Advanced Engineering Mathematics
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Lecture No. # 09
Orthogonality,
Gram-Schmidt Orthogonalization Process

In this lecture, we shall discuss about orthogonality and Gram-Schmidt orthogonalization process. Here we shall discuss about orthogonality of vectors in an inner product space. Recall that in the previous lecture, we have generalized the distance or length concept of vectors that is, we have defined that length of a vector and distance between two vectors.

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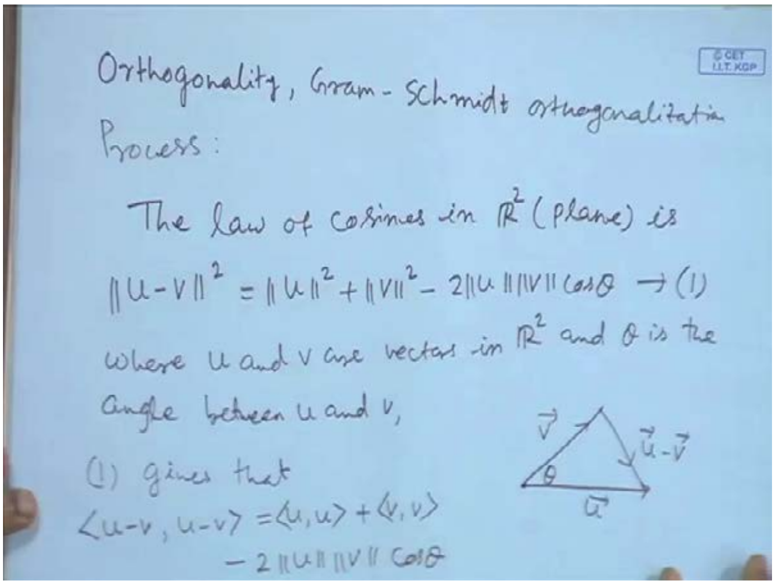

So, here in this lecture we shall discuss about that angle between vectors or that is orthogonality. So, this orthogonality concept is motivated by the law of cosines in plane R 2. So, let us see this. So, here we shall discuss about this orthogonality of vectors and this Gram-Schmidt orthogonalization process orthogonalization process. So, this orthogonalization concept has been motivated by the law of cosines in R 2. So the law of cosines in R 2 or in a plane Euclidean plane is given by this u minus v norm square is equal to norm of u square that square of norm of v minus twice norm of u norm of v cos theta, say this be equation one. Here the vectors u and v, where vectors in R 2 u and v are vectors in R 2, and theta is the angle between u and v.

So this norm we have considered with respect to the inner product in R 2. So the vectors u and v are like this, that if this is the vector u and this is the vector v, then this u minus v will be this vector, or in other words and this angle theta is angle between u and v, or in other words this is this a three sides of a triangle and this cosine law that holds for this triangle. So this one, if we write their in terms of inner product, then this one gives that one gives that norms square of u minus v this vector that is equal to inner product of u minus v with itself and right hand side is inner product of u with itself plus inner product of v with itself minus 2 norm of u norm of v cos theta.

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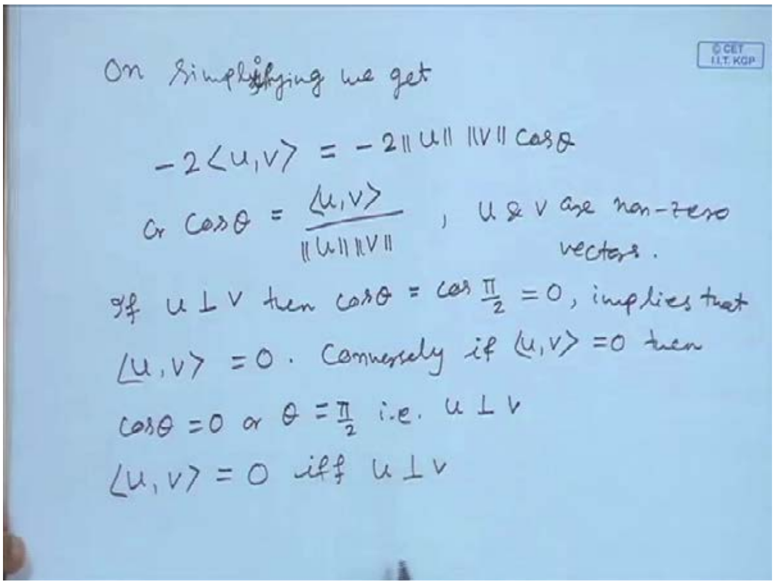

So, on simplifying this we get on simplifying on simplifying we get minus 2 times inner product of u and v that is equal to minus 2 times norm of u norm of v cos theta or cos-theta is equal to inner product of u and v divided by norm of u times norm of v. Of course, here u and v are non zero vectors; that u and v are non zero vectors. So, this says that, if u and v are orthogonal that is if u and v are orthogonal or u is perpendicular to v, then we get then value of this costheta that is cos pie by 2 and this is equal to 0. So, this implies that this implies that right hand side this inner product of u, v is equal to u and v is equal to0. And conversely, conversely if this inner product of u and v is equal to0, then this value of cos theta will be equal to 0 or theta is equal to pie by 2 that is, u is orthogonal to v.

