral is whatever we have obtained is not the complete analytical, so there is a kind of numerical process involved which is bringing in some kind of error. So, I would like to use this word for the first time, so why do we say error, it is obvious we have used only three terms.

So, then we got some value, but your friend used ten terms and got little refinement, so that means you can compare, if use three terms, the solution and then use ten terms. There is some difference, so that is contribution of the error if you stop after three terms, so you can refine and then improve. Now, is there any particular name given to this method, of course yes, so it is popularly known as Picard's method of successive approximation. Now, the next question comes definitely who would ensure that the method gives solutions which converge to the exact solution.

(Refer Slide Time: 33:10)

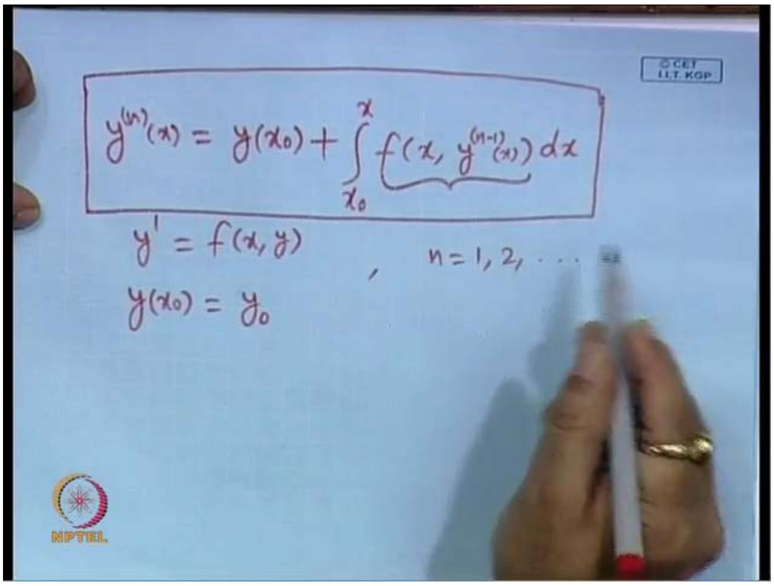


So, that is the Picard's existence and uniqueness theorem, who would ensure the convergence solution that is Picard's existence and uniqueness theorem, so I am not stating the theorem how are you can refer the standard books given in this course. So, you can refer to know what is Picard's existence and uniqueness theorem, so this theorem ensures that the solution converges. Now, what y 3 we obtained is x plus by 3 plus 2 x 5 by 15 plus x 7 by 63, so an intelligent person would try to say can we close it like this appears to be a nice series. So, can we get a close form expression at we immediately conclude that so this could be our exact solution.

Fortunately, the example which we have taken it is supporting this is tan x, so situations where the series supports a close form expression. Then we will be tempted to say this is a solution, however numerically 10 x 1, it is expanded in any computer core. Definitely, one has to use terms up to certain number and then rest will be through that is where I am defining those chances ever.

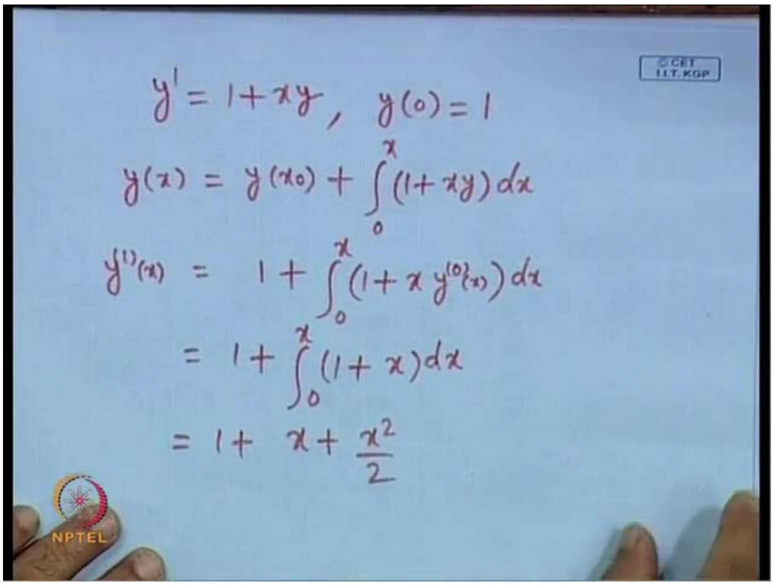
We will discuss in more detail about what do you mean by error, now in situations where we cannot get such close form obviously we have to say that this is the solution. For example, now somebody would like to get at 6. So, we substitute in the corresponding expression excreta, now are we in a position to define general yes of course we are in a position to define the general process.

