Advanced Engineering Mathematics

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Lecture No. # 06
Method to Find Eigenvalues and Eigenvectors
Diagonalization of Matrices

So, here we will discuss about a method to find eigenvalues and eigenvectors of matrices.

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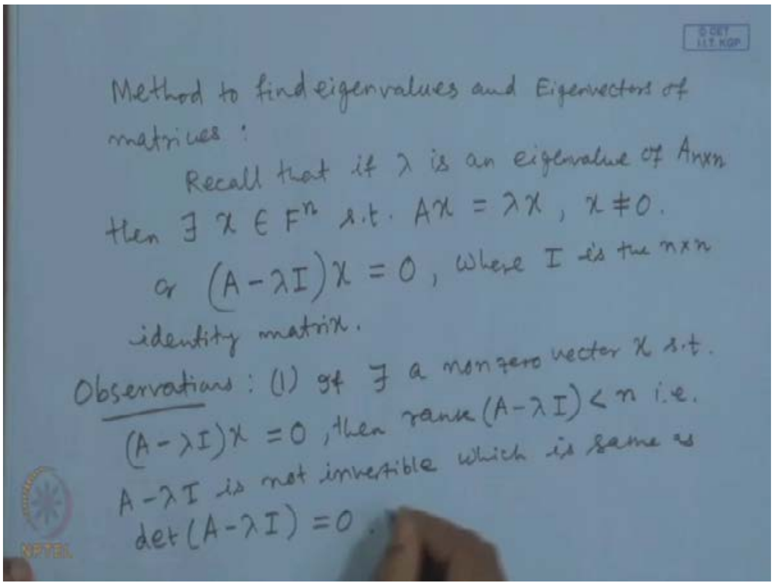

So, we will discuss about this method to find Eigen values and eigenvectors of matrices. Recall that if it become that if lambda is an Eigen value of a matrix A of size n by n, then there exist a vector x in the field, in this vector space F n such that A x is equal to lambda x, here x is not the 0 vector; or you can write this as A minus lambda times I x is equal to 0, where I is at the n by n identity matrix. So, we have the following observations, that first observation is like this, if there is a non-zero solution of, if there exist a non zero vector x such that. A minus lambda I x is equal to 0, then rank of this A minus lambda I is less than n, that this number of variables. So, in other words, that is A minus lambda I is not invertible and that is same as which is same as determinant of A minus lambda I is equal to 0. So, from this equation we find all Eigen values of the matrix A. So, this equation.

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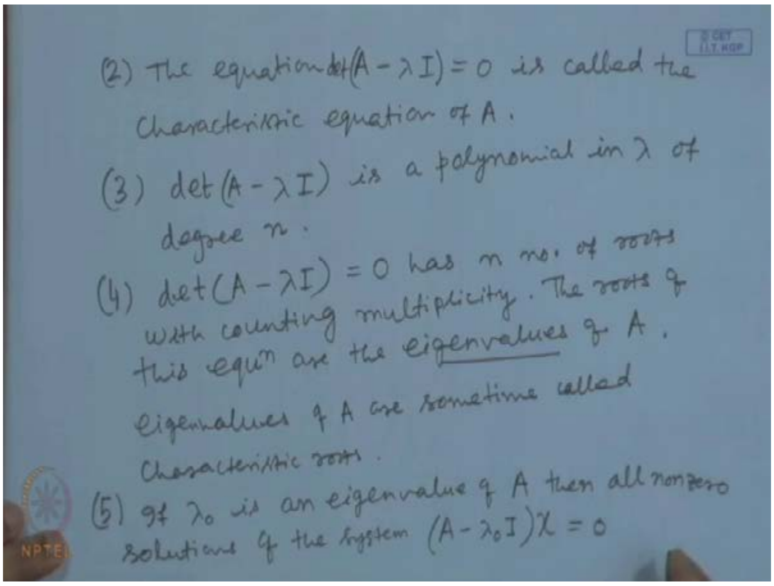

So, this the equation A minus lambda I is equal to 0. So, the determinant of A minus lambda I is equal to 0 is called the characteristic equation of A. And this is, this determinant of A minus lambda I is a polynomial in lambda of degree n this is or degree n polynomial. So, this equation is therefore, this determinant of A minus lambda I equal to 0 has at most n distinct solutions or roots and distinct roots are with counting multiplicity, this has n number of roots with counting multiplicity. And the roots of this equation are the Eigen values of A.