So, this says that if u and v are orthogonal to each other, then this inner product is equal to 0 and the converge is also equal to 0. So, it says that. So, on summarizing we can write this that u inner product of u and v that is equal to 0, if and only if u is orthogonal to v, so motivated by this concept that one defines this orthogonality for vectors in an inner product space.

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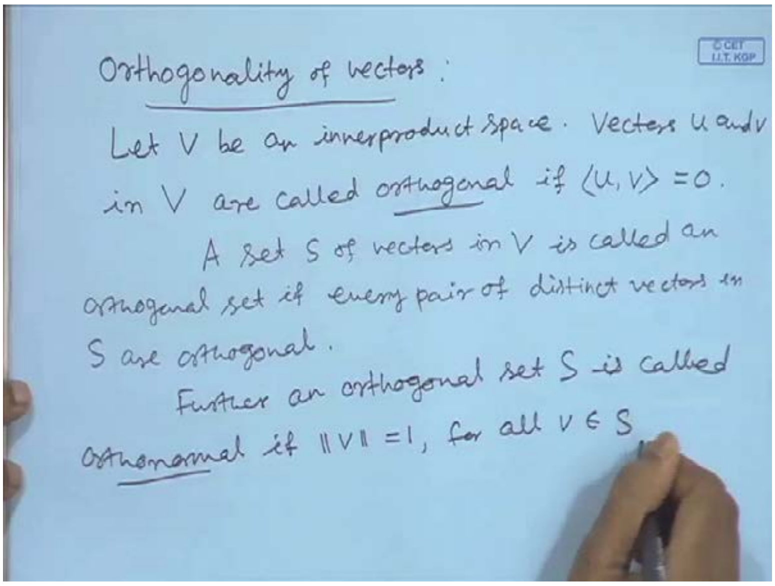

So, here we shall define this orthogonality of vectors orthogonality of vectors in an inner product space. So, let us take that V be let V be an inner product space be an inner product space, then vectors vectors u and v in V in this inner product space are called orthogonal, if this inner product of u and v is equal to 0. We also define that orthogonality of a set of vectors. So, we say that a set S of vectors in V is called an orthogonal set is called an orthogonal set, if every pair of vectors every pair of distinct vectors pair of pair of distinct vectors in S are orthogonal. Further we say that a orthogonal set is orthonormal, if it satisfiy the condition. Further an orthogonal set an orthogonal set S is called orthonormal, if norm of every vector is equal to 1; in this set is, if norm of v is equal to 1 for all vectors v in S.

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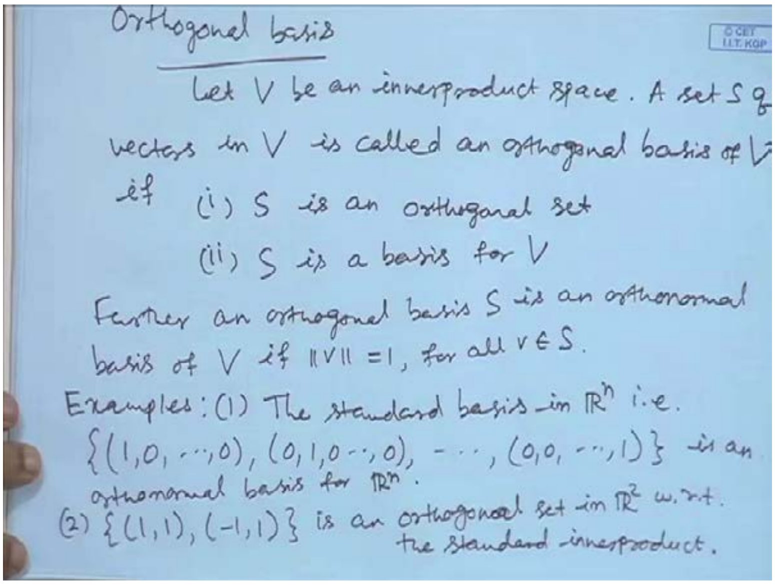

So, next we shall discuss about an orthogonal basis. So, earlier in case of vector space we have seen a basis. So now, we shall define an orthogonal basis. So, this orthogonal basis; here we say, again we consider an inner product space. So, let V be an inner product space. A set S of vectors in V is called an orthogonal basis orthogonal basis of V, if the following conditions hold: That first condition is that, S is an orthogonal set S is an orthogonal set and second condition is that, S is a basis for V that is a set of vectors in an inner product product space is an orthogonal basis, if it is basis and every pair of distinct vertex in S are orthogonal. So, further we say that S is an orthogonal basis, if that norm of every vector in is is equal to one.