(Refer Slide Time: 35:21)



This term is always y of x 0, so this is any f of x y, you consider what is this corresponds to IVP y dash equals to f of x y with initial condition. So, what is n there n is 1, 2 so on. So, this is the Picard's method of successive approximation and we ensure the convergence under Picard's existence and uniqueness theorem.

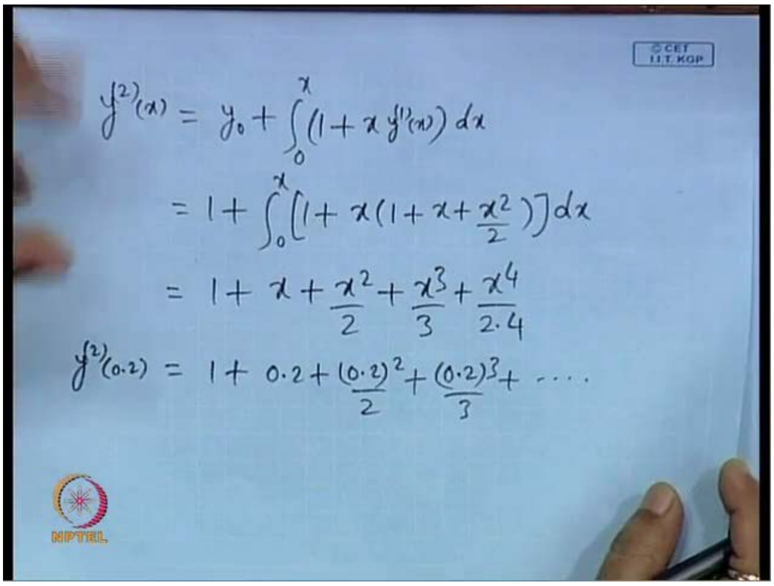
(Refer Slide Time: 36:42)



So, let us try another example, so let us say your problem is y dash equals to y dash equals to say 1 plus x y and let us say that cautions y of 0 equals to 1. So, we defined first approximation y of x is y of x naught plus integral 0 to x, so y 1 of x would be y of x

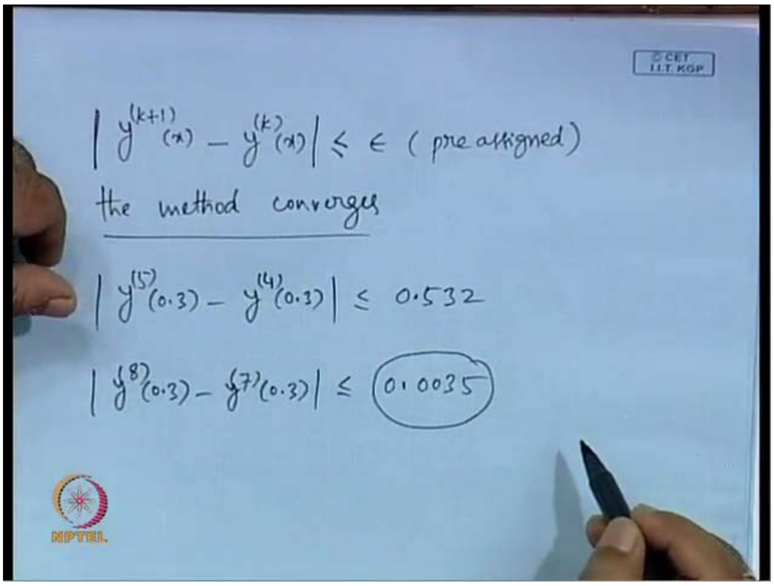
0 is 1, 0 to x 1 plus x y 0 x d x 1 plus 0 to x 1 plus x. So, it our y 0 x, we have to consider the initial condition as first approximation, therefore this is 1, so it is so plus so this is our y 1. Now, we would improve up on this what do we improve up on this, now y 2 is y 0 1 plus x y 1 x.

