Advanced Engineering Mathematics

Prof. Pratima Panigrahi
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture No. #07
Jordan Canonical Form
Cayley Hamilton Theorem

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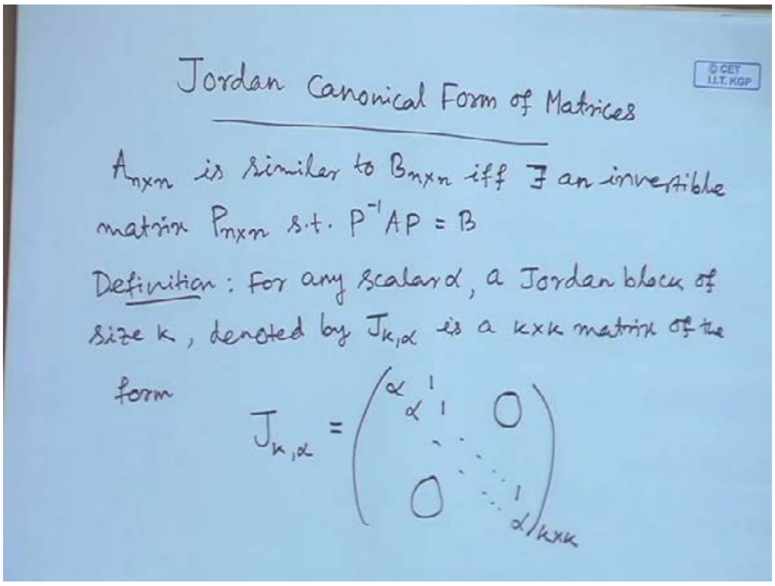

So, we shall start today about this Jordan canonical form Jordan canonical form. (No audio from 00:32 to 00:47) So, remember that, a square matrix of size n by n is similar to a matrix B of size n by n, if and only if there exist an invertible matrix invertible matrix P of size n by n, such that P inverse A P is equal to B. So, in the previous lecture, we have seen that not all square matrices are similar to diagonal matrices; or in other words, not all square matrices are diagonalizable. But here we shall see that every square matrix is similar to a matrix that is nearly diagonal.

So, nearly diagonal means; it is of the form that the diagonal entries besides the diagonal entries, the super diagonal entries are not zero, and they are equal to 1. At the most that is, we are talking about that every square matrix is similar to a matrix that is in Jordan canonical form**.** So, here we shall define a Jordan block that for any scalar alpha, a

Jordan block of size k denoted by J k alpha is a matrix is a k by k matrix of the form; this J k alpha, it is of the form that the diagonal entries are alpha and the super diagonal entries they are equal to 1 that is, the super diagonal entries are equal to 1. So, this called a Jordan block. And the next, we shall define a Jordan block matrix or a Jordan canonical form.

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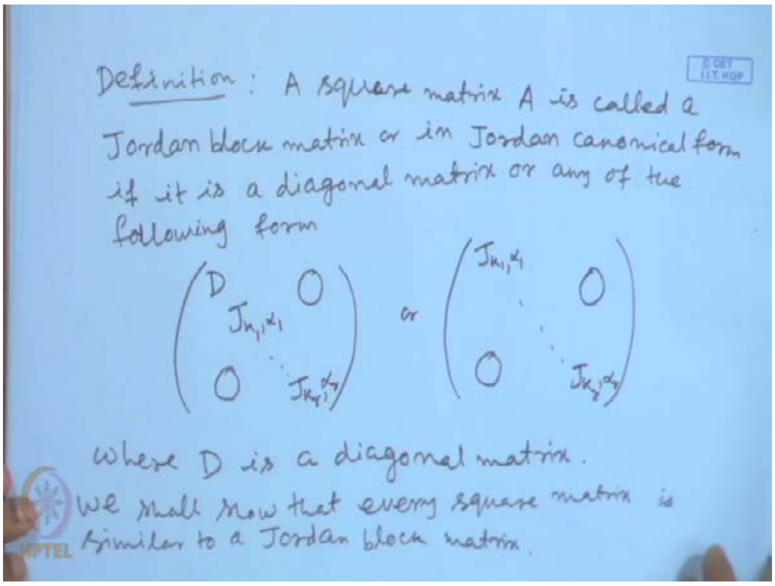

Our next definition is this, that a square matrix a square matrix A is called a Jordan block matrix Jordan block or in Jordan canonical form Jordan canonical form, if it is it is a diagonal matrix diagonal matrix or any one of any of the following part that is, here this is a diagonal matrix, and this J k 1 alpha 1; like this J k r alpha r or it is of this form that j k 1 alpha 1 up to this J k r alpha r. where this D is a diagonal matrix of course, the half diagonal entries, or this blocks half diagonal blocks, they are all zero blocks. Basically this is, this two are block matrices that the diagonal blocks are like this and half diagonal blocks are zero. And here D is a diagonal matrix. So, we shall show that here we shall show that every square matrix square matrix is similar to a Jordan block matrix