So, further an orthogonal basis an orthogonal basis S is an orthonormal orthonormal basis of V, if norm of every vector is equal to 1. So, let us see few examples of orthogonal basis. So, here first example is that trivial one. The standard basis in R(n), the standard basis in R(n) that is this set that, 1 0 0, 0 1 0 0, and this 0 0 1. This is an orthonormal basis this is an orthonormal basis for R(n). So, second example is like this; here we shall see that this set consisting of vector 1 1 minus 1 1 is an orthogonal set is an orthogonal set orthogonal set in R 2 with respect to the standard inner product with respect to the standard inner product. So, that we shall see that if we change this inner product and take different one, then this vectors need not be orthogonal. So, it is of course, important to say that vectors are orthogonal with respect to each basis that with respect to each inner product that mentioning the inner product is also important.

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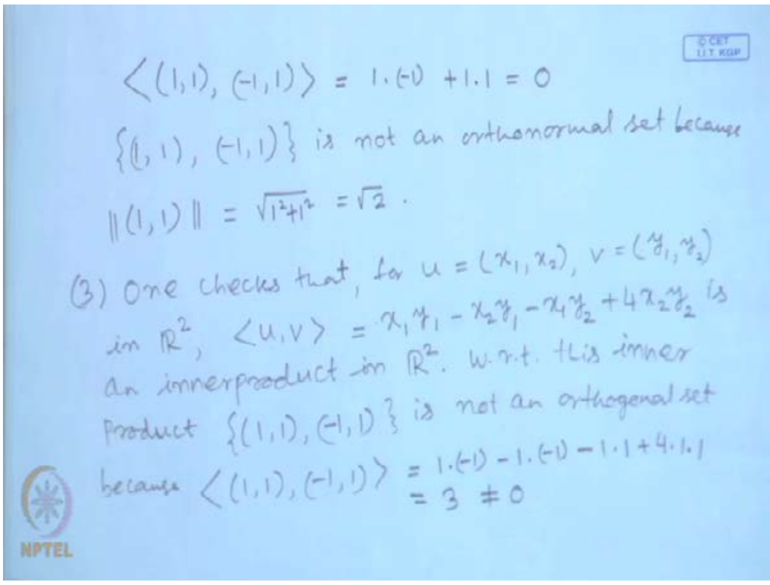

So, take this inner product of this two vectors 1, 1 and minus 1, 1, and this we get 1 into minus 1 plus 1 in to 1, and that is equal to 0. So, this implies that this two vectors are orthogonal. But this set is not an orthonormal set. This 1, 1, minus 1, 1 is not an orthonormal set because that norm of this vector 1, 1 that is equal to the positive square root of inner product of this vector with itself, and that is equal to square root of 2. So, next we have another example that is, we consider an different kind of inner product in R 2. So, one checks that this mapping checks that this checks that for the vectors u, say that is x 1, x 2 and v is vector y 1,y 2 in R 2.

This mapping or this function, that u, v defined like this: x 1 y 1 minus x 2 y 1 minus x 1 y 2 plus 4 x 2 y 2 is an inner product is an inner product in R 2. So, with respect to this inner product the above set is not an orthogonal set. With respect to this inner product this inner product, this set consisting of vectors 1, 1, minus 1, 1 is not an orthogonal set, because here this inner product of this two vectors is given by; here we get this 1 into minus 1 minus 1 into minus 1 minus 1 into1 plus 4 into 1 into 1. And here we one can see that its value is equal to 3 that is not equal to 0. So, therefore, this two vectors 1, 1 and minus 1,1 is not an orthogonal set with respect to this inner product here we have defined. So, when we say that two vectors are orthogonal with respect to which inner product we are saying that is important. In some inner product vectors may be orthogonal but with respect to some other inner product they may not be orthogonal.

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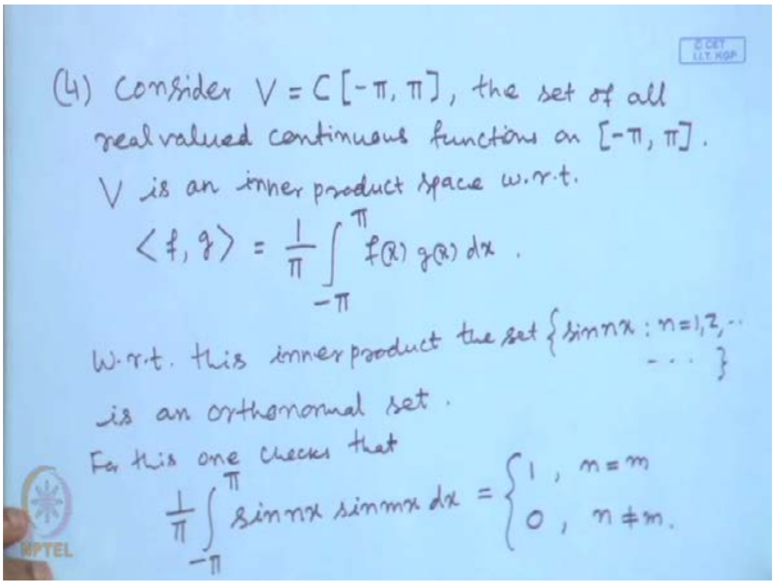

