Advanced Engineering Mathematics
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Lecture No. #08
Inner Product Spaces, Cauchy-Schwarz Inequality

So, now we shall start a new topic that is called inner product spaces. And this is also very important concept in linear algebra.

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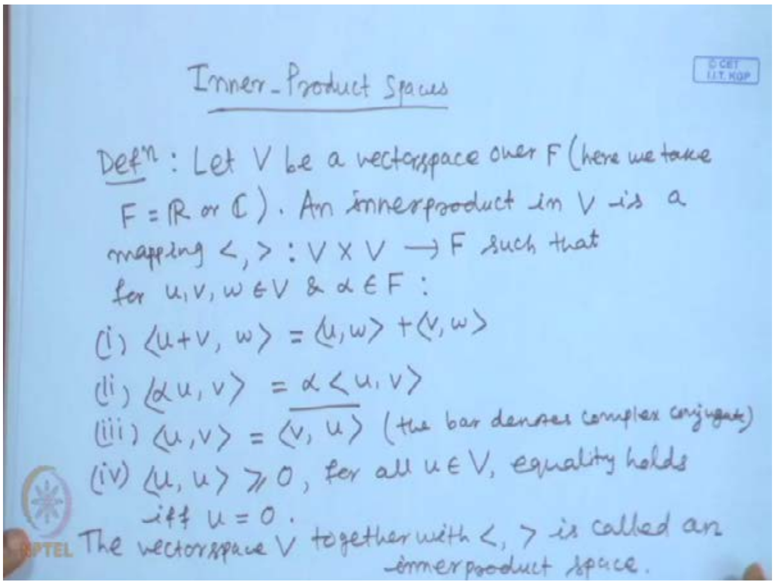

So, recall that the vector spaces are usually algebraic extension of Euclidean spaces, in other words that algebraic properties of the Euclidean spaces have been generalized, and got that vector spaces. So, in the next, we shall generalize the distance and length of vectors concept of Euclidean spaces. So, we shall impose some more structures to vector spaces and get a new space that is called inner product space. In case of Euclidean spaces vector of length, and we find distance between any two vectors. So, that concept we shall generalize to a vector space.

Remember that, in case of Euclidean spaces we have those scalar product; and in terms of scalar product, we find that length of a vector and distance between any two vectors.

So, something similar to that scalar product; we shall define in a vector space or derived vector space, and that is called inner product. So, that we define an inner product in a vector space. So, let V be a vector space over a field F of course, here we take this field F as set of real numbers, or real field or complex field. An inner product in that is, in V an inner product product in V is a mapping, we denote by this from this V cross V to this field F, such that for vectors u, v, w in the vector space and scalar alpha in the field F; First condition is that u plus v comma w are this inner product of u plus v comma w is equal to u, w plus v, w.

Second axiom is that: alpha u, v this is equal to alpha times u, v; third axiom is inner product of u, v is equal to if we check the order of the vectors; inner product of v, u its complex conjugate. So, here this the bar is the bar denotes complex conjugate. So, fourth axiom is for any vector u in v, u, u inner product of u with itself that is greater than or equal to 0 for all u belongs to this vector space. Equality holds equality holds if and only if u is equal to the zero vector. So, any vectors space together within mapping, this inner product that is called an inner product.

And we satisfy this four axioms will be called an inner product and the vector space together with an inner product is called an inner product space. So, the vector space the vector space V together with this inner product is called an inner product space called an inner product space. So, here we must notice one thing that the fourth Axiom; that inner product of u itself, this has to be a real number. Otherwise we cannot say that this comparison that greater than or equal to 0. So, this actually follows this from this third axiom.