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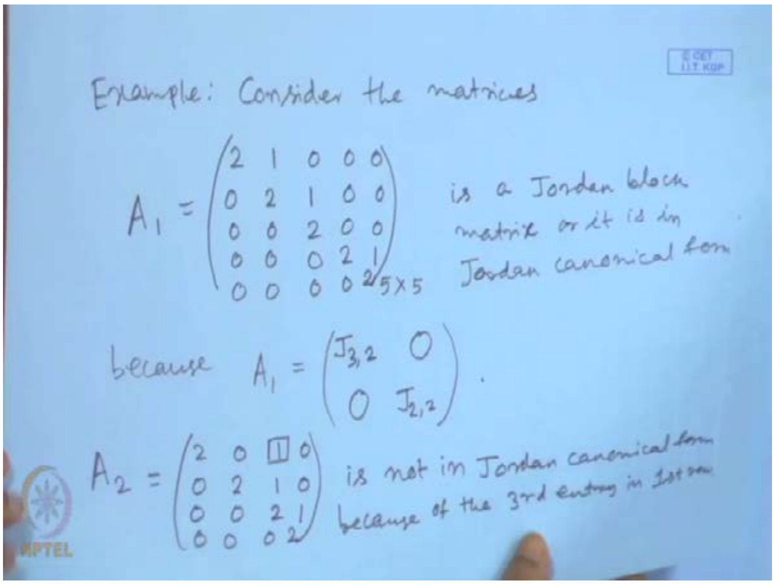

So, let us see some example of Jordan block matrices; that look at this example. Here we consider the matrices consider the matrices A 1 is like this, it is of size 5 by 5, and entries are 2 1 0 0 0 0 2 1 0 0 0 0 2 0 0 0 0 0 2 1 and this 0 0 0 0 and this 2. So, here this matrix A 1 is in Jordan form, that is a Jordan block matrix, or it is in matrix or it is in Jordan canonical form canonical form because it is of the form A 1 is of the form that, here we can say this as first this 3 by 3 sub-matrix that forms j 3 2 and this this one 2 by 2 sub-matrix that forms j 2 2, and the remaining are zero blocks.

But this matrix, if we consider A 2 be like this, this is 2 0 1 0 0 2 1 0 0 0 2 1 0 0 0 2 is not in Jordan canonical form in Jordan canonical form, because of this entry that because of because of the third entry in first row. So, this is not a super diagonal entry and this not equal to zero. So, therefore, this is not in Jordan canonical form. So, here we shall discuss that how to find Jordan canonical form of every square matrix. And for that, we shall define another terminology that is called generalized eigenvector.

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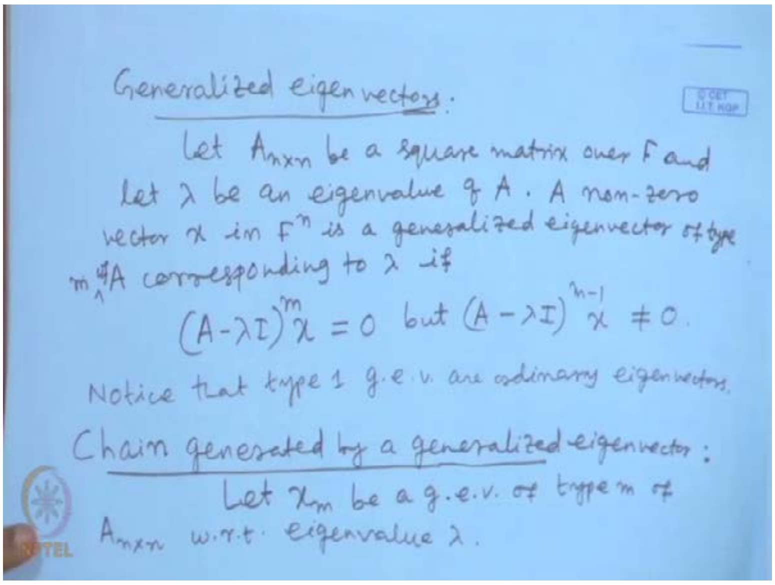

So, here we define generalized eigenvectors of square matrices. So, here all we Consider, let A be a square matrix over a field F and let lambda be an eigenvalue of A, then non-zero vector or a non-zero vector x in F n is a generalized generalized eigenvector of A corresponding to lambda corresponding to lambda, if of course this x is a generalized eigenvectors of we take type m type m of A corresponding to lambda, if this A minus lambda I whole to the power m into this x, this is equal to zero but A minus lambda I whole to the power m minus 1 x is not equal to zero. So, this is called type m generalized eigenvectors of A corresponding to lambda.