So, given any matrix A therefore, we form the matrix A minus lambda I first and then take its determinant and equal to 0 and get all roots of this equation. The roots of this equation are exactly the Eigen values of the matrix A. Eigen values are also called characteristic root sometime, because of this second. So, Eigen values of A are sometime called characteristic equations sorry characteristic roots, Eigen values of A are sometimes called characteristic roots. Then next we will see how to find Eigen vectors corresponding to an Eigen value of the matrix A. If lambda 0 is an Eigen value of A then all non zero solutions of the system A minus lambda 0 I x is equal to 0 are the eigenvectors corresponding to.

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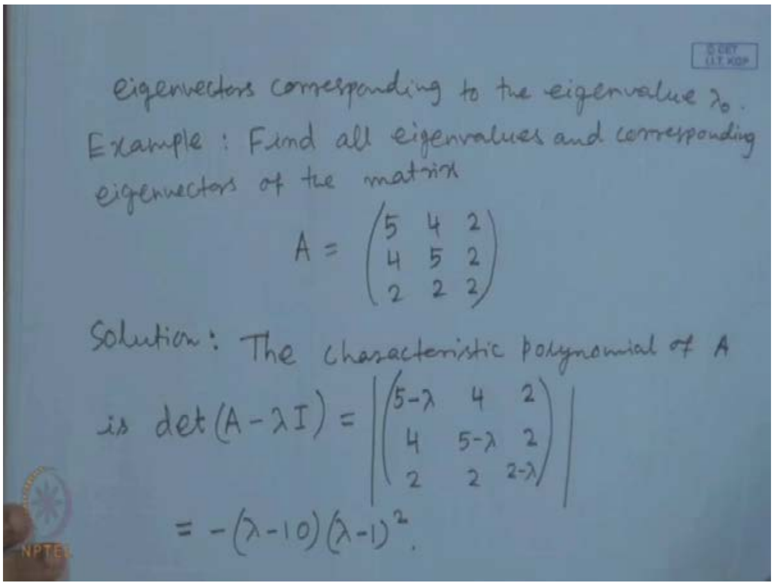

Are the eigenvectors corresponding to the Eigen value lambda 0. So, next let us see one example, where you find Eigen values and eigenvectors of matrices. So, find all Eigen values and Eigen and corresponding eigenvectors of the matrix A, that is given by 5 4 2 4 5 2 2 2 2. So, we solve like this, solution: first we find characteristic polynomial of A, the characteristic polynomial of A is determinant of A minus lambda I, here we consider I be 3 by 3 identity matrix. So, the given matrix thus A minus lambda I is therefore, 5 minus lambda 4 2 4 5 minus lambda 2 2 2 2 minus lambda and here to find out it is determinant, determinant of this matrix. And that is one gets like this after computation, one can find the determinant will like this, minus lambda minus 10 into lambda minus 1 whole square.

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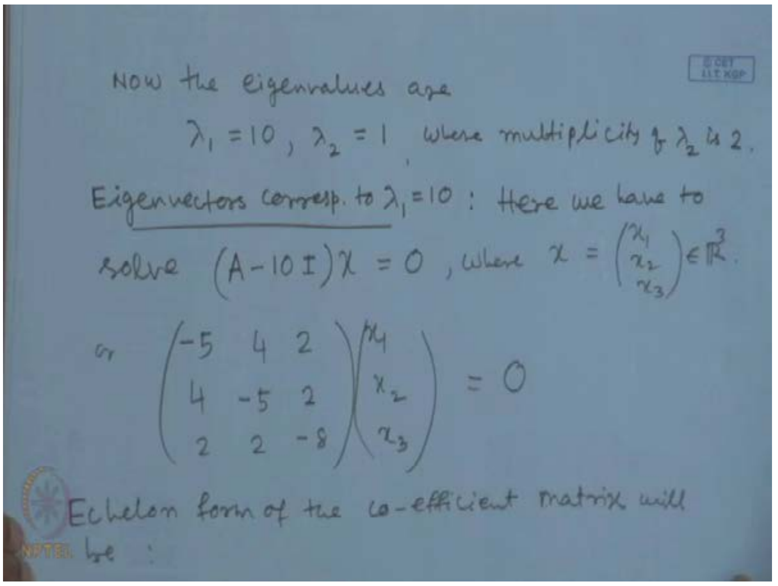