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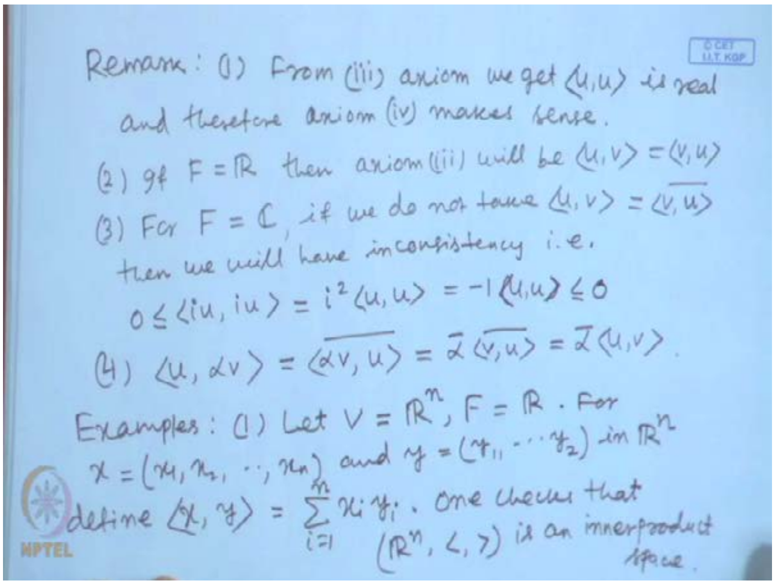

Let us see let us write this as a remark. So, we have this remark: First one is that from third axiom third axiom we get u, u inner product of u with itself is real. And therefore, and therefore, axiom four makes sense this makes sense. Second remark is that: If the field F is the real field, then axiom third will be axiom third will be inner product of u, v is equal to inner product of v, u, or in other words it is symmetric. Then third axiom so, the third remark: In third remark we have for F be the complex field, if we do not take u, v is equal to the conjugate of v, u, then we will get some inconsistency we will have inconsistency, that is i u, i u that is greater than or equal to 0 from axiom four and this equal to i square u, u and this equal to minus 1 u, u. So, this is less than or equal to so this 0 less than or equal to 0. So, we have this inconsistency.

So, fourth remark is that: Inner product of u with alpha v; that scalar is multiplied with second component. Then we get from axiom three alpha v, u its complex conjugate, and this is equal to alpha bar v, u its complex conjugate. So, alpha bar u, v; that means, if we multiply a scalar with second component, then when it comes out, we get its complex conjugate of this scalar times u, v. So, let us see some example of example of inner product spaces. So, first one is this, this is very natural one that we take. Let v be the vector space R n, F be the field set of all real numbers. So, for x is consisting of x 1, x 2 to x n and vector y with components y 1, y 2 to y n in R n. We define inner product of x and y is summation x i y i; i from 1 to n. In fact, this is the scalar product of two vectors in R n. So, one can check that this mapping will be an inner product and the space R n together with this usual scalar products is an inner product space. So, one checks that this R n together with this inner product is an inner product space an inner product space. So, similarly we will have an inner product in set of complex num of case set of complex in the vector space c to the power n.

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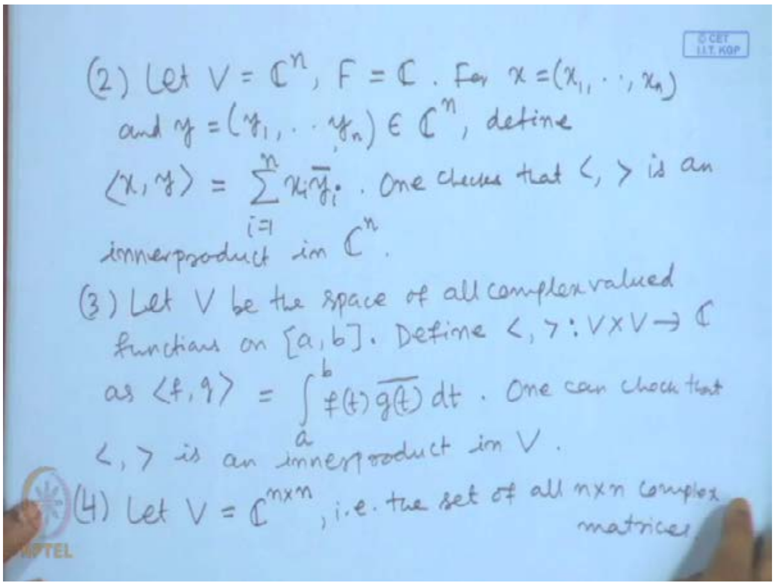

So, this second example is V be the C n, F is the field C for x vector x with components x 1, x 2 to x n, and y with components y 1, y 2 to y n. In C n, define this mapping is summation x i y i bar, this is complex conjugate again; i from 1 to n. So, again one checks that, one checks that this mapping is an inner product in C n. So, therefore this C n is also an inner product space. Third example is here we consider V be the space of all complex valued functions on this interval closed interval a b.