So, notice that type one generalized eigenvectors are ordinary vectors. So, notice that type one generalized eigenvectors are ordinary eigenvectors. So, this is why this name generalized is their. Then we shall see here a chain generated by a generalized eigenvectors. So, that is important here for finding Jordan canonical form. So, we get this chain generated by a generalized eigenvector. And here we consider that, we consider x m let x m be a generalized eigenvector of type m of a matrix, square matrix A with respect to eigenvalue lambda. Then we get a chain that is generated by x m; this generalized eigenvector x m. And here we define the chain be like this.

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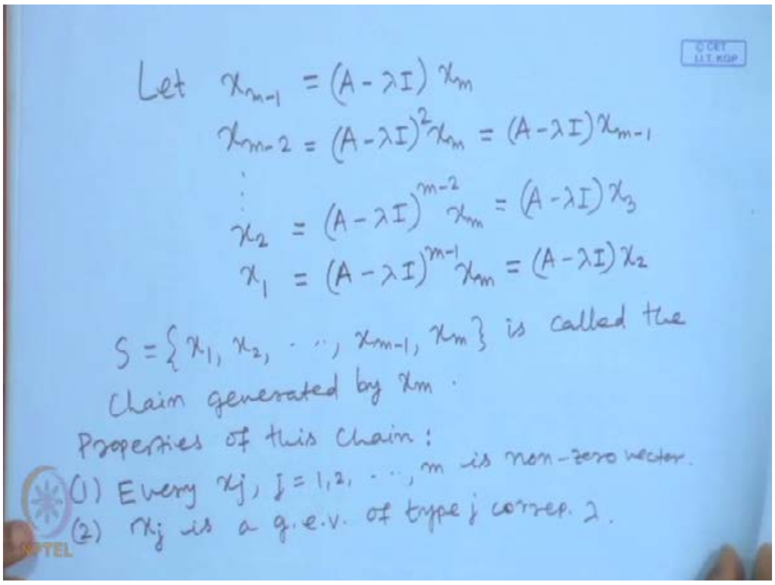

Let x m minus 1 is equal to A minus lambda I x m. x m minus 2 is A minus lambda I whole square times x m, and that is equal to A minus lambda times x m minus 1. In this fashion we get this x 2 be A minus lambda I whole to the power m minus 2 x m, or this is equal to A minus lambda I times x 3. And finally, we get this x 1; x 1 is equal to A minus lambda I whole to the power m minus 1 times x m, and that is equal to A minus lambda I times x 2. So, this set S consisting of vectors x 1, x 2 up to x m minus 1, x m is called the chain generated by x m.

So, this chain generated by x m satisfy some properties. And we shall list those properties here. So, properties of this chain are like this: First Properties is that every x j well every x j is non-zero every x j, j from 1, 2 up to m is non-zero vector is non-zero vector. This follows from definition of this generalized eigenvalue x m, because x m is a generalized eigenvalue of type m. So this is important. And this second property is that every x j here this x j x j is a generalized eigenvector of type j of the matrix A; type j corresponding to lambda. So, every x j is a generalized eigenvector of type j corresponding to lambda.

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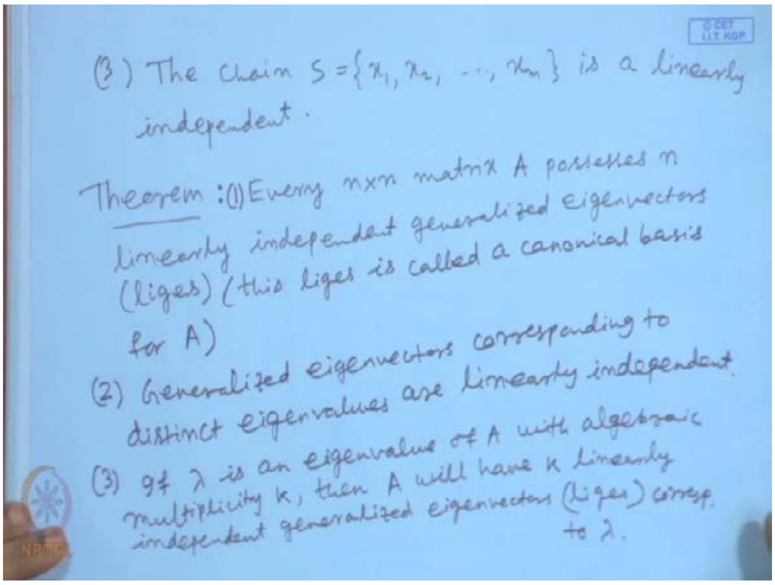