(Refer Slide Time: 38:20)



So, this is 1 plus x, so just we have plained y 1, so that is 1 plus x plus x square by 2, so this is x cube by 3, so this is y 2 x. Now, if one would get the simple substitutions, now the question is how long we continue, so we can refine, so as I mentioned what are the conditions under which convergence. Then when do you stop the conditions under which is converges there are restrictions on f, so I would request you to refer to the book given in the references for Picard’s existence and uniqueness. Now, with reference to this example, I would like to say when we stop, so when do we stop if we look at this contains five terms, whereas earlier one contains three terms. So, y 2 is this plus some additional terms, so naturally if you further refine if you get some additional terms so keeping this you when we stop.

(Refer Slide Time: 40:53)

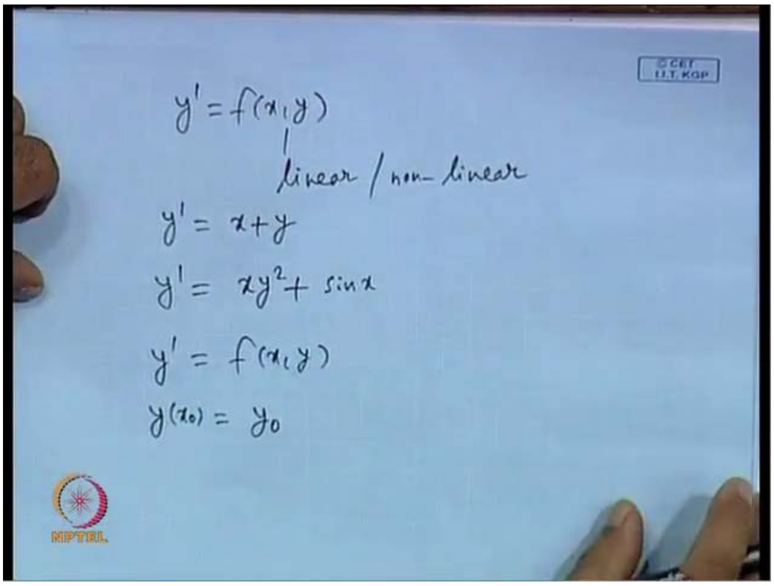


You consider at some k plus 1 stage, then you compare with the value the previous step if this is less than or equals to some excel on where excel on is to be assigned. So, then we conclude that the method converges up to over excel on pre assigned. So, for example let us say mod of y 5 of 0.3 minus y 4 of 0.3 is less than or equals to say for example, now the question is whether we stop our alteration here or we proceed. Further, somebody expect this is not a fair enough accuracy, what do you mean by this? The difference between the solution obtained fourth step and the solution obtained at fifth step.

So, the difference suppose we are not happy, so then we have to proceed further and let us say we proceeded further less than or equals to say 0.35, that means the solution obtains seventh step, eighth step. They are agreed up to two decimal places, therefore now it depends what is our requirement, so if somebody expects that I need solution up to two decimals.