We define a mapping F, so this mapping from V cross V to set of complex numbers as f, g is equal to inner this integral a to b f(t) g(t) bar, this is the complex conjugate of g(t) d t. Again one can check that, one can check that this mapping is an inner product in V. So, let us see another example that here we consider V be the set of all n by n matrices with complex entries set of all n by n matrices with complex entries entries that is, the set of all n by n matrices with complex matrices.

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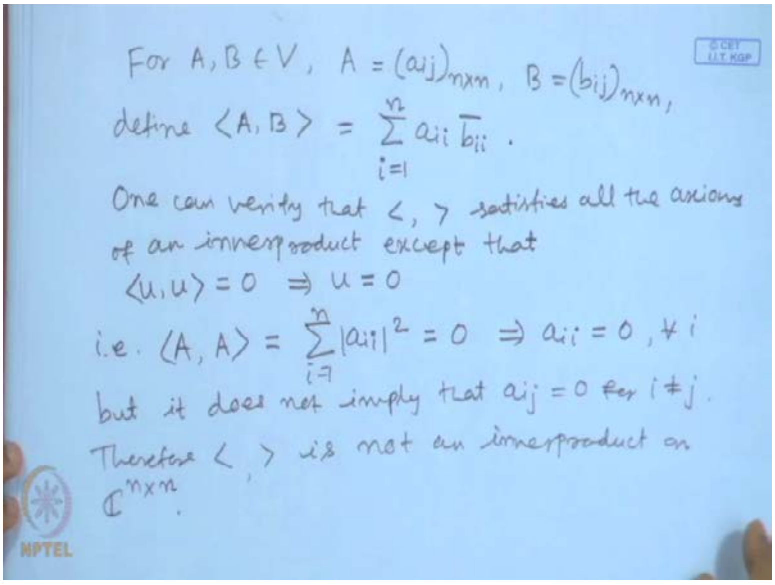

So, here we define for A, B belongs to V with entries of A be a i j; entries of b be b i j. Define this mapping A,B is equal to summation a i i b i i its complex conjugate; i from 1 to n. Here ones checks that this mapping; one can verify that this mapping satisfies all the axioms of an inner product except that u except that inner product of u with itself is equal to 0 implies that u equal to 0; that is inner product of A with itself is equal to sum of this a i i this mode square; i from 1 to n, this is equal to 0 this implies that the diagonal entries a i i are 0 for all i but it does not imply that a i j is equal to 0 for i not equal to j. Therefore, this mapping we defined is not an inner product on this set of all n by n complex matrices. So, next we shall see some important properties of this inner product. Before that we have another terminology that. Let this V be an inner product space.

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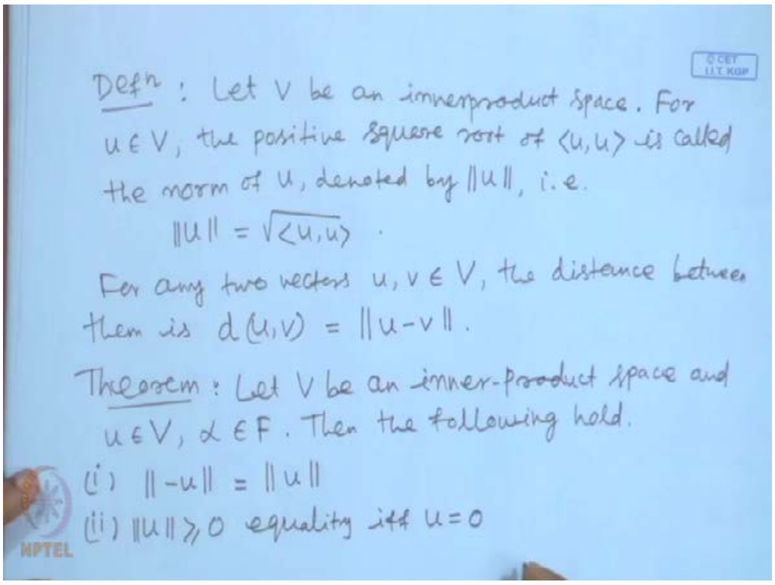

Here, we write simply V whenever we say that V this is an inner product space, we understand that there is an inner product to find on V. So, for any vector u in V, the positive square root of this inner product of u with itself is called the norm of u and we denote this by this norm of u that is, this norm of u is the positive square root of u, u, For any two vectors for any two vectors u, v in this inner product space V, the distance between them is d u, v is given by norm of u minus v. Then here we shall see some properties of this norm. We shall write this as a result the theorem. So, let V be an inner product space an inner product space and u belongs to u, v of the vector in v, and alpha be a scalar, then the following hold; first result is that norm of minus u is same as u. Second result is that: Norm of u is greater than or equal to 0. Equality if and only if u is zero; of course, this follows from fourth axioms of an inner product.