So, the roots of this equation determinant of A minus lambda I are this lambda 1 is 10 lambda 2. So, now the Eigen values are lambda 1 equal to 10 lambda 2 is equal to 1 with multiplicity 2, where multiplicity of lambda 2 is 2. So, next we shall find out eigenvectors corresponding to this Eigen values. So, eigenvectors corresponding to lambda 1 that is equal to 10 so, we find like this here, we have to solve the system, we have to solve this homogenous system A minus 10 I into this x is equal to 0, where x is this x 1 x 2 x 3 this is 3 components this is a vector in R 3 you can take. So, that is the system is like this A minus 10 I is like this, minus 5 4 2 4 minus 5 2 2 2 minus 8 and this x 1 x 2 x 3 so, this system is like this, this is equal to 0. Now, we consider echelon form of the coefficient matrix, are to find all solutions. So, the echelon form of echelon form of the co-efficient matrix will be like this.

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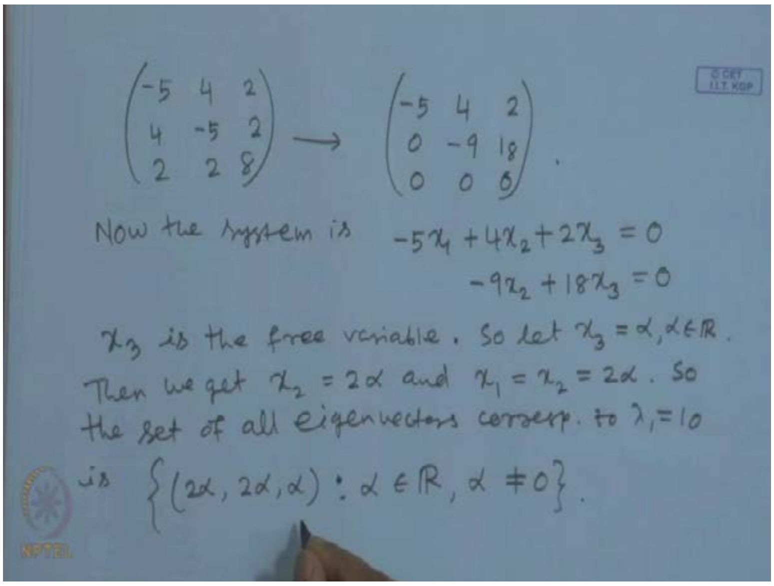

Here, this coefficient matrix is minus 5 4 2 4 minus 5 2 2 2 8 and on the echelon form of this is like this, minus 5 4 2 0 minus 9 18 0 0. So, the system of the equation is now the system is minus 5 x 1 plus 4 x 2 plus 2 x 3 is equal to 0, minus 9 x 2 plus 18 x 3 is equal to 0. Here, we consider x 3 is the free variable that is a variable which is free to take any value in the field. So, let x 3 is equal to alpha, alpha belongs to R, then we get x 2 is equal to twice alpha and x 1 is also equal to that is x 2 and it is equal to twice alpha. So, the set of all eigenvectors corresponding to lambda 1 is equal to 10 is this set, where it is consist of twice alpha, twice alpha, alpha; alpha belongs to R and alpha is not equal to 0. If alpha equal to 0 then we get this vector be a zero vector and zero vector cannot be an eigenvector. So, this is the set of all eigenvectors, corresponding to the Eigen value lambda 1 is equal to 10.

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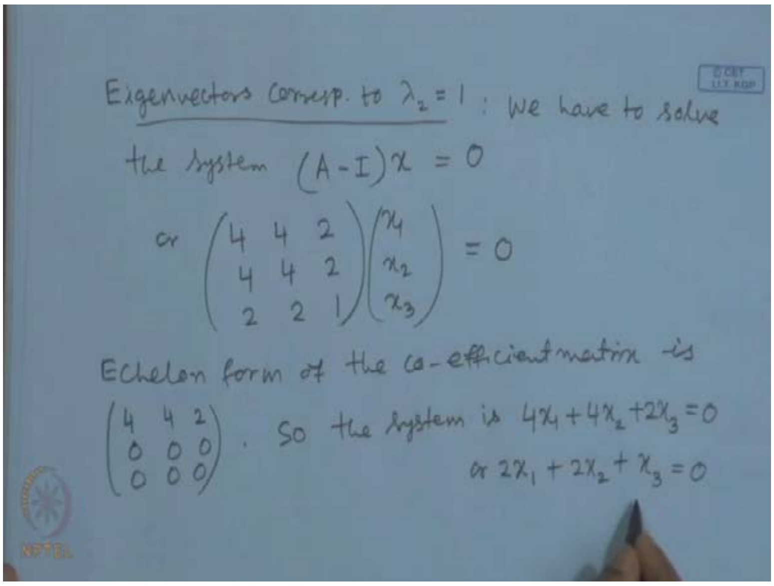