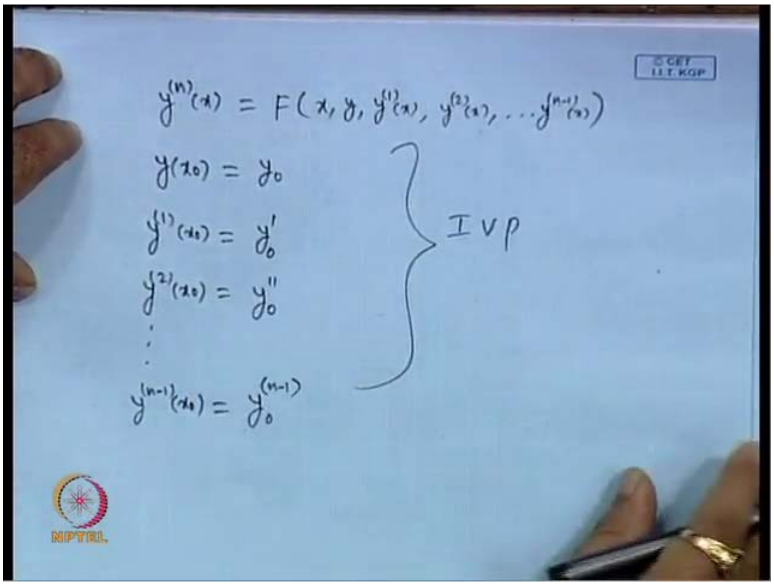
So, then we stop suppose somebody says we have proceed further, so definitely we will compute further, so this is Picard's method of successive approximations. I hope you got a clear idea of what is an iterative process and then how do we stop at the particular nitration level. So, that is sorely depends on our requirement when I said it is on this quantity, now having learnt the semi analytical method we most proceed for more numerical methods, so called numerical methods. So, before we proceed further defined formally what is a initial value problem, let us define more formally what is an initial value problem, so the simplest is y dash equals to f of x y.

(Refer Slide Time: 44:19)



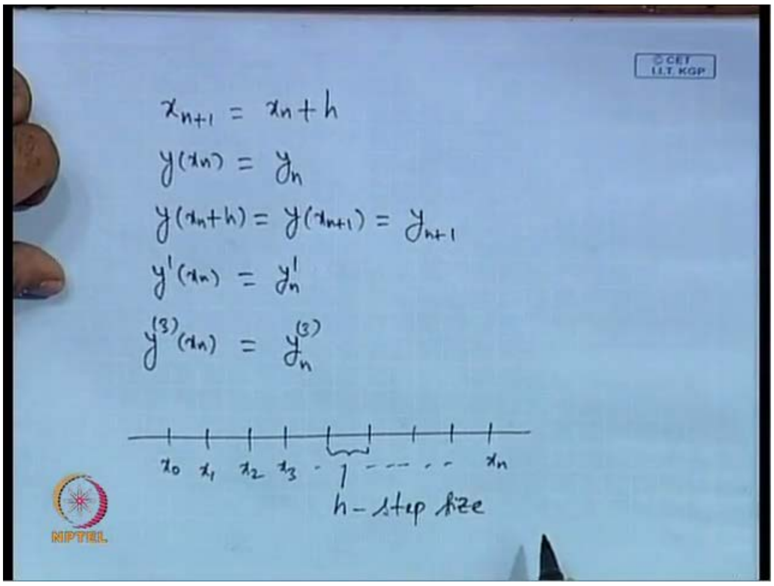
So, here f can be linear, non-linear, so example y dash equals to x plus y dash equals to x y square plus sin x, as this kind as I mentioned this is a first order equation, therefore we defined more general. So, for we had been discussing only first order IVP, so can we generalize it, yes of course we generalized.

(Refer Slide Time: 45:23)



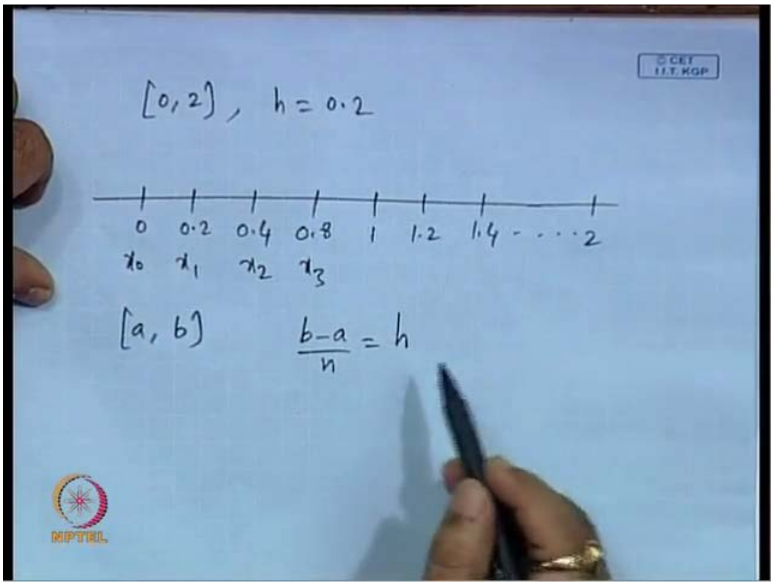
For example, you defined x, y, y 1, y 2, n x, so that means nth derivative IVP, it is expecting up to n minus 1, so right hand side is processor which expect these are the values. So, first consider y, y at x naught is given than y 1 at x naught still y 1 next y 2 still at x naught this is like this. So, what is the remark, all the values are defined at only one point, all the values if you see there been defined at only one point. So, this problem is called initial value problem, the reason is all the values have been defined at only one point.