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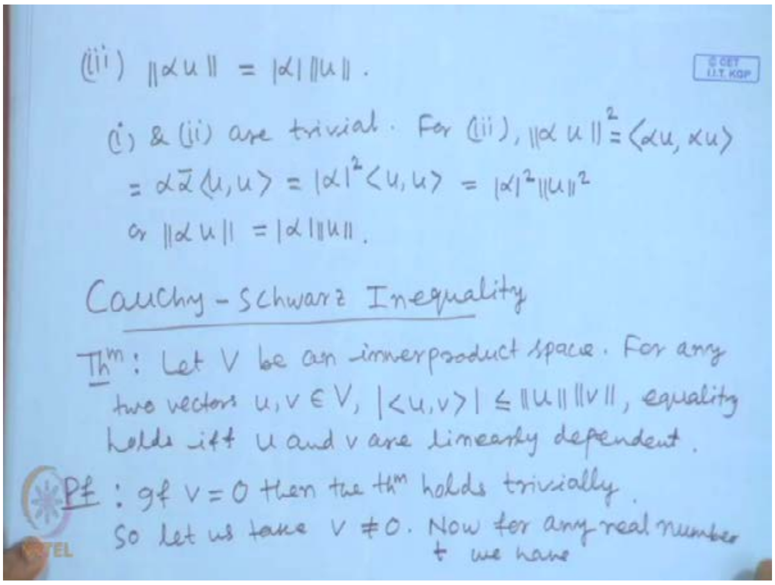

Then this third result we get here is this: Norm of alpha u is equal to absolute value of alpha times norm of u. So, this first and second are trivial. first and second are trivial For third result we have norm of alpha u square is equal to from the definition of norm inner product of alpha u with itself, and this will be alpha, alpha bar inner product of u with itself and that is equal to absolute value of square of absolute value of alpha times u, u, or from here we get this norm of alpha u is equal to that absolute value of alpha, and this of course, this is equal to this mode alpha square and norm u square. So, that is why we get here we get here that norm of alpha u is equal to absolute value of alpha times norm of u.

Next, we shall discuss an important inequality in an inner product space that is known as this Cauchy-Schwarz inequality Cauchy-Schwarz inequality. This is very important, so here it states that, we consider again that V be an inner product space an inner product space for any two vectors any two vectors u, v in this inner product space V; that absolute value of inner product of u, v, that is less than or equal to norm of u times norm of v. Further equality holds equality holds if and only if u and v are linearly dependent. So, since this is an important result. We shall see a proof of this theorem. Notice that if V is the zero-vector, then the theorem holds trivially theorem holds trivially. So, let us take V be not not the zero vector v be a V be a non-zero vector. Now, for any real number t, we have the following: We consider the norm of u minus t times inner product of u, v times v.

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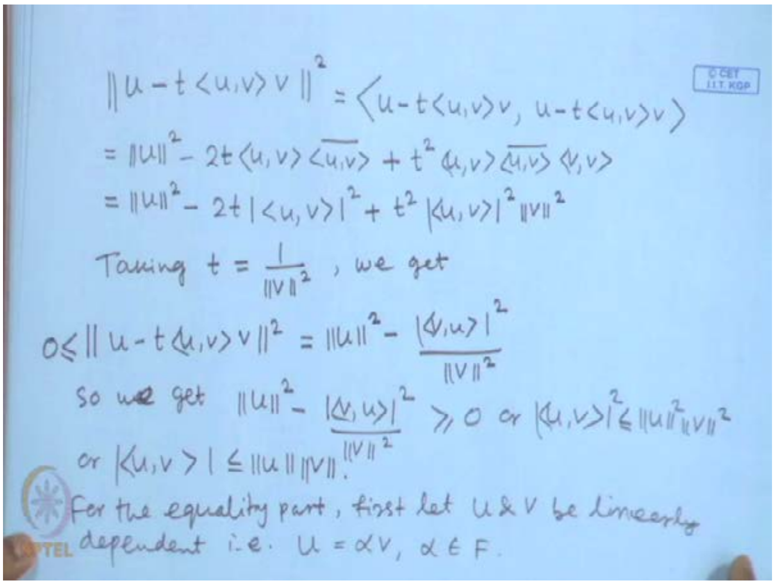