So then another important property here is this one that, this sequence consisting of this sequence of vectors x 1, x 2 to x m or this set S, this is a linearly independent set. The chain S of vectors of x 1, x 2 up to x m is a linearly independent set linearly independent set and this property one can also check easily. Then next, we shall see using this generalized eigenvectors, how to get a basis for a matrix A. That is called canonical basis for a matrix. So, here we shall write those results, known results; we shall use these results for finding Jordan canonical form of a matrix. So, it says that every n by n matrix A possessed n linearly independent generalized eigenvectors, n linearly independent generalized generalized eigenvectors and in short forms we write them as liges - linearly independent generalized eigenvectors. This liges this liges is called a canonical basis for A a canonical basis for this matrix A

So, let us take this be first result. Then the second result is like this. A generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. So, it says that generalized eigenvectors corresponding to corresponding to distinct eigenvalues distinct eigenvalues are linearly independent. This property is like this properties of ordinary eigenvectors. So, another important property is that if lambda is an eigenvalue of A with algebraic algebraic multiplicity multiplicity k, then A will have k liges or k linearly independent k linearly independent generalized eigenvectors or liges. So, corresponding to every eigenvalue; that means, if the algebraic multiplicity of lambda is k, then we get k linearly independent generalized eigenvectors corresponding to lambda here this independent generalized eigenvectors corresponding to lambda. So, this is very useful.

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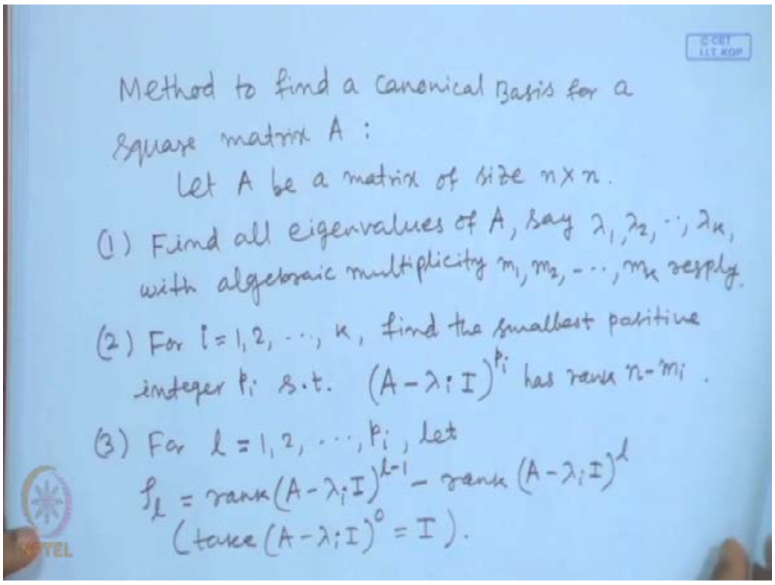

So, next let us see how to find a canonical basis for a matrix a square matrix and this is important method to it find a canonical basis for square matrix A. So, here we consider that A be a matrix of size n by n. So, here first step is, we find all the eigenvalues of A find all the eigenvalues of A, say lambda 1, lambda 2 up to lambda k with algebraic multiplicity algebraic multiplicity m 1, m 2 up to m k respectively. So, then next for this every i from 1, 2 up to k, we find find the smallest positive integer p i, such that this A minus lambda i I whole to the power p i has rank n minus m i, so after getting this p i for every k, or for every l from 1, 2 up to this p i.

Let row l be defined like this that row l is defined like this this is the equal to rank of a minus lambda i multiplied by I whole to the power l minus 1 minus rank of A minus lambda i I whole to the power l. of course, here we consider this we take this A minus lambda i I whole to the power zero be the identity matrix. So, this row l we have defined this plays important role, that this then this matrix A; then in the canonical basis canonical basis of A. There are row l liges - linearly independent generalized eigenvectors of type l corresponding to corresponding to eigenvalue lambda i.

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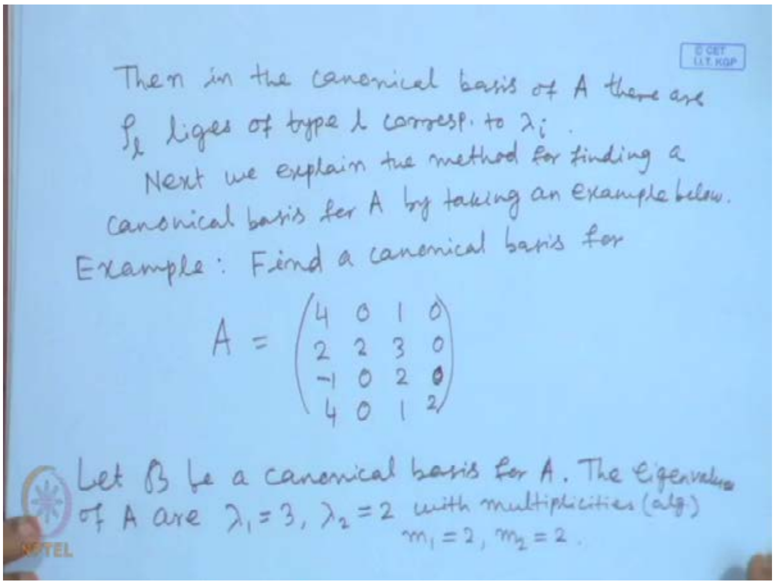