(Refer Slide Time: 47:38)



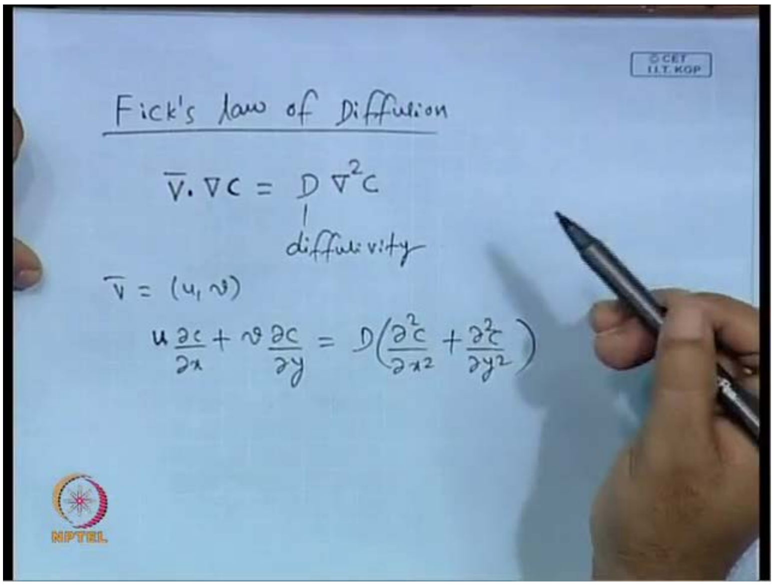
So, I must explain this notation little bit, so let us understand that the notation is x n plus 1 is x n plus h x n plus h, so y of x n is y n y of x n plus h is y of x n plus 1. This is y n plus 1, and then y dash the x n is y n prime, therefore y 3 of x n is this is at x n. Now, when we solve a numerical solutions, when we solve when we IVP numerically, we try to get the solution at particular points, therefore more formal I will give you more formal definition little later. So, we try to get at a particular point, therefore our accesses, suppose this is x 0 like interpolation we divide into parts. So, this is x 1, this is x 2, x 3 and x n, now each this is our h step size this is the step size.

(Refer Slide Time: 49:38)



So, for example we are given interval is 0 to 2 and say h is 0, 0.2, 0.4, 1, 1.2 so on, so this is our x 0 x 1 x 2 x 3 excreta. So, in general if a, b is an interval, so then if you make it into n points, so then b minus a by n is h. So, with this modulation, we try to solve numerical methods, so I would like to define more formulae IVP in my next lecture. So, let us see some more physical examples where we come across ODE or p s. So, one is conservational mass and then law we have discussed.

(Refer Slide Time: 50:57)



Now, I would like to mention another example, Fick's law of diffusion, so this is, if v is any flow quantity say velocity, then v dot grad c where c is the n mass transfer concentration where c is the concentration and d is the diffusivity. So, v dot grad c is d trance square c, so this is conservation of mass, so for example if you consider in two dimensions, so say v bar is u v, then we have u tau c by tau x plus v tau c by tau y equals d. So, this is PDE second order PDE for c, however one must know v for given u and v this defines a second order PDE and one can solve.

So, we discuss various methods to solve these second order methods as well, so first we concentrate on ordinary differential equations. Then we switch over to partial differential equations, so we have learnt couple of physical examples, conservation mass when conservation of heat conduction equation. Then conservation of any concentration, so in next lecture, we defined mode formulae what is an initial value problem and what is a boundary value problem and then start with at least one or two methods.

Thank you.