This square of this norm and from the definition of norm, we get this inner product of u minus t times norm of u, v times v with itself u minus t u, v v. On simplifying, that we simply using the axioms of an inner product space, and get that norm u square minus 2 t inner product of u, v inner product of u, v bar plus t square times inner product of u, v inner product of u, v its conjugate and inner product v with itself, or this is equal to norm u square minus 2 t absolute value of norm of u, v square. This is from the property of complex numbers; t square that absolute value square of absolute value of this inner product u, v square of norm of v. So, this is true for every real value t. So, here we consider a particular value of t, that we can take any real value for t.

So, in particular let us t be 1 open norm of square of norm V. And we get this norm of u minus t u, v inner product of u, v times v this square is equal to, on simplifying, we get norm of this u whole square minus inner product of v, u its absolute value and square of this divided by square of norm v. So, notice that this left hand side is this is greater than equal to 0 from the definition of this norm, so we get this. So, we get that this norm of u this square minus this absolute value of v, u this square divided by square of norm v this is greater than or equal to 0, or this inner product of u, v its absolute value whole square, this is less than or equal to norm u square times this square of norm v. Then taking positive square root on both sides we get the inequality in the theorem. So the next for the second part that equality part, so for the equality part for the equality part. First let u and v be linearly dependent linearly dependent that is, u is equal to alpha times v for scalar alpha.

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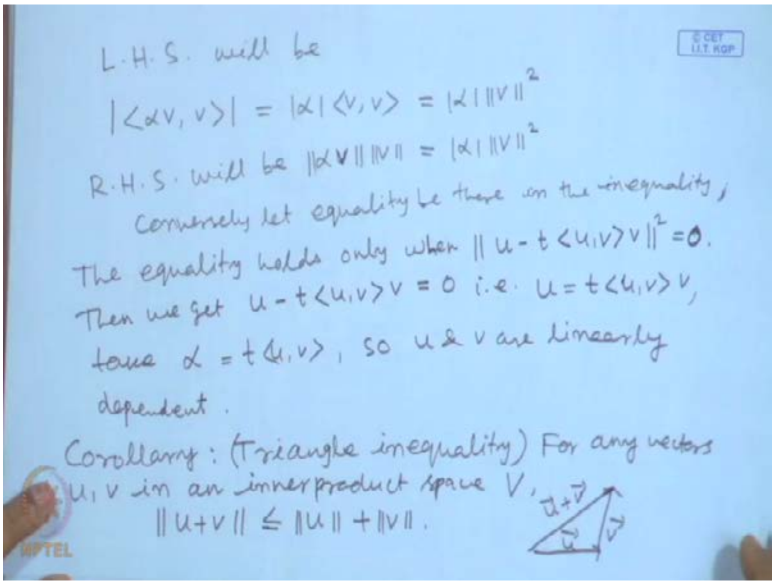

So, the LHS is LHS will be absolute value of inner product of alpha be (v, v), this is equal to absolute value of alpha inner product of v with itself. So, this is equal to absolute value of alpha and square of norm of v. And RHS will be this norm of alpha v times norm of v, this is equal to absolute value of alpha norm of v square. So, here we can see that LHS is equal to RHS. So, equality holds in Cauchy-Schwarz inequality. So, conversely suppose equality occurs, conversely let equality be there in Cauchy- Schwarz inequality in the inequality that is. So, this equality occurs only one while proving this theorem. Just recall that the equality holds only when this norm of u minus t inner product of u, v times this v this square is equal to 0.

So, this is true, if then we get this u minus t times this inner product u, v into v is equal to 0 that is, u is equal to t inner product of u, v times v. So, take alpha is t times u, v. So, u and v are linearly dependent. Next, we shall see some consequences of or this Cauchy-Schwarz inequality. And an important consequence that we get is triangle inequality. So, that will write as a corollary, and this is the triangle inequality. It is like this for any vectors u, v in an inner product space V in an inner product space V, norm of u plus v is less than or equal to norm of u plus norm of v. So, it is like that geometric in geometric we have seen that triangle inequality, that some of length two sides is greater than or equal to the length of the third side; that means, if we take the vectors u, v like this: This is u; this v. Then here this third side of this triangle will be u plus v. So, this is the geometrical interpretation of triangle inequality. That length of two sides; sum of length of two sides is greater than or equal to the length of the third side.