Then next we explain next we explain the method for finding a canonical basis for A by taking an example by taking an example below, and this example in this example we consider, find a canonical basis for this matrix A is like this: 4 0 1 0 2 2 3 0 minus 1 0 2 0 4 0 1 2, so this is zero. So, here we find a canonical basis for A in the following manner. Let B be a canonical basis for A. So, first we compute the eigenvalues the eigenvalues of this matrix A are lambda 1 is 3, lambda 2 is 2 with multiplicity with multiplicities that is algebraic only algebraic multiplicity m 1 is 2, m 2 is 2.

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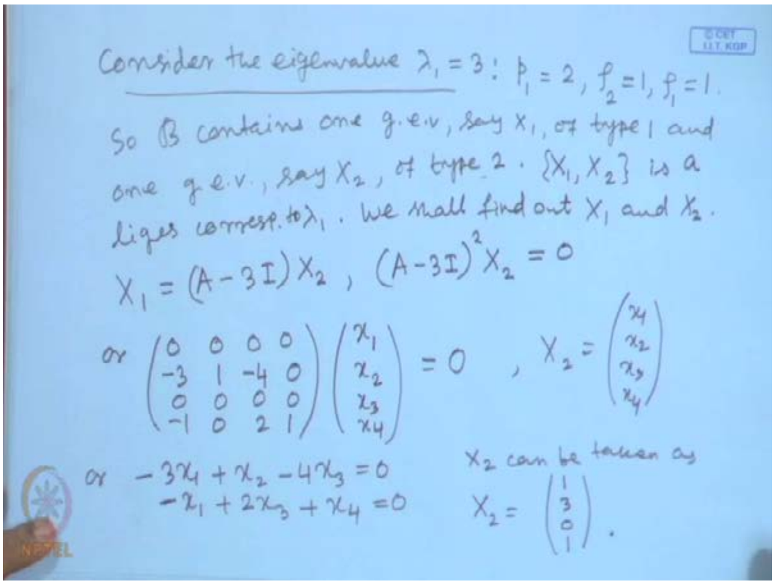

Here we shall find out this generalized eigenvectors x 1 and x 2. So, in this manner, well x 1 is actually A minus 3 I times this x 2 and this A minus 3 I whole square x 2 that is equal to zero; this x 2 is a generalized eigenvector of type 2 this. So, therefore, this x two satisfies this equation. And here we consider x 1 be this eigenvector generalized eigenvector of type 1. Then for finding this x 2 we get this or we get this A minus 3 I whole square, one can compute is like this it is the matrix first row is all zero , second row is minus 3 1 minus 4 0 then 0 0 0 0 minus 1 0 2 1. And here for x 2 we consider the entries from x 1 x 2 x 3 x 4, this is equal to zero.

Here we are considering this x 2 is this vector x 1 x 2 x 3 x 4, or we get this system is like this or we get two equation in this system that: minus 3 x 1 plus x 2 minus 4 x 3 is equal to zero, and this minus x 1 plus 2 x 3 plus x 4 is equal to zero. So, x 2 is a solution of this system. So, we can take or also we can find x 2 can be taken as this x 2 satisfies these two equations, and we can consider this or we can solve for this also. So, let us take this x 2 be like this, that it is the vector 1 3 0 1.

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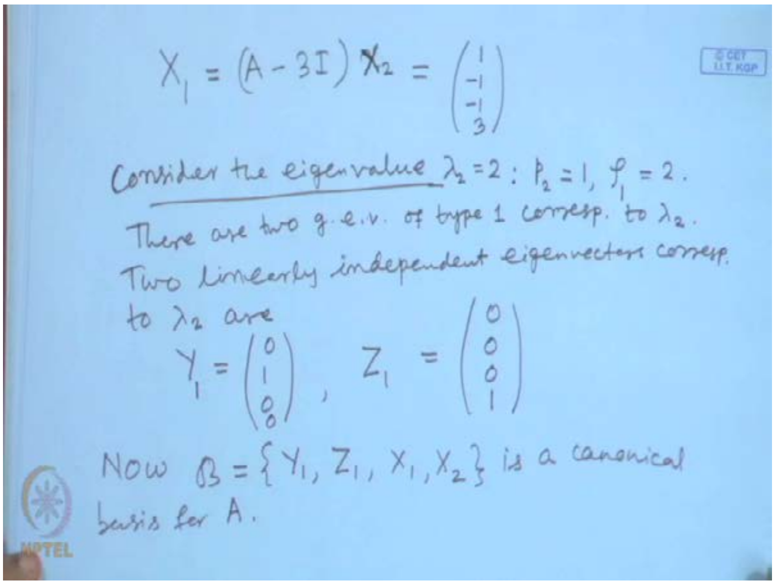