So, next we shall see another example, that fourth example. Here we consider the inner product space consider V be the inner product space that is consists of the set of all real valued continuous functions on this interval minus 1, 1. So, here this is the set of all real valued continuous functions continuous functions on this interval minus pie to pie. So, V is an inner product space with respect to the inner product to this inner product inner product of f and g is equal to 1 open pie integral minus pie to pie f(x) g(x) and dx. So, with respect to this inner product we shall show that with respect to this inner product the set of all functions the set that is, sin n(x); n from one two upto this infinity. This is an orthonormal set.

So, to check this that this set of functions sin n(x); n from 1, 2, 3 up to infinity; this is an orthonormal set. One should prove that, for this one checks that this integral value of the integral, one open pie integral minus pie to pie sin n(x) into sin m(x) dx; this value is equal to 1, if n is equal to m, and this equal to 0 for n not equal to m.

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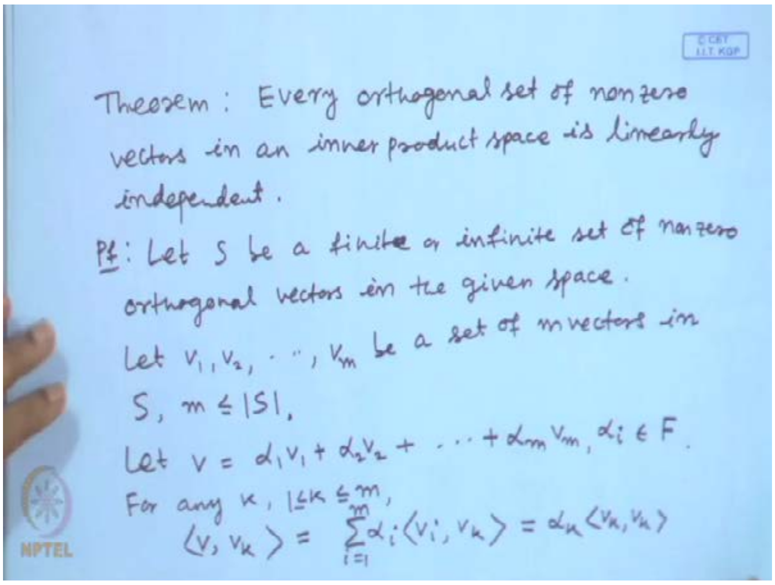

So, the next we shall see another important property of orthogonal vectors or a set in orthogonal set of vectors orthogonal set of vectors is that they are linearly independent. So, this is an important result of this orthogonality that every orthogonal set every orthogonal set of non zero vectors in an inner product space is linearly independent is linearly independent. So, here this set of vectors that consisting of orthogonal vectors may be finite or infinite. But in any case this will be linearly independent set. So, let us consider. So, let S be a finite finite or infinite set of non-zero. Here we are considering considering a non-zero orthogonal vectors in the given space. While here we are considering non-zero orthogonal vectors, one thing can be noticed that the vector zero zero vector is orthogonal to vectors and whenever this zero vectors belong to a set, that set cannot be linearly independent. Therefore, we are considering a non-zero set of vectors. Well to prove that this S is a linearly independent set; here we consider a set of m vectors.

Let v 1, v 2 to v m be a set of m vectors in S. Of course, this m is less than or equal to cardinality of S, and the vectors v 1 v 2 to v m are all distinct. So, let V be the linear combination that alpha 1 v 1 plus alpha 2 v 2 plus alpha m v m, where this alpha i's that belong to their field or their scalars. So next for any k, for any k k lies in between 1 to m. We can consider this inner product of v with v k, and this inner product is from the property of the inner product, we can see that this is equal to sum of this summation alpha i v i v k, inner product of v i v k; i from 1 to m. And since this v 1 v 2 to v n are orthogonal vectors. So, for i not equal to k this inner product is equal to 0. So, therefore, we get this is equal to alpha k times this inner product of v k with itself. This follows from orthogonality of the vectors v 1, v 2 to v n.