So, next we will find this eigenvector corresponding to lambda 2 that is equal to 1. So, here to solve the system A minus I times x is equal to 0 or it is 4 4 2 4 4 2 2 2 1 x 1 x 2 x 3 that is equal to 0. And we get the echelon form of the coefficient matrix, we will like this echelon form of the coefficient matrix is given by 4 4 2 0 0 0 0 0 0. So, the system will be, the system is given by 4 x 1 plus 4 x 2 plus 2 x 3 is equal to 0 or 2 x 1 plus 2 x 2 plus x 3 is equal to 0.

So, here we are having only one equation and three variables. So, two variables are free variables and x 2 and x 3 are therefore, free variables here, x 2 and x 3 are free variables. So, let us take that x 2 is equal to alpha and x 3 is equal to beta where alpha and beta they come from this certain real numbers, because we are considering real system, system over real numbers.

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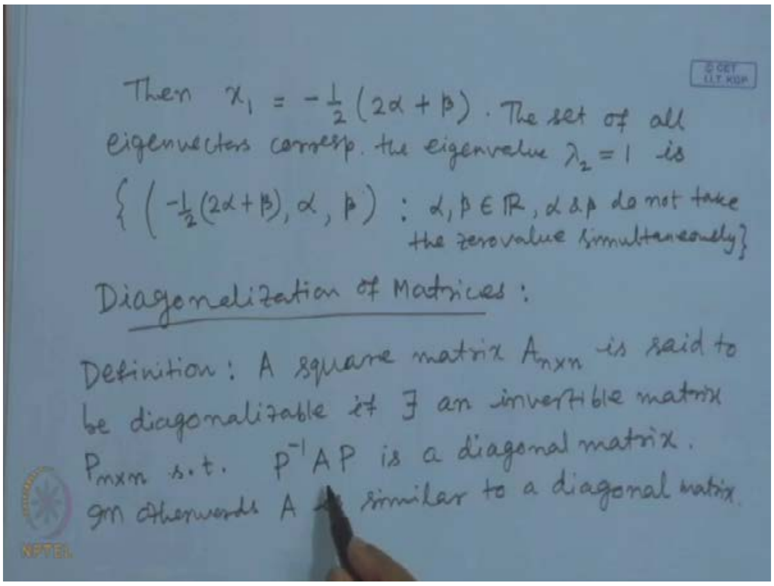

So, then we get x 1 be like this, that is minus half into twice alpha plus beta. So, now the set of all eigenvectors corresponding to the Eigen value lambda 2 is equal to 1 is given by this minus half into twice alpha plus beta, alpha, beta. Where alpha beta belongs to the set of real numbers and alpha and beta do not take the zero value simultaneously otherwise this vector will be a zero vector and that cannot be an Eigen vector. So, this is how we find Eigen values and eigenvectors of matrices. So, next we shall discuss about another important concept that is diagonalization of matrices.

Here also, we consider only square matrices that, let us see first what is the meaning of diagonalization of matrices that we given this definition, A square matrix A of size n is said to be diagonalizable if, there exist and invertible matrix P of size n by n such that, P inverse A P is a diagonal matrix. In other words this A is similar to a diagonal matrix. So, not all matrices; not all square matrices are diagonal matrices, that is not for any diagonal sorry not for any square matrix A we get an invertible matrix P so that we will have this P inverse A P will be similar, this will be a diagonal matrix. So, here we shall see have two more on definitions or terminologies for discussion of this diagonalization of matrices.