So, next we can find this X 1 is from X 2 and it is like this X 1 is A minus 3 I times X 2, so, one compute it well. We are considering this for X 2. So, we get this X 1 be the vector 1 minus 1 minus 1 3. So, now we have got this X 1 and X 2. So, next we shall find out generalized eigenvectors for the second eigenvalue; consider the eigenvalue lambda 2, and that is 2. So, for this we can see that one can check this value of p 2 is equal to

1and value of rho 1 is equal to 2. So, this means that; there are generalized eigenvectors of this means that, there are two generalized eigenvectors of type 1 corresponding to lambda 2 corresponding to lambda 2 and since the generalized eigenvectors are type 1, they are ordinary eigenvectors corresponding to lambda 2.

So, we find two linearly independent eigenvectors corresponding to lambda 2, so two linearly independent two linearly independent corresponding to lambda 2 one can find this is., Say, one is Y 1 and one may take this Y 1 as 0 1 0 0; say this one is Y 1, and this Z 1 that one take as 0 0 0 1. So, now this canonical basis corresponding to A be like this; this B consisting of Y 1, Z 1, X 1, X 2 is a canonical basis for the matrix A; of course,,, there may be several canonical basis corresponding to a matrix A .

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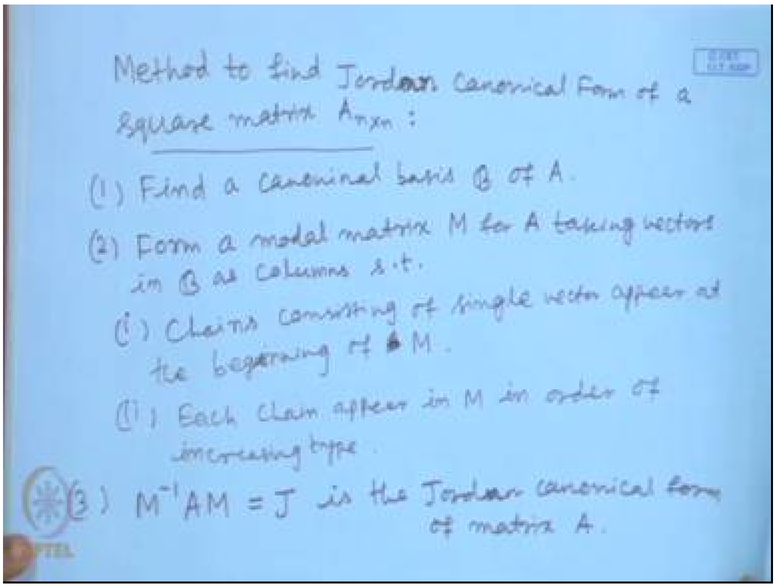

So the next we find Jordan canonical form of a square matrix. Now we are ready to find Jordan canonical form of a square matrix. So, the method is like this; method to find Jordan method to find Jordan canonical form of a square matrix A of size n. So, this steps are like this; first step is, we find a canonical basis S or may be B we can say; find a canonical basis B of this matrix A. Second step is that, we form a matrix M that we say model matrix form model matrix M for A taking vectors in B as columns, such that of course, the chains consisting of single vector appear at the beginning at the beginning at the beginning of A at the beginning of this matrix M. And second condition is that, each chain appears in M in order of increasing type. We take an example and explain this of course, so now this M inverse A M is the Jordan form of this matrix A. This is the Jordan this is the Jordan canonical form of matrix A.