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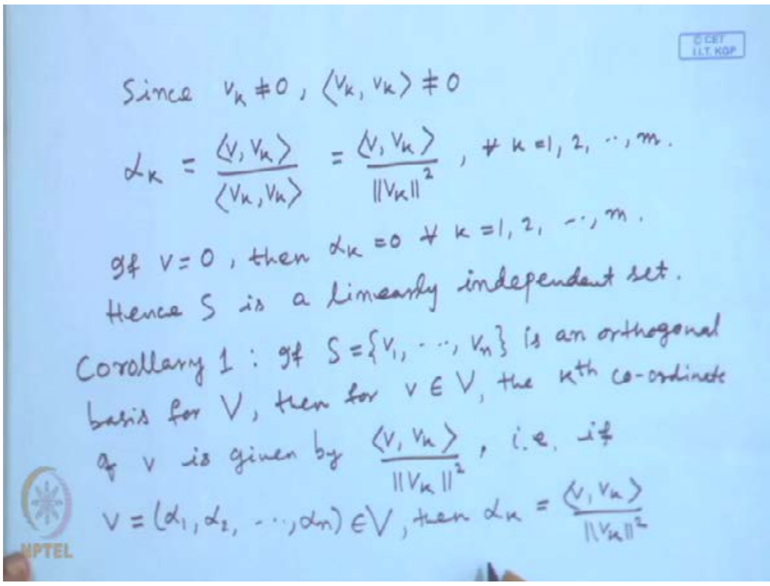

So, then, since v k is not equal to zero. Since this v k is a non zero vector, inner product of v k with itself that is also not equal to zero. Therefore, we can have this value of alpha k can be written as inner product of v and v k divided by this inner product v k with itself, or in other words, this equal to inner product of v and v k divided by norm square of this vector v k. So, this is true for all k from 1, 2 to m. So, therefore, if v is equal to zero, then alpha k is equal to zero; for all k from 1, 2 to m. Hence this S is a linearly independent set linearly independent set.

So, here we have some consequences some obvious consequences that first corollary that we have is; If this set S it is consists of set v 1, v 2 up to v n is an orthogonal basis is an orthogonal basis for this inner product space V, then it is easy to determine the coordinate of every vector in this inner product space. Then for every vector for every vector v belongs to V; the coordinates the k eth coordinate of v is given by this inner product of v with v k divided by v k norms square that is, the same thing we can write that is, if this v is equal to say alpha 1, alpha 2 to alpha n belongs to this inner product space, then this k eth coordinate alpha k is given by inner product of v and v k divided by v k norms square. So, in an inner product space whenever we have an orthogonal basis, then the coordinates can be determined by this formula.

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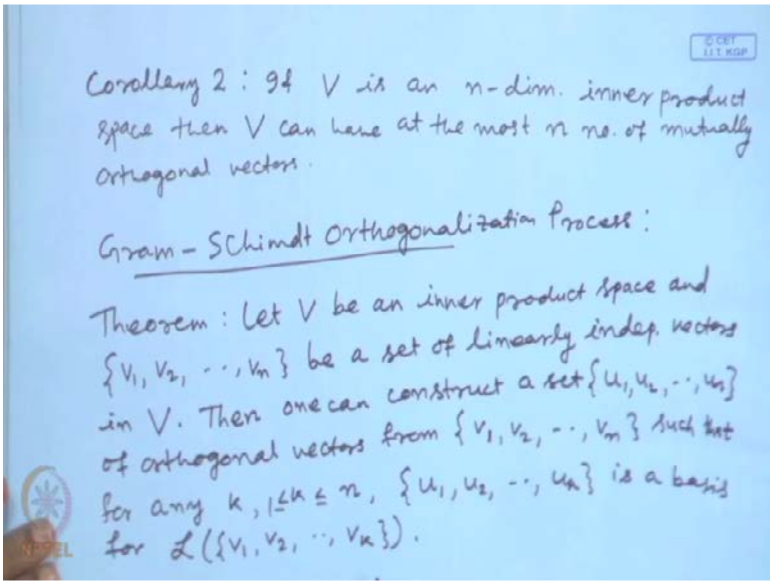

So the next we will have another consequences, that is corollary two. So this says that if we have a finite terminal inner product space and if the dimension of the inner product space be n, then at the most how many orthogonal vectors we can have. So, the answer is obviously at the most n. So, if V is an n-dimensional inner product space, n-dimensional inner product space then V can have at the most n number of orthogonal vectors, V can have at the most n number of mutually orthogonal vectors. So, this is obvious from the theorem that is, we have more than n mutually orthogonal vectors, then they are they have to be linearly independent and that contradicts to the dimension of this vector space. Next we shall see another important result of this inner product spaces is that, given any set of linearly independent vectors we can construct orthogonal set of vectors from those given linearly independent vectors, and that process is called Gram-Schmidt orthogonalization process.

So, here this method is called Gram-Schmidt orthogonalization process. So, this result we write as a theorem. So, this theorem is like this: we consider an inner product space V. Let v be an inner product space, and v 1, v 2 to v n be a set of linearly independent vectors linearly independent vectors in V, then from this linearly independent vectors we can construct a set of orthogonal vectors like this. Then one can construct one can construct a set u 1, u 2 up to this u n of orthogonal vectors from this v 1, v 2, v n, such that for any k; for any k k lies in-between 1 and n, this state of vector u 1, u 2 up to u k is a basis for this span of that is, linearly span of the vectors v 1, v 2 up to v k, or in other words this vectors u 1, u 2 u k, they depends on the vectors v 1, v 2 to v k, this is true for any k; for 1, 2 to n.