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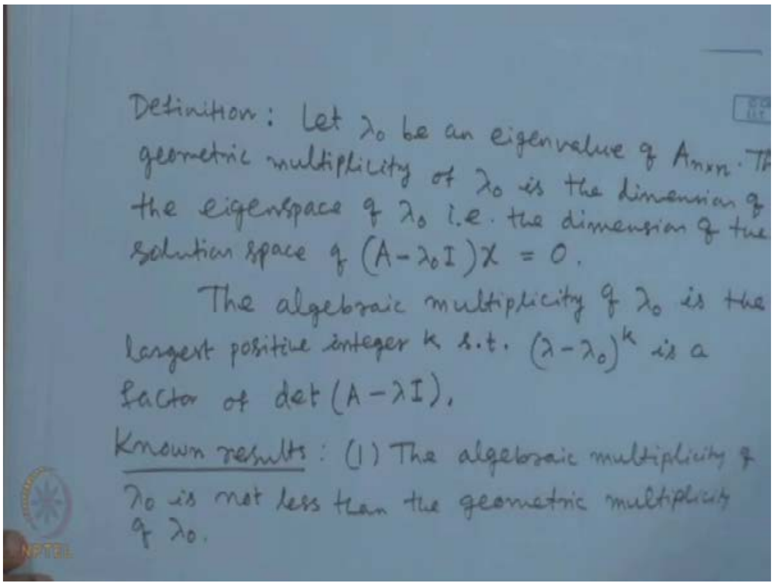

So, let us see that definition of multiplicity, geometric multiplicity and algebraic multiplicity of an Eigen value. So, let lambda 0 be an Eigen value of a matrix A of size n by n. Then the geometric multiplicity of lambda 0 is the dimension of the Eigen space of lambda 0. That is the dimension of the solution space of this homogenous system A minus lambda 0 I x equal to zero. Here also, we will find algebraic multiplicity of lambda 0, that is the algebraic multiplicity of lambda 0 is the largest positive integer k such that lambda minus lambda 0 whole to the power k is a factor of this characteristic polynomial that A minus lambda I are in other words.

This is a multiplicity lambda 0 is the multiplicity of I mean, this algebraic multiplicity of lambda 0 it is actually multiplicity of lambda 0 is a root of this characteristic equation. So, next we will just states some known results on this geometric multiplicity and algebraic multiplicities known results and also that relation among Eigen vectors corresponding to different Eigen values. So, the first one is like this the algebraic multiplicity is greater than or equal to the geometric multiplicity of an Eigen value. That the algebraic multiplicity of lambda 0 is not less than the geometric multiplicity of lambda 0, then next we have another important result that we shall use is that.

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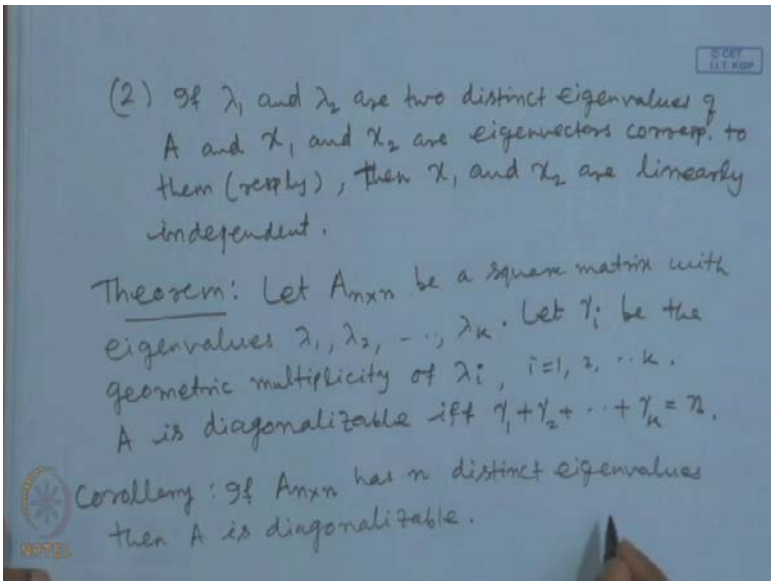

If lambda 1 and lambda 2 are two distinct Eigen values of a and x 1 and x 2 are eigenvectors corresponding to them respectively, then x 1 and x 2 are linearly independent. That is eigenvectors corresponding to distinct Eigen values are linearly independent one can prove this easily. So, next we shall use this results and a find this diagonalization of matrices of course, here important criteria is tells about diagonalization of matrices. And here is this result that tells a necessary and sufficient condition for diagonalization of matrices.

So, let this A be a matrix of size invariant and be a square matrix, the square matrix with eigenvalues lambda 1, lambda 2, lambda k. Let gamma i be the geometric multiplicity of lambda i, i from 1, 2 to k. Then A is diagonalizable if and only if, this gamma 1 plus gamma 2 plus gamma k is equal to n that is the size of this matrix. Or in other words we can also relate this A matrix is diagonalizable if and only if, this algebraic multiplicity is equal to the geometric multiplicity of every Eigen value. So, here we are having a corollary that, one gets if this matrix of size n by n has n distinct Eigen values then A is diagonalizable, because the geometric multiplicity of an Eigen value is at least 1.