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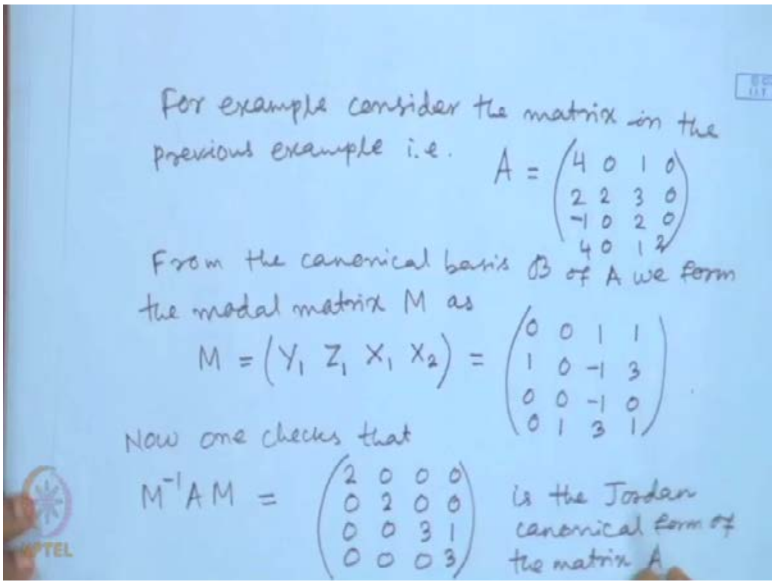

So, next, we take one example. For example, we consider consider the matrix in the previous example that is, A is the matrix this one that: 4 2 sorry 4 0 1 0 2 2 3 0 minus 1 0 2 0 4 0 1 2. So, for this matrix you recall that we have already got a canonical basis. So, remember that that the canonical basis we have got. So, now we can form model matrix from the canonical basis canonical basis B of A, we form the model matrix M as M is consist of that Y 1 because this is consist of this chain is consist of only one vector that is Y 1. So, Y 1 also only one vector in this chain, and then X 1 and X 2. So, here this X 1 and X 2 form a chain and we have to write in their increasing type; that means, X 1 is type 1 and X 2 is type 2.

So, therefore, we get this model matrix M be like this; Y 1 we have taken as 0 1 0 0 ; Z 1 we have taken that 0 0 0 1; and X 1 is 1 minus 1 minus 1; and X 2 is 1 3 0 1. So, now one can check that, now one checks that this M inverse A M is consist of the matrix of the form that it is consisting of 2 0 0 0 0 2 0 0 0 0 3 1 0 0 0 3, and this is the this matrix is the Jordan canonical form of the matrix A that is, A is similar to a Jordan block matrix. So, here we have shown that every square matrix is similar to a nearly diagonal matrix or a matrix in Jordan canonical form.

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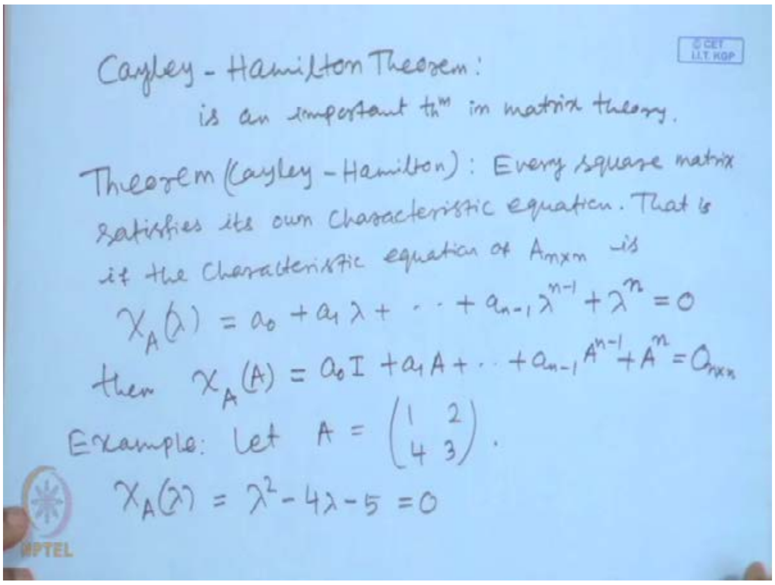

So, next, we will discuss an important theorem of this linear algebra; and in particular, this is an important theorem theorem of matrix theory to complete this vector space part. So, here we discuss about this Cayley-Hamilton theorem Cayley-Hamilton theorem. So, this is an important theorem in matrix theory this is an important theorem in matrix theory. So to complete this vector space part, we discuss about this theorem this is an important theorem in matrix theory. So, this Cayley-Hamilton theorem states like this, this is Cayley-Hamilton theorem that every square matrix every square matrix satisfies its own characteristic equation that is, if the characteristic equation of this matrix A of size n pie A lambda is a 0 plus a 1 lambda up to a n minus 1 lambda to the power n minus 1 plus lambda to the power n is equal to 0, then this A(A) is equal to 0.

This pie A(A) will be a 0 times identity matrix plus a 1 A up to a n minus 1 a to the power n minus 1 plus a to the power n is the 0 matrix; this is n by n 0 matrix. So, let us see one example. If let A be the matrix 1 2 4 3 at characteristic equation is pie A lambda; characteristic equation is lambda square minus 4 lambda minus 5 is equal to zero.