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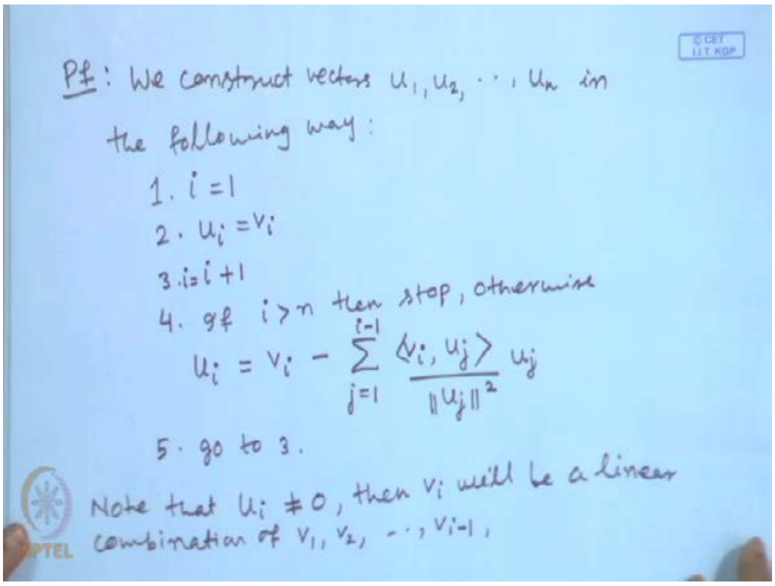

So, here while proving this theorem we use this Gram-Schmidt orthogonalization process. So, we construct the vectors u 1 we construct vectors u 1, u 2 to u n in the following way. So, that we write in this step wise. A first step is that, we consider this i is equal to 1; and second step we take this u i is equal to say, v i; and third step, we incase this i is equal to i plus 1, and then we check that if i is greater n than stop, otherwise we shall construct the vector u i in this way; this u i is equal to v i minus summation this inner product of v i with u j divided by norm u j square multiplied by this vector u j, and j runs from 1 to i minus 1. So, we continue this process that fifth step, then go to third step. So, continuing this process, that we can construct the vectors u 1, u 2 to u n. And this method of constructions is called Gram-Schmidt orthogonalization process. This is so called because the vector u i u 1, u 2 to u n will be an orthogonal set.

So, to proof this set u 1, u 2 to u n is an orthogonal set, first notice that this u i's are not 0 note that this u i is not equal to 0, because otherwise, v i will be a linear combinations of if u i equal to 0, then v i can be expressed as a linear combinations of vectors u 1, u 2 up to u i minus 1, and this u i u 1 up to u i minus 1 are expressed in terms of the vectors in terms of v 1, v 2 to v i minus 1. So, therefore, this if u i is equal to 0, then if v i will be a linear combinations of linear combinations of vectors v 1, v 2 up to v i minus 1, and this is not true, because this set is a linearly independent set; v 1, v 2 to v i is the linearly independent set. So, this not true, and hence each of this u i is a non vector.

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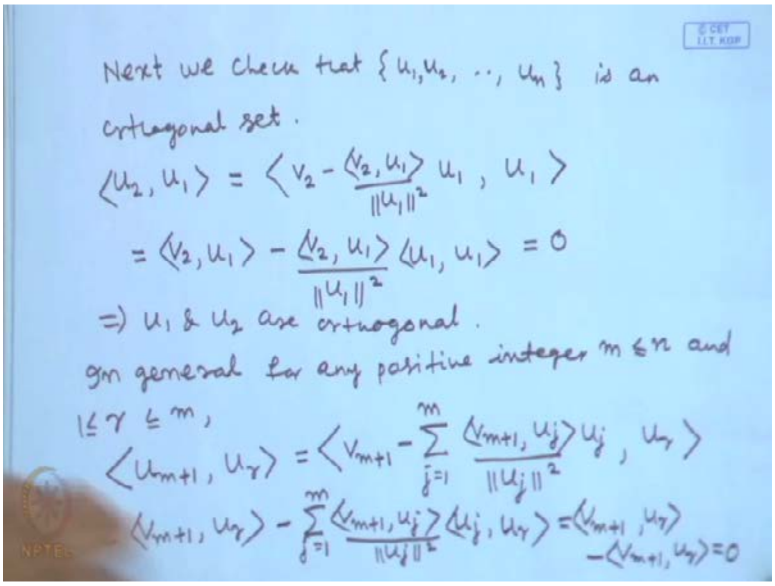

So, next we shall check that this u 1, u 2 next we shall check that this u 1, u 2 up to u n is an orthogonal set. So, for this for example, let us check this inner product u 2 with u 1. So, this u 2 is given by v 2 minus v 2 inner product of v 2 and u 1 divided by norm u 1 square multiplied by u 1 and u 1. So, on simplifying we get this inner product of v 2 with u 1 minus v 2 inner product of v 2 u 1 divided by norm u 1 square and inner product of u 1 with itself, and this is equal to 0. So, this u 1 and u 2 are u 1 and u 2 are orthogon. So, in general for any positive integer in general, we can verify this in general for any positive integer n positive integer n less than or equal to n, and this r greater than or equal to 1; less than or equal to m, we have this inner product of u m plus 1 u r.