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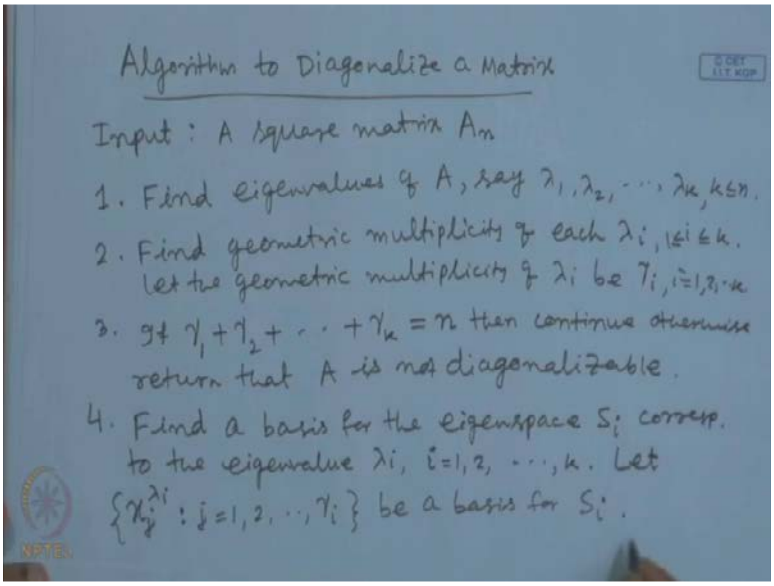

So, the next we will see, a method for diagonalization of matrices are, that we write is in algorithm to diagonalizable a matrix. So, here our input is a square matrix A of size n, then first take is this, we find eigenvalues of A say lambda 1, lambda 2, lambda k here, k is less than or equal to 1 of course. Then second step is we find geometric multiplicity of each lambda I, i from 1 to k. Let the geometric multiplicity of lambda i be gamma i, i from 1, 2 to k. Then third step is that we check, whether the sum of the geometric multiplicities that each gamma 1 plus gamma 2 up to gamma k plus gamma k is equal to n, then we continue this; then continue otherwise we return that k is not diagonalizable.

So, in fourth step we find, we know the geometric multiplicity of is eigenvalue and sum of the geometric multiplicities equal to n then here, for each Eigen value we shall find a basis for the eigenspace. So, here find this is, find a basis for the eigenspace S i corresponding to the eigenvalue lambda I, for i 1, 2 up to k. So, let the basis be; let this x j lambda i such that, j is equal to 1, 2 to gamma i be a basis for S i. So, dimension of S i is gamma i. So, therefore, in this set we have this gamma i number of vectors and this is basis for this eigenspace corresponding to the eigenvalue lambda i. So, here is i is therefore, from 1, 2 to k for each lambda i we consider a basis for it is eigenspace S i.

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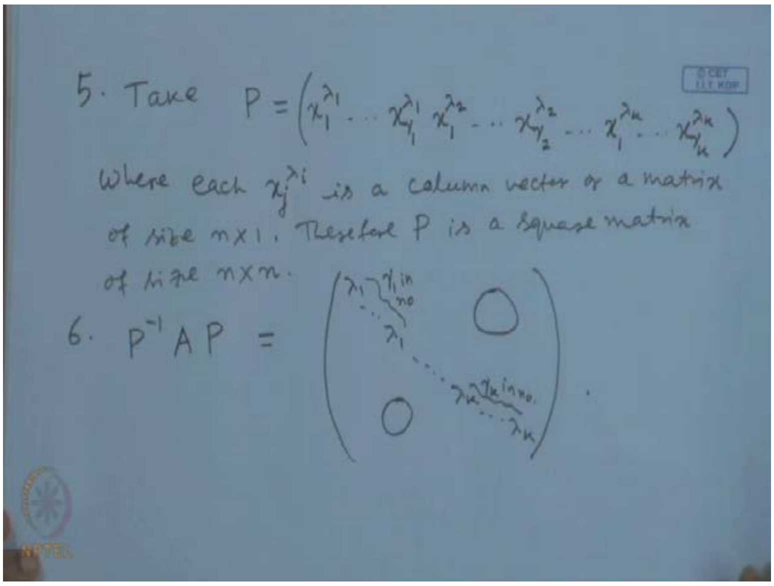

So, now this in this step, we find that invertible matrix P the like this take P be, the matrix that it is consist of x 1 lambda 1 up to this x gamma 1 lambda 1, then x 1 lambda 2 x gamma 2 lambda 2 like this, this x 1 lambda k up to this x gamma k lambda k. That is; this is a matrix of size n by n a is where each x j lambda i is a column vector or a matrix of size n by 1. Therefore, P is a square matrix of size n by n, what is that this P each invertible matrix, because all this columns in p are linearly independent and hence this rank of this P is equal to n and therefore, it is invertible.