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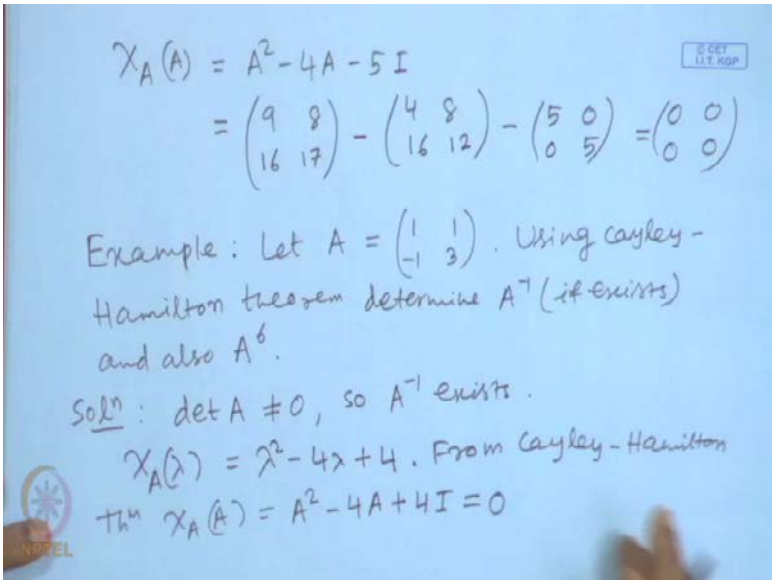

So, now, we shall find out pie A(A), and that is equal to A square minus 4 A minus 5 I, and it is the matrix that: 9 8 16 17 minus 4 8 16 12 minus 5 0 0 5, and one can check that this is equal to the 0 matrix of size two. So, using this Cayley-Hamilton theorem we can find higher power of matrices; we can also compute inverse of a matrix. So, this Cayley-Hamilton theorem is very useful. Let us see one example in support of this. So using, here we consider A be the matrix 1 1 minus 1 3.

So, using Cayley-Hamilton theorem Cayley-Hamilton theorem of determined A inverse if exists, and also sixth power of A. So, we can use this Cayley-Hamilton; note that, determinant of A is not equal to 0. So, A inverse exists. So here, characteristic polynomial of this matrix A is lambda square minus 4 lambda plus 4. So, from Cayley-Hamilton theorem from Cayley-Hamilton theorem, we get that pie A(A), that is equal to A Square minus 4 A plus 4 times identity matrix is 0.

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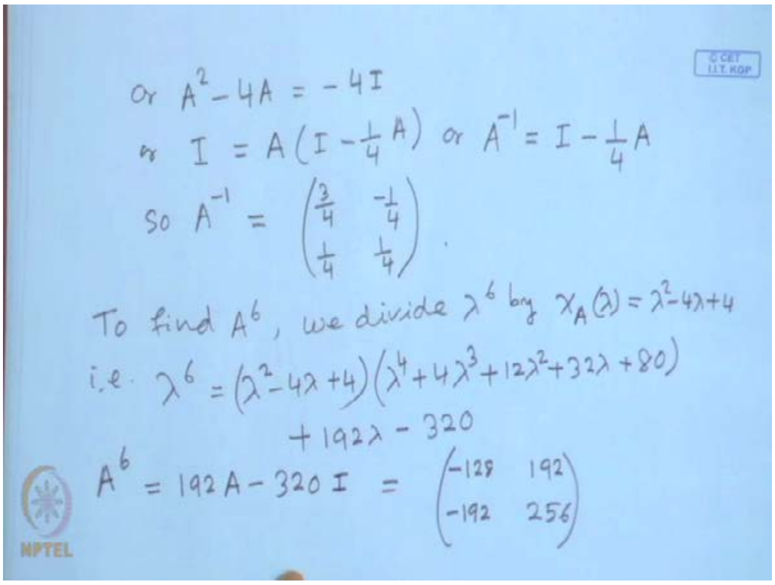

Or form here we get that, this A square minus 4 A is equal to minus 4 times identity or on simplifying we get that I is equal to A times I minus 1 by 4 A or this A inverse is the matrix I minus 1 by 4 A. So now, we know the matrix A and we know this identity matrix. So, we can find the inverse of A. And it is, one can compute like this: 3 by 4 minus 1 by 4 1 by 4 1 by 4. So, equate both the value of A in place of lambda, we get this first term equal to 0, and therefore, we get the A to the power 6 is equal to 192 times A minus 320 times identity. And one can compute that this matrix is like given by minus 128 192 minus 192 256. This is how we have computed sixth power of this matrix A. And using in this way that mean; we can use Cayley-Hamilton theorem in this fashion, and find higher power of matrices and also inverse of matrices. And this is lots of application.

So, we stop this lecture here.

And that is all thank you.