So, this can be written as u m plus 1 can be expressed as from this construction it is equal to v m plus 1 minus summation inner product of v m plus 1 with u j divided by u j norms square times u j; j from 1 to m, and inner product with u r. So, here this on simplifying we get that inner product of v m plus 1 with u r minus summation j equal to from equal to from 1 to m; v m plus 1, u j inner product of v m plus 1, u j divided by u j norms square is equal, and this inner product u j with u r, and this will be equal to Again from orthogonality that this set of vectors; this be this we can get, because it is true for all this lower values. We get this inner product of v m plus one u r and minus this v m plus 1 u r. Because this we get v m plus 1 u r, and this is equal to 0. So, this shows that this vector u 1 u 2 to u n is an orthogonal set.

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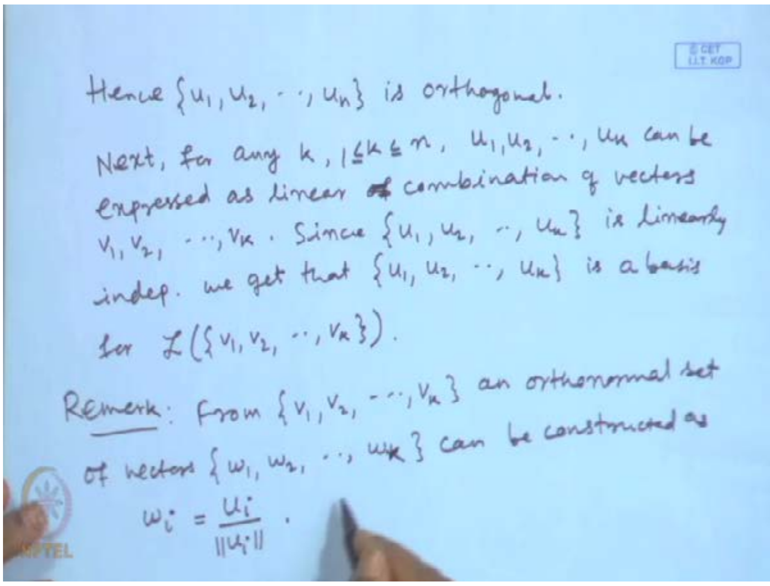

So, this how we construct that this set u 1, u 2 u n is orthogonal. Then here we have to prove the last part of this theorem that for any k the set. Next, for any k with k lies from 1, 2 to n; this u 1, u 2 to u n is a basis for this span v 1, v 2 to v k because this u 1, u 2 to u k can be expressed as a linear combinations of v 1, v 2 to v k. And since, this a linear that for any case, here this vectors u 1, u 2 to u k can be expressed can be expressed from this construction only construction of u 1, u 2 to u k can be expressed as linear combinations of can be expressed as linear combinations of vectors v 1, v 2 to v k, and since this is linearly independent set u 1, u 2 to u k is linearly independent linearly independent, we get that this set is a basis u 1, u 2 to u k is a basis for span of this vectors v 1, v 2 to v k.

So, here we can have one remark that from a given set of linearly independent vectors, not only we can construct orthogonal set of vectors, In fact, we can construct an orthonormal orthonormal set of vectors. So, here from this set of from this set of vectors from this set of vectors v 1, v 2 to v k and orthonormal set of vectors set w 1, w 2 to w k can be constructed as w i can be taken as u i divided by the norm of u i. So, now this norm of all this vectors w 1, w 2 to w k will be equal to 1 and they are also orthogonal. So, this will be Orthonormal set, so this suggest that every finite dimensional inner product spaces orthonormal basis.

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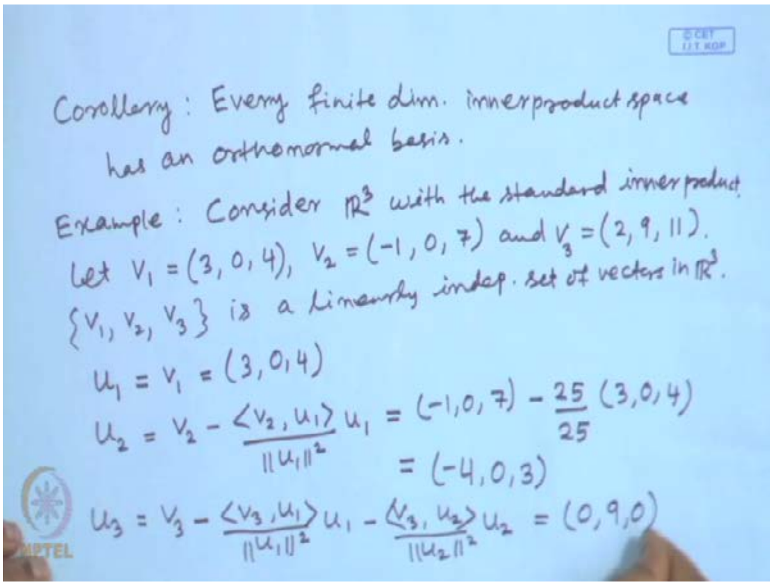