So, now we get that this P inverse A P is equal to the matrix where in this diagonal [host/first] this gamma 1 in number they will be gamma 1 in number or lambda 1 then lambda 2 like this, this lambda k there gamma k in number until rest are the zero matrices.

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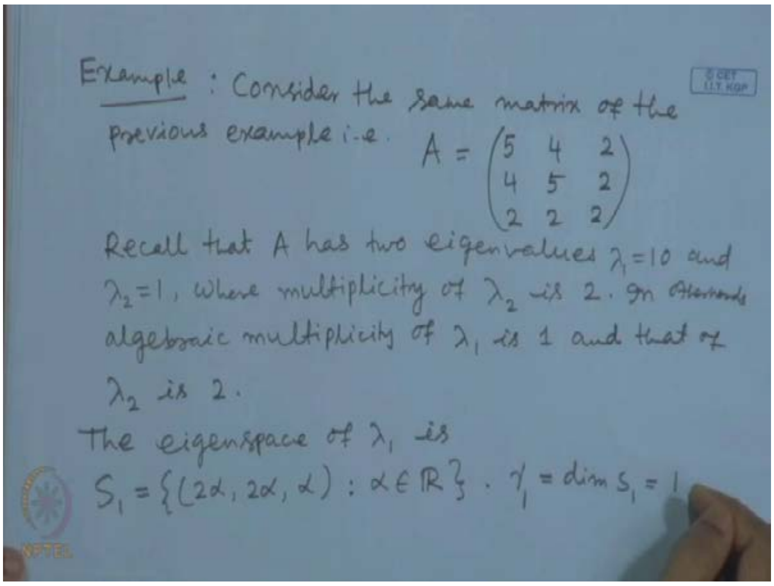

So, this is how we find a; that diagonalization of this matrix A. Let us see this example to support this algorithm, let us help this example here, we consider the same matrix of the previous example. That is A is this matrix 5 4 2 4 5 2 2 2 2 and also recall that this is two eigenvalues; recall that A has two eigen values lambda 1 is equal to 10 and lambda 2 is equal to 1, where lambda multiplicity of lambda 2 is 2 or in other words algebraic multiplicity of lambda 2 is 2. In other words algebraic multiplicity of lambda 1 is 1 and that of lamda2 is 2. So, the next we shall find geometric multiplicity of lambda 1 and lambda 2.

So, again recall that the Eigen space of lambda 1 is the set S 1 that is consist of twice alpha, twice alpha, alpha. So, is that alpha belongs to R. So, here including the zero vector so therefore, we do not take alpha is non zero. So, notice that dimension of that is gamma 1 is dimension of S 1 that is equal to 1. This also we get again from that it value take here on. So, next we consider the eigenspace of lambda 2. So, the eigenspace of lambda 2 is the set S 2, it is consist of minus half into twice alpha plus beta, alpha, beta here, alpha beta are real numbers and the dimension of S 2, that is geometric multiplicity of lambda 2.

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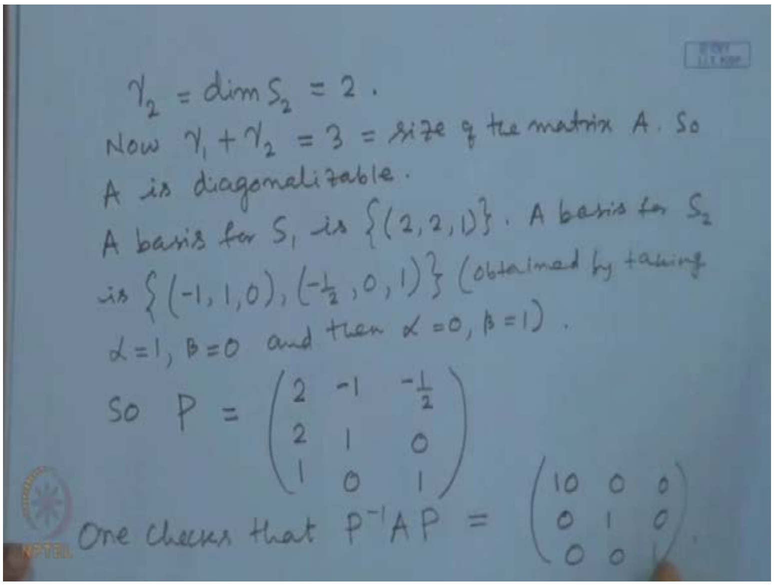