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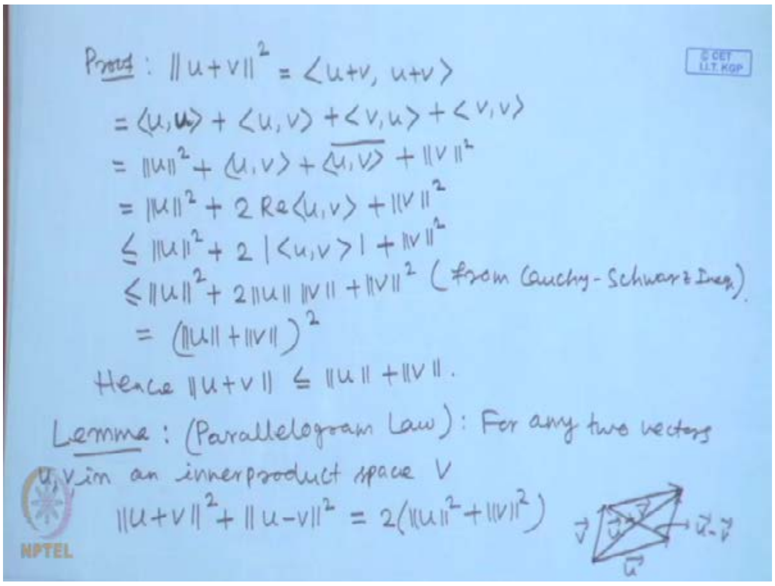

Using Cauchy-Schwarz inequality, we can give a proof of this, that we obtain by expanding this norm of u plus v this square is equal to inner product of u plus v, u plus v. And on simplifying, using axioms of an inner product space we can get that inner product of u with itself u, u plus inner product of u, v plus inner product v, u plus inner product of v with itself; and this can be written as square of norm u plus inner product of u, v plus inner product of u, v its complex conjugate plus norm v square; and this is same as square of norm of u plus 2 times real part of this u inner product of u and v plus norm square of norm of v; and this is less than or equal to square of norm of u plus 2 times absolute value of inner product of u, v.

So, this follows from that property of complex numbers again plus square of norm of v. And Cauchy-Schwarz inequality we get that this is equal to square of norm u plus 2 norm of u into norm of v plus square of norm of v. So, this is from Cauchy-Schwarz Inequality Cauchy-Schwarz Inequality. So now, this is equal to norm of u plus norm of v whole square. So, hence we get this triangle inequality by taking positive square root. This is less than or equal to norm of u plus norm of v. So, using here this inner product space also satisfies an important property that is called the parallelogram law. So this, another important property here we get is this parallelogram law. So, it says that for any two vectors in an inner product space two vectors in an inner product space. Any two vectors u and v in an inner product space V we have that u plus v its norm whole square plus square of norm of u minus v, this is equal to 2 of square of norm u plus square of norm v.

And this has also geometrical interpretation that, if we have a parallelogram with sides, that this is vector u and vector v. And if we complete this parallelogram, that the diagonal will represent the vector u plus v and the other diagonal that represent this represent u minus v. The vector it represent u minus v. So, this parallelogram law says that this sum of squares of two diagonals is equal to sum of the square of sides; all four sides; that mean this sum of square of length of two diagonals is equal to the sum of squares of length of all four sides. This is called the parallelogram law.