From here we get result that every finite dimensional inner product spaces inner product spaces has an orthonormal basis. So, this we can construct from this Gram-Schmidt orthogonalization process. And then from this remark, we can get orthonormal vectors So, let us see one example that applying Gram-Schmidt Orthogonalization process; how to construct orthogonal vectors from a given set of linearly independent vectors. So, here we consider R 3 with standard with standard basis with standard inner product with standard inner product. And vectors be like this: Let v 1 be the vector in R 3; 3 0 4, then v 2 be the vector minus 1 0 7 and v 3 be the vector 2 9 11. So, easily one can see that v 1, v 2, v 3 is a linearly independent is linearly independent set of vectors in R 3. So, here we shall construct the corresponding orthogonal set of vectors u 1, u 2 to u 3.

So, applying Gram-Schmidt orthogonalization process we first take this u 1 be the vector v 1. So, that is 3 0 4, and u 2 will be v 2 minus inner product of v two with u 1 divided by norm u 1 square multiplied by the u 1. So, this is equal to minus 1 0 7 minus this inner product can be calculated, and it is equal to 25, and norm of u 1 square is also equal to 25 and u 1 is 3 0 4. On simplifying we get this u 2 is equal to minus 4 0 3. Similarly, this vector u 3 can be calculated from this Gram-Schmidt orthogonalization process like this is equal to v 3 minus inner product of v 3 u 1 divided by the norm u 1 square u 1 minus v 3, u 2 this inner product divided by norm u 2 square multiplied by u 2. And this one can compute as 0 9 0 this vector.

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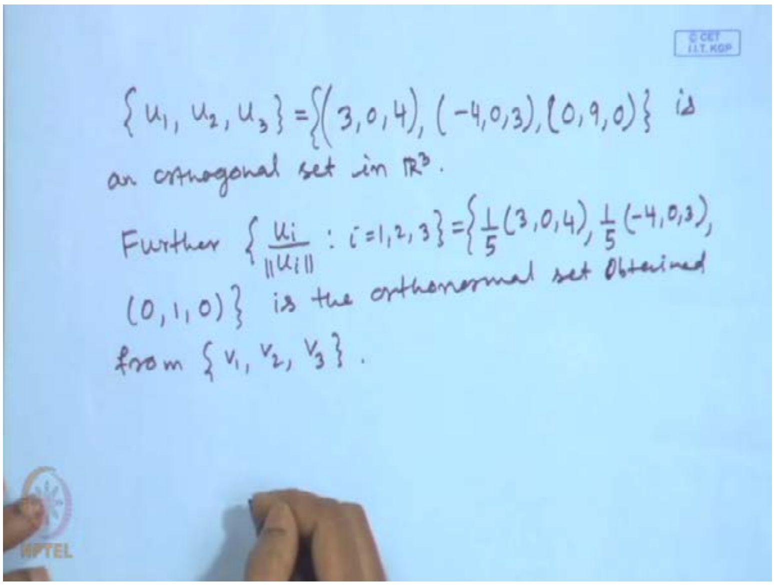

So, now, on applying this Gram-Schmidt orthogonalization process we get this u 1 u 2, u3 that u 1, u 2, u 3 that is, the vectors consist of 3 0 4; and the vector minus 4 0 3; and this vector 0 9 0 is a this is an orthogonal set of vectors set of vectors in R 3 with respect to this standard inner product in R 3. So, from here also one can get the corresponding orthonormal set. Further this set that u i divide by norm of u i; i from 1, 2 up to 3; this set that is, norm of u 1 will be 5, 3 0 4, norm of u 2 is also equal to 5 and that minus 4 0 3, and this vector that is, 0 1 0 is the orthonormal set obtained from orthonormal set obtained from the vectors v 1, v 2, v 3. So, this how one can apply that Gram-Schmidt orthogonalization process, and get a set of orthogonal vectors.

And also further, one can get a set of orthonormal vectors from the given set of linearly independent vectors. Also this Gram-Schmidt orthogonalization process one can apply and test the linearly dependency or independency of vectors. Suppose the given set of vectors v 1, v 2 to v n are given, and they are not known whether they are linearly independent. Then we consider the non-zero vectors v 1, v 2 to v n, and then apply this orthogonalization process. Then at some set we get the some u i will be equal to zero.

Then a set of vectors v 1, v 2 v n, the given set will be linearly independent. Otherwise, we will able to construct an orthogonal set of non-zero vectors v 1, v 2 to v n.

That is all for the lecture.

Thank you.

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