Gamma 2 is dimension of s 2 that is equal to 2. Now, this gamma 1 plus gamma 2 that is equal to 3, that is size of matrix a size of the matrix A. So, this matrix A is diagonalizable. So, next we shall find that invertible matrix P. So, for that to you have to find a basis for S 1. So, a basis for S 1 is its dimension is 1 so, we can take any non zero vector in this one and that will be a basis for S 1. Then a basis for S 2 is that its dimension is 2 so therefore, it is consist of two linearly independent vectors minus 1, 1, 0 and minus half, 0, 1.

That we have obtained by taking, that has been obtained by taking alpha equal to 1 beta equal to 0 and then alpha equal to 0 beta equal to 1. Now, the matrix P will be like this, we take the basis in S 1 and S 2 is columns of this matrix P. So therefore, first column will be basis for S 1 that 2 2 1 and then we write basis of S 2. So, that is minus 1 1 0 minus half 0 1. So now, one can check that this P inverse A P is the matrix this, the diagonal matrix 10 0 0 0 1 0 0 0 1, this is we find this matrix P to diagonalizable the given matrix A.

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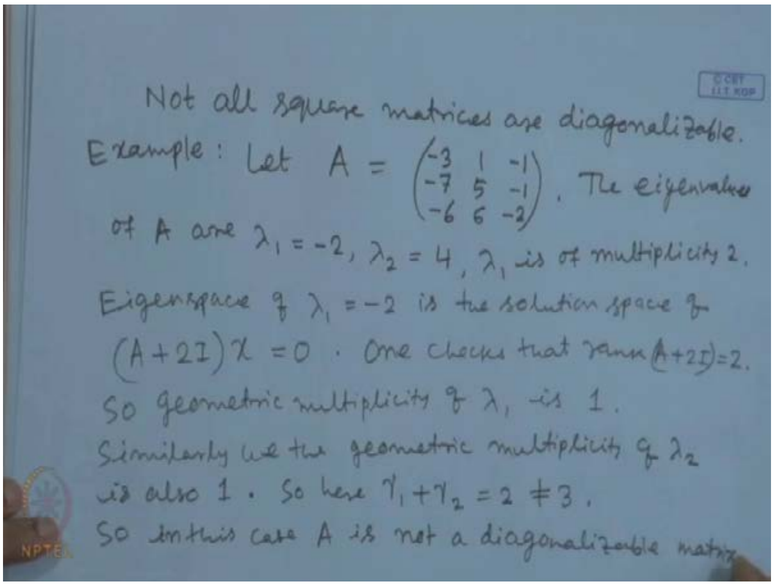

Next will see one example that, not all matrices are diagonalizable, not all square matrices are diagonalizable. So, let us see one example of it that here we consider the matrix A be like this, that minus 3 1 minus 1 minus 7 5 minus 1 minus 6 6 minus 2. So, here one checks that the eigenvalues will like this eigenvalues of A are lambda 1 is equal to minus 2, lambda 2 is equal to 4 and lambda 1 is of multiplicity 2. So, eigenspace of lambda 1 is the solution space of here, eigenspace of lambda 1 is the solutions space of the system A plus twice I x equal to 0. Here this rank of 1 can see that, this rank of A plus 2 I is equal to 2.

So, one checks that rank of A plus twice I is equal to 2. So, geometric multiplicity of lambda 1 is 1 similarly, we get the geometric multiplicity of lambda 2 is also 1. Therefore so here, gamma 1 plus gamma 2 is equal to 2 and that is not equal to 3. So, in this case A is not a diagonalizable matrix, and that (( )) one, this lecture we start here and this diagonalization of matrices there also useful. And in the next lecture, we shall discuss that using diagonalizable matrices, we can also compute higher power of matrices that (( )) for this lecture. Thank you.