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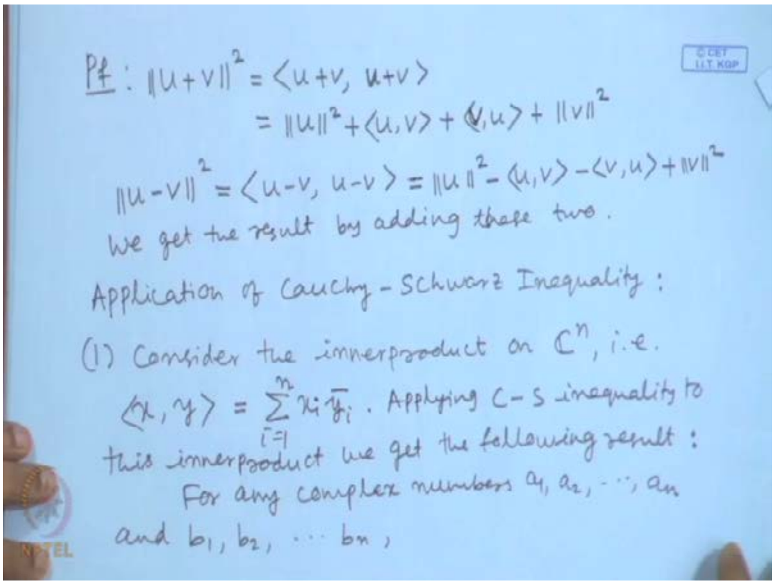

So, this parallelogram law can also be proved easily. That we can find the square of norm of u plus v be like this; u plus v inner product of u plus v with itself and on expanding we get that norm of u square plus this inner product of u, v plus this inner product of this v, u plus square of norm of v. And similarly this square of norm of u minus v is equal to inner product of u minus v with itself, and that is equal to norm of u this square minus inner product of u, v minus this inner product of v, u plus square of norm v; and we get the result by adding this two we get the result by adding this two. So, this Cauchy-Schwarz inequality has many applications. So, let us see few of them. So, application of Cauchy-Schwarz inequality here we consider, first we consider that, inner product we have defined on C n and in the example in the second example, consider the inner product consider the inner product on C n that is, inner product of x, y is equal to summation x i y i bar; i from 1 to n. Applying Cauchy-Schwarz inequality c-s inequality to this inner product, we get the following result: That for any complex numbers any complex numbers a 1 a 2 a n and b 1 b 2 b n, or any complex numbers a 1 a 2 a n and b 1 b 2 b n.

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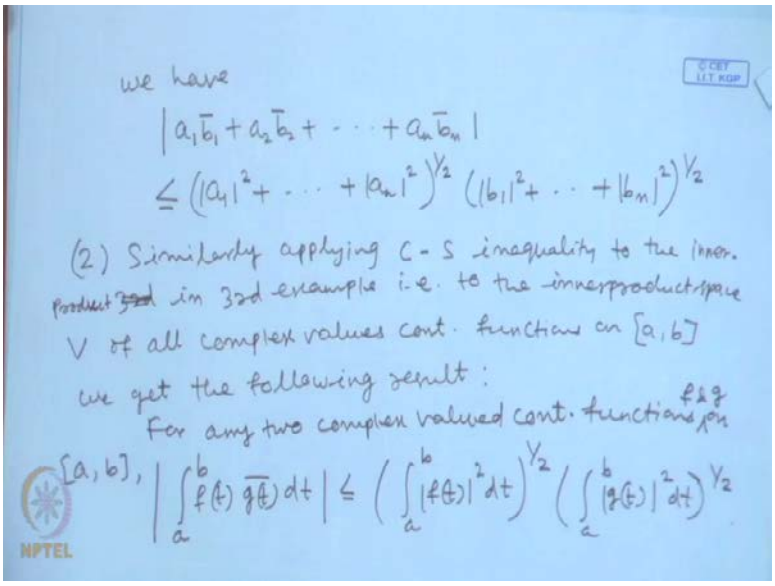

We have that absolute value of a 1 into b 1 bar plus a 2 into b 2 bar plus a n into b n bar is less than or equal to absolute value of a 1 square plus absolute value of a n square whole to the power half into absolute value of b 1 square plus absolute value of b n square whole to the power half. Similarly if we apply Cauchy-Schwarz inequality Similarly applying Cauchy-Schwarz inequality to the third example; to the inner product inner product in third example, that is to the inner product space inner product space inner product space V of all complex valued continuous functions on a,b we get the following result we get the following result that is for any two complex valued continuous functions a,b; for any complex valued continuous functions for any two complex valued continuous functions on a, b. for any two complex valued continuous functions f and g on a, b this absolute value of integration f(t) g(t) bar d t from a to b is less than or equal to integral a to b, f(t) mode square d t whole to the power half into integral a to b absolute value of g(t) its square d t whole to the power half. So, this is how applying Cauchy-Schwarz inequality, we can prove many things.

And that is all for this lecture. We stop here.

Thank you.